Equação de Langevin

$$\begin{cases} m \dfrac{dv}{dt} = -\alpha v + F_a(t) & \text{I} \quad (2^a \text{ Lei}) \\ \dfrac{dx}{dt} = v \end{cases}$$

$F_a(t)$: força aleatória (ruído branco gaussiano) de média zero.

- $\langle F_a(t) \rangle = 0$
- $\langle F_a(t) F_a(t') \rangle = B\,\delta(t-t')$

a determinar.

$$\text{I} \Rightarrow \frac{dv}{dt} = -\gamma v + \xi(t) \,, \quad \text{II}$$

$\gamma = \dfrac{\alpha}{m}$; $\langle \xi(t) \rangle = 0$

$\langle \xi(t)\xi(t') \rangle = \Gamma\,\delta(t-t')$

onde $\Gamma = \dfrac{B}{m^2}$

↳ Resolução da equação II.

Transformada de Laplace:

$$F(s) = \mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt$$

Aplica em

$$\frac{dx}{dt} = a\,x(t) + b(t)$$

mesma forma de II

$$\mathcal{L}\left\{\frac{dx}{dt}\right\} = \mathcal{L}\left\{a\,x(t) + b(t)\right\}$$

$$= \mathcal{L}\{a x(t)\} + \mathcal{L}\{b(t)\}$$

$$= a\,\mathcal{L}\{x(t)\} + B(s)$$

$$= a\,\bar{X}(s) + B(s)$$

$$\rightarrow \mathcal{L}\left\{\frac{dx}{dt}\right\} = \int_0^\infty \underbrace{e^{-st}}_{v} \underbrace{\left(\frac{dx}{dt}\right) dt}_{du}$$

$$= \Big[x(t)\,e^{-st}\Big]_{t=0}^{t=\infty} - \int_0^\infty (-s)e^{-st}\,x\,dt$$

$$= -x(0) + s\int_0^\infty x\,e^{-st}\,dt$$

$$= -x(0) + s\bar{X}(s)$$

$$\Rightarrow s\bar{X}(s) - x(0) = a\bar{X}(s) + B(s)$$

$$(s-a)\bar{X}(s) = B(s) + x(0).$$

$$\bar{X}(s) = \left(\frac{1}{s-a}\right)B(s) + \frac{x(0)}{s-a}$$

$$x(t) = \mathcal{L}^{-1}\{\bar{X}(s)\}$$

$$= \mathcal{L}^{-1}\left\{\frac{1}{s-a}\cdot B(s)\right\} + \mathcal{L}^{-1}\left\{\frac{1}{s-a}\right\}x(0)$$

convolución

$$= \int_0^t e^{as}\,b(t-s)\,ds + x(0)\,e^{at}$$

$$\boxed{x(t) = x(0)\,e^{at} + \int_0^t e^{as}\,b(t-s)\,ds}$$

Solução de II :

$$v(t) = v(0)\, e^{-\gamma t} + \int_0^t e^{-\gamma t}\, \xi(t-t')\, dt' \qquad s \to t'$$

$$\boxed{v(t) = v(0)\, e^{-\gamma t} + \int_0^t e^{-\gamma(t-t')}\, \xi(t')\, dt'} \quad \text{III}$$

Velocidade média:

$$\langle v(t)\rangle = \langle v(0)\, e^{-\gamma t}\rangle + \int_0^t e^{-\gamma t}\, \cancel{\langle \xi(t-t')\rangle}\, dt' \to 0$$

$$\boxed{\langle v(t)\rangle = v(0)\, e^{-\gamma t}} \xrightarrow{t\to\infty} 0$$

III $\Rightarrow$ $v(t) = \langle v\rangle + e^{-\gamma t}\int_0^t \xi(t-t')\, dt'$

$$v(t) - \langle v\rangle = e^{-\gamma t}\int_0^t \xi(t-t')\, dt'$$

$$(v(t) - \langle v\rangle)^2 = \left(e^{-\gamma t}\int_0^t e^{+\gamma t'}\xi(t')\, dt'\right)\left(e^{-\gamma t}\int_0^t e^{+\gamma t''}\xi(t'')\, dt''\right)$$

$$= e^{-2\gamma t}\int_0^t dt' \int_0^t dt''\, e^{+\gamma(t'+t'')}\, \xi(t')\,\xi(t'')$$

$$\langle [v(t) - \langle v\rangle]^2\rangle = e^{-2\gamma t}\int_0^t dt' \int dt''\, e^{+\gamma(t'+t'')}\, \underbrace{\langle \xi(t')\,\xi(t'')\rangle}$$

$$= e^{-2\gamma t}\int_0^t \int_0^t e^{+\gamma(t'+t'')}\, \underbrace{\Gamma\, \delta(t'-t'')}\, dt''\, dt'$$

Seleciona $t'' = t'$ da integral em t''

$$= \Gamma\, e^{-2\gamma t}\int_0^t dt'\, e^{+2\gamma t'}$$

$$= \Gamma\, e^{-2\gamma t}\left[\frac{1}{2\alpha}\, e^{+2\gamma t'}\right]_{t=0}^{t'=t}$$

$$\langle [v(t)-\langle v\rangle]^2\rangle = \frac{\Gamma e^{-2\gamma t}}{2\gamma}\left[e^{2\gamma t} - e^{0}\right]$$

$$\boxed{\langle v^2\rangle - \langle v\rangle^2 = \frac{\Gamma}{2\gamma}\left[1 - e^{-2\gamma t}\right]}.$$

↳ No regime estacionário, sem influência externa $(t\to\infty)$

$$\langle v\rangle \to 0$$

$$\boxed{\langle v^2\rangle = \frac{\Gamma}{2\gamma}}$$

Conexão com a física

→ Equilíbrio térmico (equipartição de energia)

Para o fluido (teoria cinética)

$$\frac{1}{2} m\langle v^2\rangle = \frac{k_B T}{2}$$

↳ $$\boxed{\langle v^2\rangle = \frac{k_B T}{m}}$$

$$\Rightarrow \Gamma = 2\gamma \frac{k_B T}{m}$$

$$\frac{B}{m^2} = 2\gamma\frac{k_B T}{m} \to \boxed{B = 2\gamma m k_B T}$$

$$\Rightarrow \quad \Gamma = 2\gamma \frac{k_B T}{m}$$

$$\frac{B}{m^2} = 2\gamma \frac{k_B T}{m} \Rightarrow \boxed{B = 2\gamma m k_B T}$$

Onde

$$\langle F_a(t) F_a(t') \rangle = B\,\delta(t-t')$$

$$= 2\gamma m k_B T\, \delta(t-t')$$

$$= 2 \frac{\alpha}{m} m k_B T\, \delta(t-t')$$

$\rightarrow$ Eq. Langevin

$$m \frac{dv}{dt} = -\alpha v + F_a(t)$$

$$\boxed{m \frac{dv}{dt} = -\alpha v + \sqrt{2\alpha k_B T}\, \eta(t)}$$

Gaussiana de média 0 e variância 1

Teorema de Flutuação - Dissipação

$$\boxed{\langle F_a(t) F_a(t') \rangle = 2\alpha k_B T\, \delta(t-t')}$$