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Dynamics of ferromagnetic spherical spin models with power law interactions: exact solution

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Abstract

We solve the Langevin dynamics of d-dimensional ferromagnetic spherical models with interactions that decay with distance as $r^{-(d+\sigma)}$. The long time dynamics of correlations and responses are studied in detail in the different dynamical regimes and the validity of fluctuation–dissipation relations (or its violation) are shown. In particular, we show that the fluctuation–dissipation ratio $X(t+t_w, t_w)$ is asymptotically a function only of the waiting time t_w in the aging regime and that $X \to 0$ as $t_w \to \infty$. The results are valid in any finite dimension d and for $0<\sigma<2$ where short-range behavior is recovered. We also solve the $T=0$ Cahn–Hilliard dynamics of this model (conserved order parameter). An analysis of the multiscaling behavior of the auto-correlation function is presented.

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1. Introduction

Systems with long-range interactions are very common in nature. Some important examples are dipolar systems in which the interaction decays with distance as $1/r^3$

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[1–3]; charged systems with Coulomb interactions $\propto 1/r$ [4]; spin glasses characterized by RKKY interactions [5]; block copolymers [6,7] and models of structural glasses [8], to name a few. In spite of their ubiquity these kind of systems normally receive much less attention than short-range ones, perhaps because of its greater analytical complexity. Power law decaying interactions interpolate between the much studied, although not realistic, mean field limit and the strictly local nearest neighbor interactions. Although short-range interactions are assumed to be dominant in a great variety of systems, this may not be so in many others, spin glasses being a known and controversial example.

In pure systems, the dynamics after a quench in temperature from the high temperature phase to the low ordered one proceeds by a slow coarsening of domains characteristic of the low energy excitations of the system. This coarsening process is characterized by growth laws which show how the typical length scales associated with the domains grow with time. Typical growing laws are $l(t) \propto t^{1/z}$ with z a dynamic exponent characteristic of the universality class of the system. Order parameters, correlation and response functions also show typical scaling behavior. Exponents and scaling functions also depend on the dynamics being with conserved or non conserved order parameter and also on the nature of the order parameter. From here on we will discuss only results for the vector order parameter case ($n > 1$). In the case of long-range forces decaying as $r^{-(d+\sigma)}$, a calculation of the $n \to \infty$ limit of the $O(n)$ model [9] gives a growth law $l(t) \propto t^{1/\sigma}$, for nonconserved order parameter. In the conserved case there appears the phenomenon of "multiscaling" with two characteristic length scales: $l_1 \propto t^{1/(2+\sigma)}$ and $l_2 \propto (t/\ln t)^{1/(2+\sigma)}$. Here we rederive these results working directly on the spherical model at finite temperature and for general spatial dimension d. We will also derive the scaling forms for the structure factor, two times autocorrelations and associated responses. In recent years much attention has been devoted to a possible extension of the fluctuation–dissipation relations to the case of nonequilibrium dynamics. It turns out that, in many cases, the fluctuation–dissipation theorem (FDT) may be generalized by introducing the concept of time dependent "effective temperatures" [10]. These are difficult to obtain both analytically and also to be measured in experiments. The simplicity of the spherical model permits us to analyze the character of the violation of the FDT during the coarsening process and to obtain explicitly the effective temperature.

It is known that the ferromagnetic spherical model with long-range interactions has a phase transition in all dimensions d (contrary to the short-range model which only has transition for $d > 2$) [11]. In $d = 1$ a ferromagnetic phase is present at finite temperature provided that $0 < \sigma < 1$ and also in $d = 2$ provided that $0 < \sigma < 2$. Our results concerning the ordering dynamics after a quench from the high temperature phase to the low temperature one are valid for every σ such that $0 < \sigma < s$, where $s = d$ for $d \leqslant 2$ and $s = 2$ for $d > 2$. The case of $\sigma = 2$ will be excluded as the systems in $d = 1$ and 2 have no phase transitions in this case. For $d > 2$ and $\sigma > 2$ the system has the same critical properties of the model with short-range interactions.

2. The model

We consider spherical spin models consisting of N continuous spins $s_i(t)\,(i=1,\dots,N)$ which satisfy for all times t the spherical constraint

$$\sum_{i=1}^{N} s_i^2(t)=N\,. \tag{1}$$

The Hamiltonian of the system is given by

$$\mathscr{H}=-\sum_{(i,j)} J(\mathbf{r}_{ij})s_i s_j\,, \tag{2}$$

where the sum runs over all distinct pairs (i,j) of spins of a d-dimensional hypercubic lattice and $\mathbf{r}_{ij}$ is the distance vector between sites i and j. The interactions $J(\mathbf{r}_{ij})$ decay as a power law of the distance between a pair of spins in the following way:

$$J(\mathbf{r}_{ij})=J_0\frac{r_{ij}^{-(d+\sigma)}}{\sum_j' r_{ij}^{-(d+\sigma)}} \quad \text{for } i\neq j\,, \tag{3}$$

where $\sum_j'$ runs over all sites $j\neq i$; d is the space dimension, $\sigma>0$, $J_0>0$ and $J(0)=0$. Let us now introduce the Fourier transform of the spin variables $s_i(t)$

$$s_{\mathbf{k}}(t)=\frac{1}{\sqrt{N}}\sum_j s_j(t)\,\mathrm{e}^{\mathrm{i}\mathbf{k}\cdot\mathbf{r}_j} \tag{4}$$

with $s_{-\mathbf{k}}(t)=s_{\mathbf{k}}^*(t)$. Considering periodic boundary conditions the wave vector is $\mathbf{k}=(k_1,\dots,k_d)$ with $k_i=2\pi n_i/L$, $n_i=0,\pm1,\pm2,\dots,\pm(L/2-1),\pm(L/2)$ $(i=1,\dots,d)$ and $N=L^d$. Now defining

$$J(\mathbf{k})\equiv\sum_{\mathbf{r}} J(\mathbf{r})\,\mathrm{e}^{\mathrm{i}\mathbf{k}\cdot\mathbf{r}} \tag{5}$$

the Hamiltonian (2) is diagonalized,

$$\mathscr{H}=-\sum_{\mathbf{k}} J(\mathbf{k})|s_{\mathbf{k}}(t)|^2 \tag{6}$$

with [11,12]

$$J(\mathbf{k})=J_0\frac{\sum_{\mathbf{l}}'|\mathbf{l}|^{-(d+\sigma)}\cos(\mathbf{k}\cdot\mathbf{l})}{\sum_{\mathbf{l}}'|\mathbf{l}|^{-(d+\sigma)}}\,, \tag{7}$$

where the sum $\sum_{\mathbf{l}}'$ is over all lattice vectors $\mathbf{l}\neq0$. The critical temperature of this model is given by [11]

$$\beta_c J_0=\frac{1}{N}\sum_{\mathbf{k}}\frac{1}{1-J(\mathbf{k})/J_0}\,. \tag{8}$$

In the long wavelength scaling limit $k\ll1$ Eq. (7) behaves as [13],

$$J(\mathbf{k})\approx J_0(1-Ck^\sigma+O(k^2))\,, \tag{9}$$

where

$$C = \frac{I_{d,\sigma}}{v_a \Omega_{d,\sigma}(\mathscr{L})} , \tag{10}$$

$$I_{d,\sigma} = \frac{2^{1-\sigma}\pi^{d/2}\Gamma(1-\sigma/2)}{\sigma\Gamma(d/2+\sigma/2)} . \tag{11}$$

v_a is the volume of a unit cell in a d-dimensional Bravais lattice $\mathscr{L}$ and $\Omega_{d,\sigma}(\mathscr{L}) = \sum_{\mathbf{l}}' |\mathbf{l}|^{-(d+\sigma)}$. This approximation is valid for $0<\sigma<2$. At $\sigma=2$ logarithmic corrections are present in $d=2$. In d dimensions and for $\sigma>2$ the leading term in the above expansion is $O(k^2)$ and so the critical properties of the model are those corresponding to short-range interactions. The dynamical properties in this case are also well known (see e.g. Refs. [14,15]). The analysis we present in this work can be extended to the mean field regime $-d \leqslant \sigma<0$. However, several works on related models [16–20] suggest that the critical properties (both statical and dynamical) in the whole interval are the same as those for the case $\sigma=-d$ (mean field). The dynamics for this last case can be almost trivially solved, showing no coarsening effects (for instance, the two-time autocorrelation function always decays exponentially with the difference of times). Hence, we will not discuss this further in this paper.

We will concentrate on the analysis of the correlation and response functions. The two-time autocorrelation function is defined as

$$C(t,t') = \frac{1}{N}\sum_{\mathbf{r}} \langle s_{\mathbf{r}}(t) s_{\mathbf{r}}(t')\rangle \tag{12}$$

$$= \frac{1}{N}\sum_{\mathbf{k}} C_{\mathbf{k}}(t,t') \tag{13}$$

with $C_{\mathbf{k}}(t,t') \equiv \langle s_{\mathbf{k}}(t)s_{-\mathbf{k}}(t')\rangle$. Note that $C_{\mathbf{k}}(t,t) = \sum_{\mathbf{r}} C(\mathbf{r},t)\,\mathrm{e}^{-\mathrm{i}\mathbf{k}\cdot\mathbf{r}}$ is the dynamic structure factor while $C(\mathbf{r},t) \equiv \langle s_0(t)s_{\mathbf{r}}(t)\rangle$ is the spatial correlation function. The two-time response function is defined as

$$G(t,t') \equiv \frac{1}{N}\sum_{\mathbf{r}} \left.\frac{\delta\langle s_{\mathbf{r}}(t)\rangle}{\delta h_{\mathbf{r}}(t')}\right|_{h=0} \tag{14}$$

$$= \frac{1}{N}\sum_{\mathbf{k}} G_{\mathbf{k}}(t,t') \tag{15}$$

with

$$G_{\mathbf{k}}(t,t') = \frac{\delta\langle s_{\mathbf{k}}(t)\rangle}{\delta h_{\mathbf{k}}(t')} , \tag{16}$$

where $h_{\mathbf{r}}(t)$ is an inhomogeneous external magnetic field and

$$h_{\mathbf{k}}(t) = \frac{1}{\sqrt{N}}\sum_{\mathbf{r}} h_{\mathbf{r}}(t)\,\mathrm{e}^{\mathrm{i}\mathbf{k}\cdot\mathbf{r}} .$$

3. Nonconserved order parameter

In this case we consider a Langevin dynamics for the spins:

$$\frac{\partial s_i}{\partial t} = -\frac{\delta \mathscr{H}}{\delta s_i} - z(t)s_i(t) + \xi_i(t)\,, \tag{17}$$

where the Lagrange multiplier $z(t)$ enforces the spherical constraint at each instant and ξ is a Gaussian white noise with moments $\langle \xi_i(t)\rangle = 0$ and $\langle \xi_i(t)\xi_j(t')\rangle = 2T\delta_{ij}\delta(t-t')$ as usual.

The Langevin equation in Fourier space reads

$$\frac{\partial s_{\mathbf{k}}}{\partial t} = (J(\mathbf{k}) - z(t))s_{\mathbf{k}}(t) + \xi_{\mathbf{k}}(t) \tag{18}$$

with $\langle \xi_{\mathbf{k}}(t)\rangle = 0$ and $\langle \xi_{\mathbf{k}}(t)\xi_{\mathbf{k}'}(t')\rangle = 2T\delta_{\mathbf{k},-\mathbf{k}'}\delta(t-t')$. The formal solution of (18) is

$$s_{\mathbf{k}}(t) = s_{\mathbf{k}}(0)\,\mathrm{e}^{J(\mathbf{k})t - \int_0^t z(t')\,\mathrm{d}t'} + \int_0^t \mathrm{e}^{J(\mathbf{k})(t-t') - \int_{t'}^t z(t'')\,\mathrm{d}t''}\,\xi_{\mathbf{k}}(t')\,\mathrm{d}t'\,. \tag{19}$$

From (19) and setting $t > t'$ we get

$$C_{\mathbf{k}}(t,t') = \frac{1}{\sqrt{\Xi(t)\Xi(t')}}\left[C_{\mathbf{k}}(0,0)\,\mathrm{e}^{J(\mathbf{k})(t+t')} + 2T\int_0^{t'} \mathrm{e}^{J(\mathbf{k})(t+t'-2t'')}\Xi(t'')\,\mathrm{d}t''\right]\,, \tag{20}$$

where $C_{\mathbf{k}}(0,0) = \sum_{\mathbf{r}} C(\mathbf{r},0)\,\mathrm{e}^{-\mathrm{i}\mathbf{k}\cdot\mathbf{r}}$ is the initial autocorrelation and

$$\Xi(t) \equiv \mathrm{e}^{2\int_0^t z(t')\,\mathrm{d}t'}\,.$$

Again using (19) we easily obtain

$$G_{\mathbf{k}}(t,t') = \mathrm{e}^{J(\mathbf{k})(t-t')}\sqrt{\frac{\Xi(t')}{\Xi(t)}}\,. \tag{21}$$

From the spherical constraint $C(t,t) = 1\ \forall t$ we obtain a selfconsistent Volterra equation for the function $\Xi(t)$:

$$\Xi(t) = \frac{1}{N}\sum_{\mathbf{k}}\left[C_{\mathbf{k}}(0,0)\,\mathrm{e}^{2J(\mathbf{k})t} + 2T\int_0^t \mathrm{e}^{2J(\mathbf{k})(t-t')}\Xi(t')\,\mathrm{d}t'\right]\,. \tag{22}$$

In order to solve this equation for $\Xi(t)$ we must specify the initial correlation $C(\mathbf{r},0)$. For a random initial configuration with magnetization m_0 we can choose $C(\mathbf{r},0) = \delta_{\mathbf{r},0} + (1-\delta_{\mathbf{r},0})m_0^2$, so $C_{\mathbf{k}}(0,0) = (1-m_0^2) + m_0^2 N\delta_{\mathbf{k},0}$. We will choose $m_0 = 0$ which is an interesting case for studying the phase ordering dynamics after a quench from the disordered to the low temperature ordered phase [21]. With this choice $C_{\mathbf{k}}(0,0) = 1$.

The simplest and most relevant case to study is a quench to zero temperature. It has been shown that scaling functions and exponents are the same in the whole low temperature phase and the role of temperature fluctuations only amounts to a renormalization of the amplitudes [14]. We will show this result emerging clearly from the exact solution of the model at temperatures $T < T_c$.

For $T=0$ we obtain from Eq. (22) in the thermodynamic limit $N\to\infty$

$$\Xi(t)=\frac{1}{N}\sum_{\mathbf{k}} \mathrm{e}^{2J(\mathbf{k})t} \to \frac{1}{(2\pi)^d}\int \mathrm{d}\mathbf{k}\, \mathrm{e}^{2J(\mathbf{k})t} . \tag{23}$$

Using (9) and making a change of variables $u=k^\sigma t$ this integral can be explicitly evaluated as

$$\Xi(t) \propto \frac{\mathrm{e}^{2J_0 t}}{t^{d/\sigma}} \int_0^{2\pi^\sigma t} u^{\frac{d-\sigma}{\sigma}}\, \mathrm{e}^{-J_0 C u}\, \mathrm{d}u . \tag{24}$$

For $0<\sigma<d$ and $t \gg 1$ the asymptotic behavior is

$$\Xi(t) \sim A \frac{\mathrm{e}^{2J_0 t}}{t^{d/\sigma}} \tag{25}$$

with

$$A=\frac{\Omega_d}{(2\pi)^d}\left(\frac{d}{\sigma}\right) 2^{-\frac{d}{\sigma}} \int_0^\infty u^{\frac{d-\sigma}{\sigma}}\, \mathrm{e}^{-J_0 C u}\, \mathrm{d}u ,$$

where Ω_d is the volume of a d-dimensional hypersphere of unit radius. Using this result we obtain for the two time autocorrelation:

$$C_{\mathbf{k}}(t,t')=\frac{\mathrm{e}^{J_0(1-Ck^\sigma)(t+t')}}{\sqrt{\Xi(t)\Xi(t')}} \tag{26}$$

$$\sim \frac{1}{A}\, \mathrm{e}^{-J_0 C k^\sigma (t+t')} (tt')^{\frac{d}{2\sigma}} . \tag{27}$$

In particular we note that the dynamic structure factor

$$C_{\mathbf{k}}(t)=\frac{1}{A}\, \mathrm{e}^{J_0 C k^\sigma t}\, t^{\frac{d}{\sigma}} \tag{28}$$

shows the expected scaling form

$$C_{\mathbf{k}}(t)=L^d(t) f(kL(t)) \tag{29}$$

with

$$L(t)=t^{1/\sigma} , \tag{30}$$

thus recovering the $n\to\infty$ limit of the $O(n)$ model [9,22].

It is known that in the asymptotic coarsening regime a ferromagnet *ages*, that is, two-time correlation and response functions depend on both times t and t' explicitly through a nontrivial scaling form. In coarsening systems the dependence is of the form $L(t)/L(t')$ $(t,t' \gg 1)$, i.e., correlations and responses depend on both times through the ratio of the corresponding characteristic lengths. We will analyze the aging dynamics of the present system with the inclusion of finite temperature fluctuations. In doing so we will see that two-time scaling laws in the aging regime are *exactly* the same, asymptotically, for every temperature $T<T_c$. At finite temperatures the Volterra equation (22) can be solved by Laplace transforming $\Xi(t)$:

$$\tilde{\Xi}(s)=\int_0^\infty \Xi(t)\, \mathrm{e}^{-st}\, \mathrm{d}t . \tag{31}$$

Using that $C_{\mathbf{k}}(0,0)=1$ the Laplace transform $\tilde{\Xi}(s)$ of $\Xi(t)$ results in

$$\tilde{\Xi}(s)=\frac{(1/N)\sum_{\mathbf{k}} 1/(s-2J(\mathbf{k}))}{1-2T(1/N)\sum_{\mathbf{k}} 1/(s-2J(\mathbf{k}))} \tag{32}$$

In order to obtain the scaling behavior of $\tilde{\Xi}(s)$ we have to calculate first the function

$$K(s)\equiv\frac{1}{N}\sum_{\mathbf{k}}\frac{1}{s-2J(\mathbf{k})}\,. \tag{33}$$

In the thermodynamic limit $N\to\infty$

$$K(s)=\frac{1}{(2\pi)^d}\int\frac{\mathrm{d}\mathbf{k}}{s-2J(\mathbf{k})}\,. \tag{34}$$

Introducing (9) and after some algebra one obtains [11,12] in the scaling regime $k\ll 1$:

$$K_c-K\propto\begin{cases}\varepsilon^{(d-\sigma)/\sigma} & \text{if } d/2<\sigma<d\,,\\ \varepsilon\ln\varepsilon & \text{if } \sigma=d/2\,,\\ \varepsilon & \text{if } 0<\sigma<d/2\,,\end{cases} \tag{35}$$

where $\varepsilon=s-2J_0$ and $K_c=K(2J_0)=\beta_c/2$ (see Eq. (8)). From now on one would proceed by different routes depending on the value of σ. Nevertheless, it can be shown that the final results for the scaling behaviors of the correlations and responses are the same for all $0<\sigma<2$. Hence, we will only present here the calculations for $d/2<\sigma<d$.

With the help of (32), (34) and (35) we find

$$\tilde{\Xi}(s)=\frac{\beta_c/2-B(s-2J_0)^{d/\sigma-1}}{1-\beta_c/\beta+(2B/\beta)(s-2J_0)^{d/\sigma-1}} \tag{36}$$

with $B=A|\Gamma(1-d/\sigma)|$, and $\Gamma(x)$ being the gamma function. Working in the low temperature limit $\beta\gg\beta_c$ and for long times we can anti-transform the above expression obtaining

$$\Xi(t)=\frac{A}{(1-\beta_c/\beta)^2}\frac{\mathrm{e}^{2J_0t}}{t^{d/\sigma}}\,. \tag{37}$$

One can see that the only difference with the $T=0$ expression (25) is the factor $(1-\beta_c/\beta)^{-2}$. The time dependence and exponents were not affected by finite temperature. Now we can go back and calculate the two-time correlation (20). It is important to note that for the evaluation of $C_{\mathbf{k}}(t,t')$ we need $\Xi(t)$ for all times and not only for long ones. A detailed analysis of the behavior of $\Xi(t)$ for short as well as long times shows that its contribution to the long time solution of $C_{\mathbf{k}}(t,t')$ will only be finite for times $t>t_{mic}$, t_{mic} being some microscopic timescale. One then obtains

$$C_{\mathbf{k}}(t,t')\sim\mathrm{e}^{-J_0Ck^\sigma(t+t')}\left[\frac{(1-\beta/\beta_c)^2}{A}(tt')^{d/2\sigma}+2Tt'\left(\frac{t}{t'}\right)^{d/2\sigma}\right.$$
$$\left.\times\int_{t_{mic}/t'}^{1}\mathrm{e}^{2J_0Ck^\sigma t'u}u^{-d/\sigma}\,\mathrm{d}u\right]\,. \tag{38}$$

The behavior of the autocorrelation depends on the integral which is a function of $k^\sigma t'$.

Now we will analyze the two different dynamical regimes $k^\sigma t \gg 1$ (fluctuations inside the domains) and $k^\sigma t \ll 1$ (coarsening regime) and show how this reflects in the fluctuation–dissipation relations.

3.1. $k^\sigma t' \gg 1$: bulk fluctuations or quasi-equilibrium behavior

In this case the two-time autocorrelation (38) shows the asymptotic behavior

$$C_{\mathbf{k}}(t,t') \sim e^{-J_0 C k^\sigma (t+t')} \left[\frac{(1-\beta_c/\beta)^2}{A} e^{-J_0 C k^\sigma t'} (tt')^{d/2\sigma} + \frac{T}{J_0 C k^\sigma} \left(\frac{t}{t'}\right)^{d/2\sigma} e^{2 J_0 C k^\sigma t'} \right] \tag{39}$$

and for long times $t \gg t'$ we have

$$C_{\mathbf{k}}(t,t') \sim \frac{T}{J_0 C k^\sigma} \left(\frac{t}{t'}\right)^{d/2\sigma} e^{-J_0 C k^\sigma (t-t')} , \tag{40}$$

that is, the dynamics becomes stationary with exponential decay of correlations. It is simple to obtain the two-time response function (21) which gives

$$G_{\mathbf{k}}(t,t') = \left(\frac{t'}{t}\right)^{-d/2\sigma} e^{-J_0 C k^\sigma (t-t')} . \tag{41}$$

Comparing Eqs. (40) and (41) we see that the fluctuation–dissipation theorem:

$$T G_{\mathbf{k}}(t,t') = \frac{\partial C_{\mathbf{k}}(t,t')}{\partial t'} \tag{42}$$

is obeyed in this regime. That is, short wavelength fluctuations $k \gg 1/L(t)$ reflect the (local) equilibrium inside the domains. We now analyze the more interesting coarsening regime.

3.2. $k^\sigma t' \ll 1$: coarsening, nonequilibrium behavior

The autocorrelation behaves in this regime as

$$C_{\mathbf{k}}(t,t') \sim e^{-J_0 C k^\sigma (t+t')} \left[\frac{(1-\beta_c/\beta)^2}{A} (tt')^{d/2\sigma} + \frac{2T}{(1-d/\sigma)} t' \left(\frac{t}{t'}\right)^{d/2\sigma} + \mathcal{O}(k^\sigma t') \right] . \tag{43}$$

We see that the only effect of temperature on the structure factor is a renormalization of the amplitude:

$$C_{\mathbf{k}}(t) = \frac{(1-\beta_c/\beta)^2}{A} t^{d/\sigma} e^{-2 J_0 C k^\sigma t} . \tag{44}$$

This has the same scaling as the zero temperature limit (28).

In order to analyze the two-time scalings in the aging regime we integrate all modes $\mathbf{k}$ in the autocorrelation:

$$C(t,t') = \frac{1}{(2\pi)^d} \int d\mathbf{k} C_{\mathbf{k}}(t,t') . \tag{45}$$

This gives

$$C(t,t') = \frac{2^{d/\sigma}}{(t+t')^{d/\sigma}} \left[(1-\beta_c/\beta)^2 (tt')^{d/2\sigma} + \frac{2AT}{(1-d/\sigma)} t' \left(\frac{t}{t'}\right)^{d/2\sigma} \right] . \tag{46}$$

In the aging regime $t \gg t'$ this simplifies to

$$C(t,t') = 2^{d/\sigma}(1-\beta_c/\beta)^2 \left(\frac{t'}{t}\right)^{d/2\sigma} \tag{47}$$

$$\propto f\left(\frac{L(t')}{L(t)}\right) \tag{48}$$

which shows the aging scaling typical of coarsening systems with $L(t) = t^{1/\sigma}$ valid for general vector models. A similar computation for the response function gives

$$G(t,t') = \frac{2^{d/\sigma} A}{(t-t')^{d/\sigma}} \left(\frac{t}{t'}\right)^{d/2\sigma} . \tag{49}$$

It can be readily seen that the fluctuation–dissipation theorem (42) is not obeyed in this regime. Nevertheless, it can be extended to this nonequilibrium regime defining the so called "fluctuation–dissipation ratio" [10] as

$$X(t,t') = \frac{TG_{\mathbf{k}}(t,t')}{\partial C_{\mathbf{k}}(t,t')/\partial t'} . \tag{50}$$

This function is particularly interesting in the case of complex systems such as spin glasses or structural glasses where it can be associated with physical effective temperatures [10]. Its time dependence encodes information on the time scales structure of the system [23]. In the case of coarsening systems the response is weak and asymptotically goes to zero, signalling the weakness of memory effects in these systems [24,3]. This can be explicitly seen in this model where at long times

$$X(t,t') \propto (t')^{1-d/\sigma} \to 0 . \tag{51}$$

We see in the above result that $X(t,t')$ is, asymptotically, a function only of t', as is observed in models of structural glasses. This implies that, in particular for fixed t', the fluctuation–dissipation ratio is a constant. The value of the exponent $1-d/\sigma$ also shows that the dynamics becomes faster, and memory effects weaker, as the interactions become more long ranged. This means that ferromagnetic domain walls move faster as long-range interactions become more important.

4. Conserved order parameter

We now consider the continuous limit of the spherical model, that is, the spin variables $s_i(t)$ are replaced by a field $s(\mathbf{r},t)$, $\mathbf{r}$ now being a d-dimensional continuous position vector, and where the field is subject for all times t to the spherical constraint:

$$\int \mathrm{d}\mathbf{r}[s(\mathbf{r},t)]^2 = V ,$$

where the integral is carried out over a hypercube of side L with $V = L^d$. We will restrict the analysis to zero temperature. In the case of conserved order parameter the dynamics is governed by the Cahn–Hilliard equation:

$$\frac{\partial s(\mathbf{r},t)}{\partial t} = \nabla^2 \frac{\delta \mathscr{H}'}{\delta s(\mathbf{r},t)} , \tag{52}$$

where the Hamiltonian now takes the form

$$\mathscr{H}' = -\int \mathrm{d}\mathbf{r} \int \mathrm{d}\mathbf{r}' J(|\mathbf{r}-\mathbf{r}'|) s(\mathbf{r},t) s(\mathbf{r}',t) + z(t) \int \mathrm{d}\mathbf{r} [s(\mathbf{r},t)]^2 . \tag{53}$$

Transforming Fourier we arrive at

$$\frac{\partial s_{\mathbf{k}}}{\partial t} = -k^2 \frac{\delta \mathscr{H}}{\delta s_{\mathbf{k}}} = k^2 \{[J(k) - z(t)] s_{\mathbf{k}}\} , \tag{54}$$

where the asymptotic form (9) is assumed for $J(k)$. The formal solution of Eq. (54) is now

$$s_{\mathbf{k}}(t) = s_{\mathbf{k}}(0) \mathrm{e}^{k^2 J(\mathbf{k}) t - k^2 \int_0^t z(t')\,\mathrm{d}t'} \tag{55}$$

and the structure factor reads:

$$C_{\mathbf{k}}(t) = C_{\mathbf{k}}(0) \mathrm{e}^{2k^2 J(k) t - 2k^2 g(t)} \tag{56}$$

with $g(t) \equiv \int_0^t z(t')\,\mathrm{d}t'$ and considering again $C_{\mathbf{k}}(0) = 1$. Following Ref. [9] we will obtain the scaling form of the structure factor by making the reasonable assumption that $C_{\mathbf{k}}(t)$ will present a maximum as a function of k at some $k_m(t)$ and at long times it will evolve into a Bragg peak. From the spherical constraint $(2\pi)^{-d} \int \mathrm{d}\mathbf{k} C_k(t) = 1$ it follows that $C_{k_m}(t)$ scales with k_m as

$$C_{k_m}(t) \propto k_m^{-d} . \tag{57}$$

Maximizing $C_k(t)$ we obtain

$$k_m^{\sigma} = \frac{2}{2+\sigma} \frac{[J_0 t - g(t)]}{J_0 C t} . \tag{58}$$

From Eqs. (56)–(58) we get

$$\frac{k_m^{2+\sigma}}{\ln k_m} = -\frac{d}{J_0 C \sigma t} \tag{59}$$

whose asymptotic solution for long times is

$$k_m \approx \left[\frac{d}{J_0 C \sigma (2+\sigma)} \frac{\ln t}{t} \right]^{1/(2+\sigma)} . \tag{60}$$

We can now reconstruct the structure factor obtaining finally:

$$C_k(t) = [l^d(t)]^{\phi(k/k_m)} \tag{61}$$

in which

$$l(t) = t^{1/(2+\sigma)} \tag{62}$$

and the scaling function

$$\phi(x) = \frac{(2+\sigma)}{\sigma}x^2 - \frac{2}{\sigma}x^{2+\sigma} . \tag{63}$$

As in the short-range case, the phenomenon of "multiscaling" is also present with long-range interactions. In our case two characteristic length scales show up as

$$l(t) \propto t^{1/z} \quad \text{with } z = 2+\sigma \tag{64}$$

and

$$k_m^{-1}(t) \propto \left(\frac{t}{\ln t}\right)^{1/z} . \tag{65}$$

Interestingly, this second length scale grows more slowly than t and will produce a particular scaling form of the two-time autocorrelations called "sub-aging". This behavior has been studied in detail recently in Ref. [25] for the $O(n)$ model with short-range interactions in the $n \to \infty$ limit. Here we generalize the result to systems with power law interactions.

Now it is straight-forward to write the solution to (55) at long times as

$$s_{\mathbf{k}}(t) = s_{\mathbf{k}}(0) \exp\left\{ -k^{2+\sigma}J_0Ct + k^2\left[\frac{d}{2\sigma}\left(\frac{(2+\sigma)}{2}J_0Ct\right)^{2/\sigma}\ln t\right]^{\sigma/(2+\sigma)}\right\} . \tag{66}$$

From this result we obtain the two-time autocorrelation following the steps in the nonconserved case. The first important point to note is that, similar to what happens with short-range interactions, the autocorrelation $C(t'+\tau,t')$ relaxes completely in a time $\tau \sim t'$ as

$$C(2t',t') \propto (t')^{(d/(2+\sigma))(f-1)} \to 0 , \tag{67}$$

where

$$f = \left(\frac{2^{2/(2+\sigma)}+1}{3}\right)^{2/\sigma}\frac{2^{2/(2+\sigma)}+1}{2} < 1 \tag{68}$$

and we have set $J_0C = 1$. So the dynamics is considerably faster than in the nonconserved case. In the regime when $\tau \ll t'$ we obtain

$$C(t'+\tau,t') \propto (t')^{d/(2+\sigma)[(2/\sigma)^{2/\sigma}-1]} \exp\left\{-\left(\frac{2}{\sigma}\right)^{2/\sigma}\left(\frac{d}{2+\sigma}\right)\frac{\ln t'}{(2+\sigma)\sigma^2}\left(\frac{\tau}{t'}\right)^2\right\} . \tag{69}$$

It is interesting to compare this result with the corresponding one for short-range interactions [22]. First of all, if $\sigma \neq 2$, the correlation decays algebraically with t', as noted above. For fixed t' the relaxation goes as

$$C(t'+\tau,t') \propto \mathrm{e}^{-b(\tau/t_r)^2} \tag{70}$$

with $b = (2/\sigma)^{2/\sigma}(d/2+\sigma)1/(2+\sigma)\sigma^2$ and

$$t_r = \frac{t'}{\sqrt{\ln t'}} . \tag{71}$$

This relaxation time is smaller than t' and the particular aging dynamics is called "sub-aging". The same scaling is observed in the short-range case [25].

Finally, we want to point out that, although these results are interesting *per se* as they show complex behavior emerging from a closed analytic solution of the model, it is important to note that the presence of multiscaling in dynamics with conserved order parameter is limited to the spherical limit of general n-vector models. Simple scaling is recovered when corrections to order $1/n$ are considered [26].

5. Conclusions

Summarizing, we presented the exact solution for the dynamics of the ferromagnetic spherical model with power law decaying interactions in an arbitrary dimension, after a quench from infinite temperature into the ordered phase. In the case of nonconserved order parameter we analyzed the Langevin dynamics of the model, obtaining the exact long time scaling form for correlations and responses at finite temperature. In the case of conserved order parameter we analyzed a microscopic version of the Cahn–Hilliard dynamics of the model at zero temperature, obtaining the long time scaling form of the two-time autocorrelation function. These exact results allow us to check at a microscopic level several scaling hypothesis about coarsening dynamics in pure systems, which are currently derived at a coarse grained level through phenomenological Landau–Ginzburg and Cahn–Hilliard equations. In particular, we recovered some known results about the scaling behavior of the $n \to \infty$ limit of the $O(n)$ model, by working directly on the spherical model. This allowed us to extend those results by obtaining explicit scaling forms for several correlation and response functions.

One new result worth mentioning is that of the fluctuation–dissipation ratio, Eq. (51). This behavior has been observed numerically in several nondisordered systems [3,24] and it has been conjectured to be characteristic of systems which do not present replica symmetry breaking [23], i.e., systems with a single pure state. Up to now, the present result is one of the few exact solutions available confirming that conjecture.

A natural and interesting extension of this work is the study of the dynamics of frustrated systems without disorder, like models with competing short-range ferromagnetic and long-range antiferromagnetic (e.g., dipolar) interactions, in which simulations have shown a rich dynamical behavior [2,3]. In fact, there is a whole class of different physical systems with these characteristic competing interactions, two examples being charged systems with weak coulomb interactions [4] and a proposed model for the behavior of structural glasses [8].

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