THE MAGNETIC STRESS TENSOR IN MAGNETIZED MATTER

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To $\frac{B^2}{8\pi}$ or not to $\frac{B^2}{8\pi}$, *that* is the question!
(adapted from J. Horvath's Concluding Remarks, with apologies to W. Shakespeare)

We derive the form of the magnetic stress tensor in a completely general, stationary magnetic medium, with an arbitrary magnetization field $\vec{M}(\vec{r})$ and free current density $\vec{j}(\vec{r})$. We start with the magnetic force density $\vec{f}$ acting on a matter element, modelled as a collection of microscopic magnetic dipoles in addition to the free currents. We show that there is an unique tensor $\mathbf{T}$ quadratic in the magnetic flux density $\vec{B}(\vec{r})$ and the magnetic field $\vec{H}(\vec{r}) = \vec{B} - 4\pi\vec{M}$ whose divergence is $\nabla \cdot \mathbf{T} = \vec{f}$. In the limit $\vec{M} = 0$, the well-known vacuum magnetic stress tensor is recovered. However, the general form of the tensor is asymmetric, leading to a divergent angular acceleration for matter elements of vanishing size. We argue that this is not inconsistent, because it occurs only if $\vec{M}$ and $\vec{B}$ are not parallel, in which case the macroscopic field does indeed exert a torque on each of the microscopic dipoles, so this state is only possible if there are material stresses which keep the dipoles aligned with each other and misaligned with the macroscopic field. We briefly discuss the consequences for the stability of strongly magnetized stars.

1 Introduction

The equilibrium thermodynamics of magnetized matter is important to understand the structure and evolution of compact astrophysical objects such as white dwarfs and neutron stars. Of particular relevance is the pressure, since it enters in an important way in the equations that determine the structure of the star (Newtonian hydrostatic equilibrium equation or its relativistic Oppenheimer-Volkov generalization). If the gas that conforms the star is made of electromagnetically active matter (i.e., ionized matter or magnetic dipoles) in the presence of macroscopic electromagnetic fields, the total pressure will have both a kinetic and an electromagnetic components.

There seems to be some confusion and disagreement about the nature of the latter (and also about the isotropy of the total pressure). For instance, recent papers[1,2,3] claim that, for a magnetization parallel to the magnetic field, the electromagnetic contributions to the pressure take the form

$$P_{\parallel} = 0 \; ; \qquad P_{\perp} = -MB \,, \tag{1}$$

where $\parallel$ and $\perp$ refer to the direction of the magnetic field. The authors of Ref.[3] claim that, with an electromagnetic pressure of this form and some assumptions about the growth of the magnetic field from the surface of the star towards its center, magnetars with surface fields of the order of 10^{15} Gauss would be unstable to collapse.

However, the correct electromagnetic pressure in the case of no magnetization, as derived from the Maxwell stress tensor for fields in vacuum is

$$P_{\parallel}^{\mathrm{vac}} = -\frac{B^2}{8\pi}, \qquad P_{\perp}^{\mathrm{vac}} = \frac{B^2}{8\pi}, \tag{2}$$

which clearly does not correspond to the zero magnetization limit of expressions (1).

In this work we address this problem by studying the electromagnetic force on a piece of magnetized matter from a purely classical point of view and constructing thereby the corresponding magnetic stress tensor for the system. Our results imply that the electromagnetic contribution to the pressure, defined as the negative of the diagonal components of the stress tensor, in the case of parallel magnetization and magnetic field are given by

$$P_{\parallel} = -\frac{B^2}{8\pi}, \qquad P_{\perp} = \frac{B^2}{8\pi} - MB \,, \tag{3}$$

which have the correct vacuum limit (2) and, naturally, disagrees with Eq.(1).

Other works where the problem of the electromagnetic pressure has been touched upon may be found in Refs.[4,5,6].

2 Magnetic Force Density on Matter

Macroscopic matter interacts with a magnetic field through its electrical currents (due to the motion of free charges) and macroscopic magnetization (due to the alignment of microscopic magnetic dipoles, usually associated with quantized spins).

The force density (force per unit volume) on the currents is well-known to be $\vec{f}^{\rm curr} = \vec{j} \times \vec{B}/c$, where $\vec{j}(\vec{r})$ is the electric current density, and $\vec{B}(\vec{r})$ is the magnetic flux density.

The force on the dipoles can be found by considering the force on a single dipole of moment $\vec{m}$, whose Cartesian components are $F_i = m_j \nabla_i B_j$. Adding the dipole moments in a small volume, we obtain the net force density, $f_i^{mag} = M_j \nabla_i B_j$, so the total force density can be written as[7]

$$f_i = \frac{1}{c} \epsilon_{ijk} j_j B_k + M_j \nabla_i B_j \,, \tag{4}$$

where ϵ_{ijk} are the components of the totally antisymmetric (Levi-Civita) tensor, and the Einstein convention of summation over repeated indices is understood here and in what follows.

We emphasize that this force density is *not* the same as the one obtained from defining a "magnetization current" $\vec{j}^{mag} \equiv \nabla \times \vec{M}$ and writing $\vec{f}^{mag'} = \vec{j}^{mag} \times \vec{B}/c$, although they agree when integrated over a bounded material body. This is most clearly seen in the simple case of an uniformly magnetized medium in an uniform magnetic field. Our expression for f_i^{mag} gives an uniform force per unit volume, as physically expected from the uniform distribution of dipoles acted on by the magnetic field. The alternative expression, for $\vec{f}^{mag'}$, gives a vanishing force in the interior of the body (zero "magnetization current" in the interior), but a compensating force on the surface.

3 The Magnetic Stress Tensor

3.1 Derivation

The component T_{ij} of the stress tensor is generally defined as minus the flux in the $j-$direction of the $i-$component of the linear momentum. The force density $\vec{f}$ on a matter element is the time derivative of its momentum density, i.e., minus the divergence of the momentum flux, thus

$$\vec{f} = \nabla \cdot \mathbf{T}, \qquad \text{or} \qquad f_i = \nabla_j T_{ij} \,. \tag{5}$$

Using Maxwell's equations

$$\nabla \cdot \vec{B} = 0 \qquad \text{and} \qquad \nabla \times \vec{H} = \frac{4\pi}{c} \vec{j} \,, \tag{6}$$

together with the definition $\vec{H} = \vec{B} - 4\pi\vec{M}$, Eq. (4) can be manipulated to write $\vec{f}$ as the divergence of a possible stress tensor, with components

$$T_{ij} = \frac{1}{4\pi}\left[H_i B_j - \left(\vec{H}\cdot\vec{B} - \frac{1}{2}B^2\right)\delta_{ij}\right] \quad (7)$$

$$= \frac{B_i B_j}{4\pi} - \frac{B^2}{8\pi}\delta_{ij} - M_i B_j + \vec{M}\cdot\vec{B}\delta_{ij}\,. \quad (8)$$

3.2 Uniqueness?

Of course, the form just obtained is not unique, since any tensor with vanishing divergence can be added to $\mathbf{T}$ without changing Eq. (5). Thus, we search for a divergence-free tensor $\tilde{\mathbf{T}}$, quadratic in the magnetic fields ($\vec{B}$ and $\vec{H}$), whose general form is

$$\tilde{T}_{ij} = \alpha_{ijkl}H_k H_l + \beta_{ijkl}H_k B_l + \gamma_{ijkl}B_k B_l\,, \quad (9)$$

where α_{ijkl}, β_{ijkl}, and γ_{ijkl} are components of constant tensors. These tensors must be rotationally invariant (in order not to introduce preferred directions other than those defined by the magnetic fields), which constrains them to be linear combinations of $\delta_{ij}\delta_{kl}$, $\delta_{ik}\delta_{jl}$, and $\delta_{il}\delta_{jk}$. Thus, we write

$$\tilde{T}_{ij} = (\alpha\vec{H}^2 + \beta\vec{H}\cdot\vec{B} + \gamma\vec{B}^2)\delta_{ij} + \alpha' H_i H_j + \beta' H_i B_j + \beta'' B_i H_j + \gamma' B_i B_j, \quad (10)$$

where α, β, γ, α', β', β'', and γ' are constant coefficients to be adjusted so as to make the tensor divergence-free for any choice of the vector fields $\vec{H}(\vec{r})$ and $\vec{B}(\vec{r})$, with the only restriction that $\nabla\cdot\vec{B} = 0$, i.e., $\nabla_j B_j = 0$.

In particular, we may first choose $\vec{B} = 0$ and $\vec{H} = x_a\hat{e}_b$ (i.e., $H_k = x_a\delta_{kb}$), where a and b are any *given* two different Cartesian indices, which implies

$$\nabla_j\tilde{T}_{ij} = 2\alpha x_a\delta_{ia} = 0 \quad (11)$$

(no summation over a), so $\alpha = 0$. Changing $\vec{H}$ to be any other vector field with $\nabla\cdot\vec{H} = 0$ but $(\vec{H}\cdot\nabla)\vec{H} \neq 0$ gives $\alpha' = 0$. Exchanging the roles of $\vec{H}$ and $\vec{B}$ in the arguments just given, we obtain $\gamma = \gamma' = 0$. Finally, putting $\vec{H} = x_a\hat{e}_b$ and $\vec{B} = x_c\hat{e}_d$ with different combinations of values for a, b, c, d (respecting the constraint that $c \neq d$ to satisfy $\nabla\cdot\vec{B} = 0$) implies $\beta = \beta' = \beta''$.

Thus, among tensors quadratic in $\vec{H}$ and $\vec{B}$, the one we originally obtained is unique in giving the correct magnetic force density.

3.3 Vacuum Limit

In the absence of magnetization, the stress tensor given by Eq. (7) takes the form

$$T_{ij} = \frac{B_i B_j}{4\pi} - \frac{B^2}{8\pi}\delta_{ij}\,, \tag{12}$$

which is the correct, symmetric, well-known[8] form in vacuum.

3.4 Asymmetry and Magnetic Torque

For arbitrary, non-parallel $\vec{H}$ and $\vec{B}$, the stress tensor is manifestly non-symmetric, $T_{ij} \neq T_{ji}$. It is usually argued[9] that the stress tensor must be symmetric, on the ground that a non-symmetric stress tensor would produce a torque on a small matter element which decreases less quickly with decreasing volume than the moment of inertia of the element, thus producing a divergent angular acceleration. We note that this argument applies to the *total* stress tensor, which could be composed of (in principle non-symmetric) pieces of different physical origin.

In the present case, the asymmetry is present only if $\vec{H}$ and $\vec{B}$ or, equivalently, $\vec{M}$ and $\vec{B}$ are not collinear. This means that the microscopic dipoles are not aligned with the macroscopic field, which therefore indeed causes a torque on each dipole, which adds up to a torque per unit volume $\vec{\tau} = \vec{M} \times \vec{B}$. As a consistency check, we calculate a component of the torque density in terms of the stress tensor (as in Ref.[9], but note the opposite sign convention for the stress tensor),

$$\tau_i = -\epsilon_{ijk} T_{jk} = -\epsilon_{ijk} H_j B_k/(4\pi)\,, \tag{13}$$

thus

$$\vec{\tau} = -\vec{H} \times \vec{B}/(4\pi) = \vec{M} \times \vec{B}\,, \tag{14}$$

completing the consistency check.

Of course, if there is no counter-acting torque (for example due to microscopic interactions among neighboring dipoles), the dipoles will orient themselves along $\vec{B}$, in which case the stress tensor becomes symmetric and the torque disappears. If there are microscopic interactions keeping the dipoles locally aligned and at fixed positions with respect to each other, then a macroscopic matter element may act as a rigid body with a finite angular acceleration, that can not be subdivided into infinitesimal pieces with a divergent angular acceleration, because their magnetic torques are cancelled by the local microscopic interactions.

3.5 Magnetized Fluid and Anisotropic Pressure

In a stationary, magnetized fluid, there are no microscopic forces keeping the the magnetization misaligned with the magnetic field; therefore $\vec{M}(\vec{r})$, $\vec{B}(\vec{r})$, and $\vec{H}(\vec{r})$ are collinear at every point $\vec{r}$. Choosing a local Cartesian basis with the $z-$axis aligned in the same direction, the stress tensor becomes diagonal, with components $T_{xx} = T_{yy} = -P_\perp = MB - B^2/(8\pi)$ and $T_{zz} = -P_\| = B^2/(8\pi)$. Thus, the parallel pressure $P_\| < 0$ always, corresponding to the usual tension along field lines, while $P_\perp > 0$ as usual, unless $M \geq B/(8\pi)$.

4 Conclusions

On the basis of physical arguments, we have found a magnetic stress tensor in magnetized matter that, although manifestly asymmetric, has the correct vacuum limit and appears to be consistent with all conceptual tests that we have applied. For the special case of a magnetized fluid, with magnetization parallel to the macroscopic field, we show that the longitudinal pressure remains negative and, unless the magnetization is extremely strong, the perpendicular pressure remains positive. Thus, a neutron star with an extremely strong magnetic field will definitely *not* collapse to a *prolate* structure, as has been proposed[2,3], but, if anything, to an *oblate* structure, assuming that the star's fluid pressure does not impede it.

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References

1. M. Chaichian *et al.*, Phys. Rev. Lett. 84, 5261 (2000).
2. R. González Felipe *et al.*, *Quantum instability of magnetized stellar object*, astro-ph/0207150 (2002).
3. A. Pérez Martínez, H. Pérez Rojas and H. J. Mosquera Cuesta, Eur. Phys. J. **C** DOI: 10.1140/epjc/s2003-01192-6 (2003), and these Proceedings.
4. V. Canuto, Phys. Rev. **A**3, 648 (1971).
5. I. Easson and C. J .Pethick, Phys. Rev. **D**16, 275 (1977).
6. R.D. Blandford and L. Hernquist, J. Phys. **C**15, 6233 (1982).
7. This is in agreement with C. J. Pethick, in Structure and Evolution of Neutron Stars, eds. D. Pines, R. Tamagaki, & S. Tsuruta (Redwood City: Addison Wesley), p. 115 (1992).
8. J. D. Jackson, Classical Electrodynamics, 2nd. Edition, John Wiley & Sons (1975).
9. B. F. Schutz, A first course in general relativity, Cambridge University Press (1985).