Expansão em autofunções

- Matriz de Transição $\hat{T}$ (representada em uma base qualquer)
- Probabilidade inicial $\vec{P}_0$

$\rightarrow \quad \vec{P}_\ell = \hat{T}^\ell \vec{P}_0 \quad \Rightarrow \quad |P_\ell\rangle = \hat{T}^\lambda |P_0\rangle$

Autovetores à direita

$$\begin{cases} \hat{T}|\psi_i\rangle = \lambda_i |\psi_i\rangle & \text{e à esquerda} \\ \langle\phi_j|\hat{T} = \lambda_j \langle\phi_j| \end{cases}$$

$$\langle\phi_j|\hat{T}|\psi_i\rangle = \lambda_i \langle\phi_j|\psi_i\rangle$$

$$\langle\phi_j|\hat{T}|\psi_i\rangle = \lambda_j \langle\phi_j|\psi_i\rangle$$

$$\langle \phi_j | \hat{T} | \psi_i \rangle - \langle \phi_j | \hat{T} | \psi_i \rangle = (\lambda_i - \lambda_j) \langle \phi_j | \psi_i \rangle$$

$$(\lambda_i - \lambda_j) \langle \phi_j | \psi_i \rangle = 0$$

- Se $\lambda_i = \lambda_j \Rightarrow \langle \phi_j | \psi_j \rangle > 0$

  $\hookrightarrow$ posso normalizar e ter $\langle \phi_j | \psi_j \rangle = \underline{1}$

- Se $\lambda_i \neq \lambda_j \Rightarrow \langle \phi_j | \psi_i \rangle = 0$

$\Rightarrow$ Ortonormais : $\boxed{\langle \phi_j | \psi_i \rangle = \delta_{ij}}$

$\rightarrow$ Sei que existe autovalor $\lambda = 1$:

$$\langle \phi_0 | \hat{T} = 1 \langle \phi_0 | \quad \text{(eq. de autovalores)}$$

Em termos dos componentes $\hookrightarrow$

vetor x matriz

elemento da matriz

$$\phi_0(n) = \sum_m \phi_0(m)\, T(m,n)$$

(I) $\phi_0(n) = \sum_m{}' \phi_0(m)\, T(m,n)$

Como $\hat{T}$ é matriz estocástica, vale

(II) $\sum_m{}' T(m,n) = 1$

$\phi_0(n) \sum_m{}' T(m,n) = \phi_0(n)$

porque n é fixo

(II) − (I)

$\phi_0(n) - \phi_0(n) = \sum_m{}' [\phi_0(n) - \phi_0(m)]\, T(m,n)$

$0 = \sum_m{}' [\phi_0(n) - \phi_0(m)]\, T(m,n)$

↳ Em geral → $\phi_0(n) = \phi_0(m) \overset{\text{escolho}}{=} 1$

(eventualmente $T(m,n) = 0$ para alguns pares (m,n))

* Com a escolha $\langle\phi_0| = (1 \;\; 1 \;\; 1 \; \dots \; 1)$,

a equação dos autovalores é satisfeita

<u>sempre</u> com $\lambda = 1$

$$\langle\phi_0|\hat{T} = \lambda\langle\phi_0| = \langle\phi_0|$$

↳ em componentes

$$\phi_0(n) = \sum_m{}' \phi_0(m)\, T(m,n) \qquad \swarrow \phi_0(m) = 1$$

$$= \sum_m{}' T(m,n) \qquad \swarrow \hat{T} \text{ é estocástica}$$

$$= 1 .$$

<u>ou</u>

→ Probabilidade estacionária satisfaz

$$\vec{P}_\infty = \hat{T}\,\vec{P}_\infty \quad \rightarrow \quad |P_\infty\rangle = \hat{T}\,|P_\infty\rangle$$

equação de autovalores p/ $\lambda = 1$

$$\Rightarrow \quad |\psi_0\rangle = |P_\infty\rangle \quad \text{(estacionária)}$$

Verificar

$$\langle \phi_0 | \psi_0 \rangle = \sum_m \phi_0(m)\, \psi_0(m)$$

$$= \sum_m 1 \times \psi_0(m)$$

$$= \sum_m \psi_0(m)$$

$$= \sum_m P_\infty(m)$$

$$= 1 \quad \underline{\text{ok}}$$

Evolução Temporal

$$|P_l\rangle = \hat{T}^l |P_0\rangle$$

Da eq dos autovalores:

$$\hat{T}^l |\psi_k\rangle = \lambda_k^l |\psi_k\rangle$$

e como os autovetores formam uma base ortonormal.

$$|P_0\rangle = \sum_m |\psi_k\rangle\langle\phi_k|P_0\rangle = \sum_m P_m |\psi_k\rangle$$

$$= \left(\sum_m |\psi_k\rangle\langle\phi_k|\right)|P_0\rangle$$

$$= \hat{I}$$

$$\hat{I} = \sum_m |\psi_k\rangle\langle\phi_k|$$

$$|P_l\rangle = \hat{T}^l |P_0\rangle = \hat{T}^l \hat{I} |P_0\rangle$$

$$= \hat{T}^l \sum_m |\psi_m\rangle\langle\phi_m|P_0\rangle$$

$$= \sum_m \hat{T}^l |\psi_m\rangle\langle\phi_m|P_0\rangle$$

$$= \sum_m \lambda_m^l |\psi_m\rangle\langle\phi_m|P_0\rangle$$

$$|P_l\rangle = \underbrace{\left[\sum_m \lambda_m^l |\psi_m\rangle\langle\phi_m|\right]}_{\hat{T}^l} |P_0\rangle \quad *$$

$$\boxed{\hat{T}^l = \sum_m \lambda_m^l |\psi_m\rangle\langle\phi_m|}$$

Como sempre há o autovalor 1:

$$|P_\ell\rangle = |\psi_0\rangle \underbrace{\langle \phi_0 | P_0\rangle}_{1} + \sum_{m=1}^{N-1} \lambda_m^\ell |\phi_m\rangle\langle \psi_m | P_0\rangle$$

$$\langle \phi_0 | P_0\rangle = \sum_m \phi_0(m)\, P_0(m)$$

1

vetor com todos elementos iguais a 1.

$$= \sum_m P_0(m)$$

normalizado

$$= 1$$

$$|P_\ell\rangle = |\psi_0\rangle + \cdots$$

$$\boxed{|P_\ell\rangle = |P_\infty\rangle + \sum_{m=1}^{N-1} \lambda_m^\ell |\phi_m\rangle\langle \psi_m | P_0\rangle}$$

$$|P_\ell\rangle = |P_\infty\rangle + \sum_{m=1}^{N-1} \lambda_m^\ell |\phi_m\rangle\langle\psi_m|P_0\rangle$$

Como $|\lambda_m| < 1$ para $m \neq 0$

$\lambda_m^\ell \xrightarrow{\ell\to\infty} 0$ e a probabilidade converge para a probabilidade estacionária.

Exemplo 6.2

$$\hat{T} \xrightarrow{rep.} \begin{pmatrix} 1-b & q \\ b & 1-q \end{pmatrix}$$

Eq. autovalores

$$\hat{T}\vec{v} = \lambda\vec{v} = \lambda\hat{I}\vec{v}$$

$$(\hat{T} - \lambda\hat{I})\vec{v} = \vec{0}$$

$$\det(\hat{T} - \lambda\hat{I}) = 0$$

$$\begin{vmatrix} 1-b-\lambda & q \\ b & 1-q-\lambda \end{vmatrix} = 0$$

$$(1-b-\lambda)(1-q-\lambda) - bq = 0$$

$$1-q-\lambda-b+\cancel{bq}+b\lambda$$
$$-\lambda+q\lambda+\lambda^2 \cancel{-bq} = 0$$

$$\lambda^2 - \underbrace{(2-b-q)}_{\lambda_0+\lambda_1}\lambda + \underbrace{(1-q-b)}_{\lambda_0\lambda_1} = 0$$

↳ Sei que $\lambda_0 = 1$ é solução

$$(\lambda-\lambda_0)(\lambda-\lambda_1) = 0$$

$$\lambda^2 - (\lambda_0+\lambda_1)\lambda + \lambda_0\lambda_1 = 0$$

$$\lambda^2 - S\lambda + P = 0$$

↳ $$\boxed{\begin{aligned} \lambda_0 &= 1 \\ \lambda_1 &= 1-b-q \end{aligned}}$$

$$\begin{pmatrix} 1-b-\lambda & q \\ b & 1-q-\lambda \end{pmatrix} \vec{v} = \vec{0}$$

<u>Autovetores</u>

Direita:

$\underline{\lambda_0 = 1}$: $\begin{pmatrix} -b & q \\ b & -q \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$$\begin{cases} -b v_1 + q v_2 = 0 \\ b v_1 - q v_2 = 0 \end{cases}$$

$$\boxed{b v_1 = q v_2} \quad \forall v_1$$

$\hookrightarrow$ Escolher $\begin{cases} v_1 = q \\ v_2 = b \end{cases}$

$$\vec{v} = \begin{pmatrix} q \\ b \end{pmatrix}$$

Esquerda:

$$\vec{u} \, (\hat{T} - \lambda \hat{I}) = (0, 0)$$

$$(u_1, u_2) \begin{pmatrix} -b & q \\ b & -q \end{pmatrix} = (0, 0)$$

$$\begin{cases} -b u_1 + b u_2 = 0 \\ q u_1 - q u_2 = 0 \end{cases}$$

$$\begin{cases} -b(u_1 - u_2) = 0 \\ q(u_1 - u_2) = 0 \end{cases}$$

$\rightarrow u_1 = u_2$, $\forall u_1$

Escolho $u_1 = 1$

$$\vec{u} = (1, 1)$$

$$A \, \vec{u} \cdot \vec{v} = 1$$

$$A \, (u_1 \;\; u_2) \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = 1$$

$$A \, (1, 1) \begin{pmatrix} a \\ b \end{pmatrix} = 1$$

$$A \, (a + b) = 1$$

$$A = \frac{1}{b+a}$$

Normalização:

$$|\psi_0\rangle = A \, |\vec{v}\rangle$$

$$= \frac{1}{b+a} \begin{pmatrix} a \\ b \end{pmatrix}$$

$$\boxed{\begin{aligned} \langle \phi_0 | &= (1, 1) \\ |\psi_0\rangle &= \frac{1}{a+b} \begin{pmatrix} a \\ b \end{pmatrix} \end{aligned}}$$

$$\underline{\underline{\lambda_1 = 1 - b - a}}$$

Direita.

$$\begin{pmatrix} 1 - b - \lambda & a \\ b & 1 - a - \lambda \end{pmatrix} \vec{v} = \vec{0}$$

$$\begin{pmatrix} a & a \\ b & b \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{cases} a \, (v_1 + v_2) = 0 \\ b \, (v_1 + v_2) = 0 \end{cases} \rightarrow v_2 = -v_1$$

Escolha $v_1 = 1$

$$\vec{v} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

Esquerda:

$$(u_1 \ \ u_2) \begin{pmatrix} a & a \\ b & b \end{pmatrix} = (0, 0)$$

$$\begin{cases} a u_1 + b u_2 = 0 \\ a u_1 + b u_2 = 0 \end{cases}$$

$$\boxed{a u_1 = -b u_2} \quad \forall\, v_1$$

Escolhe $u_1 = b$

$$\downarrow$$

$$u_2 = -a$$

Para normalizar

$$A \, \vec{u} \cdot \vec{v} = 1$$

$$A (b, -a) \begin{pmatrix} 1 \\ -1 \end{pmatrix} = 1$$

$$A (b + a) = 1$$

$$A = \frac{1}{b + a}$$

$$\boxed{\begin{aligned} |\psi_1\rangle &= \begin{pmatrix} 1 \\ -1 \end{pmatrix} \\ \langle \phi_1 | &= \frac{1}{b+a} (b \ \ -a) \end{aligned}}$$

Ortogonalidade:

$$\langle \phi_0 | \psi_1 \rangle = (1, 1)\begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$$= 1 - 1 = 0$$

$$\langle \phi_1 | \psi_0 \rangle = \frac{1}{(b+q)^2}(b \ -q)\begin{pmatrix} q \\ b \end{pmatrix}$$

$$= \frac{1}{(b+q)^2}(bq - qb)$$

$$= 0$$

OK

$$|P_\ell\rangle = \hat{T}^\ell |P_0\rangle$$

$$= \sum_m \lambda_m^\ell |\psi_m\rangle\langle\phi_m | P_0\rangle$$

$$= \lambda_0^\ell |\psi_0\rangle\langle\phi_0|P_0\rangle + \lambda_1^\ell|\psi_1\rangle\langle\phi_1|P_0\rangle$$

$$= |\psi_0\rangle\langle\phi_0|P_0\rangle + (1-b-q)^\ell|\psi_1\rangle\langle\phi_1|P_0\rangle$$

Na base original:

$$|\psi_0\rangle\langle\phi_0| \rightarrow \frac{1}{b+q}\begin{pmatrix} q \\ b \end{pmatrix}(1 \ 1)$$

$$= \frac{1}{b+q}\begin{pmatrix} q & q \\ b & b \end{pmatrix}$$

$$|\psi_1\rangle\langle\phi_1| \rightsquigarrow \begin{pmatrix}1\\-1\end{pmatrix}\frac{1}{q+b}\begin{pmatrix}b & -q\end{pmatrix}$$

$$= \frac{1}{q+b}\begin{pmatrix}b & -q\\ -b & q\end{pmatrix}$$

CUIDADO com A BASE

$\downarrow$

$|p_0\rangle = p_1|1\rangle + p_2|2\rangle$

$\hookrightarrow$ p_1 p_2

$$|p_\ell\rangle = |\psi_0\rangle\langle\phi_0|p_0\rangle + (1-b-q)^\ell\,|\psi_1\rangle\langle\phi_1|p_0\rangle$$

$$= \frac{1}{b+q}\begin{pmatrix}q & q\\ b & b\end{pmatrix}\begin{pmatrix}p_1\\ p_2\end{pmatrix} - \frac{(1-b-q)^\ell}{b+q}\begin{pmatrix}b & -q\\ -b & q\end{pmatrix}\begin{pmatrix}p_1\\ p_2\end{pmatrix}$$

$$= \underbrace{\left[\frac{1}{b+q}\begin{pmatrix}q & q\\ b & b\end{pmatrix} - \frac{(1-b-q)^\ell}{b+q}\begin{pmatrix}b & -q\\ -b & q\end{pmatrix}\right]}_{\text{Representação de } \hat{T}^\ell}\begin{pmatrix}p_1\\ p_2\end{pmatrix}$$

$$|P_\ell\rangle \rightarrow \frac{1}{b+q}\begin{pmatrix} q & q \\ b & b \end{pmatrix}\begin{pmatrix} P_1 \\ P_2 \end{pmatrix} - \underbrace{\frac{(1-b-q)^\ell}{b+q}\begin{pmatrix} b & -q \\ -b & q \end{pmatrix}}\begin{pmatrix} P_1 \\ P_2 \end{pmatrix}$$

$$= \frac{1}{b+q}\begin{pmatrix} qP_1 + qP_2 \\ bP_1 + bP_2 \end{pmatrix} - \ldots$$

$$= \frac{1}{b+q}\begin{pmatrix} q\,(P_1+P_2) \\ p\,(P_1+P_2) \end{pmatrix} - \ldots$$

é probabilidade

$$\hookrightarrow |P_0\rangle = P_1|1\rangle + P_2|2\rangle$$

$$P_1 + P_2 = 1$$

$|\pi_0\rangle$

$$|P_\ell\rangle = \underbrace{\frac{1}{b+q}\begin{pmatrix} q \\ p \end{pmatrix}} - \frac{(1-b-q)^\ell}{b+q}\begin{pmatrix} b & -q \\ -b & q \end{pmatrix}\begin{pmatrix} P_1 \\ P_2 \end{pmatrix}$$

Prob. estacionária

$$|P_\ell\rangle = \frac{1}{b+q}\begin{pmatrix} q \\ p \end{pmatrix} - \frac{(1-b-q)^\ell}{b+q}\begin{pmatrix} b & -q \\ -b & q \end{pmatrix}\begin{pmatrix} P_1 \\ P_2 \end{pmatrix}$$

- b e q não podem ser simultaneamente nulos, porque caso contrário, não há transição entre estados

$$\hat{T} \to \begin{pmatrix} 1-b & q \\ b & 1-q \end{pmatrix}$$

- b e q não podem ser simultaneamente iguais a 1 (matriz cíclica)

$$|\lambda_1| = |1-b-q| < 1 \quad e$$

$$|P_\ell\rangle \xrightarrow{\ell\to\infty} |P_\infty\rangle = |\pi_0\rangle \to \frac{1}{b+q}\begin{pmatrix} q \\ p \end{pmatrix}$$