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Validity of the long-range expansion in the n-vector model

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The critical behavior of an n-component system in d dimensions with long-range (LR) interactions decaying as $1/|\vec{r}|^{d+\sigma}$, for $\sigma>0$, is reexamined near the crossover to short-range (SR) behavior, where $\sigma=2$, by means of renormalized perturbation theory in $\epsilon'=2\sigma-d$. It is pointed out that $2-\sigma$ should not be viewed as an expansion parameter, and consequently the critical exponents turn out to be discontinuous at $\sigma=2$ without the intermediate region of weakly LR interactions found earlier by Sak. For any $\sigma<2$ it is shown that (i) the LR expansion is stable to weak SR perturbations and (ii) the SR expansion in $\epsilon=4-d$ breaks down under a weak LR perturbation.

The critical behavior for a d-dimensional system with an n-component order parameter and long-range (LR) interactions decaying as $1/|\vec{r}|^{d+\sigma}$, for $\sigma>0$, has already been studied for some time. Fisher, Ma, and Nickel[1] first showed, through a renormalization-group (RG) expansion in $\epsilon'=2\sigma-d$, that even in the nonclassical regime, $\epsilon'>0$, the critical exponent η "sticks" to the "classical" value $\eta=2-\sigma$ with no corrections at least to $O(\epsilon'^3)$, for any *fixed* $\sigma<2$, while the remaining exponents have a nonclassical dependence on σ. Similar results were obtained by Suzuki, Yamazaki, and Igarashi.[2] When $\sigma>2$ the exponents for a short-range (SR) system were obtained for all d, in agreement with results on the spherical model.[3] If $\eta=2-\sigma$ uniformly in ϵ' and for any $\sigma<2$, this would imply the existence of a discontinuity at $\sigma=2$, since $\eta_{\mathrm{SR}}\neq 0$. Soon thereafter Sak[4,5] suggested that $\eta=2-\sigma$ only as long as $2-\sigma>\eta_{\mathrm{SR}}$ and that, for $2-\sigma\leq\eta_{\mathrm{SR}}$, there should be a region of weakly LR potentials for which $\eta=\eta_{\mathrm{SR}}$, implying a continuity in all critical exponents.

In the case of LR interactions there are two independent parameters corresponding to σ and d, one of which is ϵ', and the other one can be chosen as $\epsilon=4-d$. The purpose of this paper is to draw attention to the fact that if one wants to study the crossover to SR behavior under varying σ close to 2 one has to fix d, i.e., keep ϵ fixed. Then it follows from

$$\sigma=2-\frac{\epsilon}{2}+\frac{\epsilon'}{2}\ , \tag{1}$$

in which $\epsilon'\leq\epsilon$, that by letting ϵ' become arbitrarily small in order of the RG expansion in ϵ' to make sense, $2-\sigma$ will become bounded by the fixed $\epsilon/2$. The crucial point that we make here, distinct from most previous authors,[2,4,6] is that $2-\sigma$ is *not* an expansion parameter, although it can be made arbitrarily small with ϵ, as long as $\epsilon\geq\epsilon'$. For a *given* d one can, of course, choose ϵ' such that $2-\sigma=0(\epsilon'^2)$. However, when ϵ' is made arbitrarily small, $2-\sigma$ will not follow the variation of ϵ'. This means, as will be discussed next, that the LR expansion that one has for any fixed $\sigma<2$ remains valid up to $\sigma=2^-=\lim(2-\delta)$ as $\delta\to 0^+$, with $\eta(2^-)=0$, whereas $\eta=\eta_{\mathrm{SR}}$ when σ is replaced by 2, in accordance with the old conjecture of a discontinuity in η at $\sigma=2$. We can show explicitly that the LR expansion in ϵ' is stable to weak SR perturbations for any $0<\sigma<2$, no matter how close σ comes to 2, whereas already a weak LR perturbation yields infrared divergences in the free-field limit which signals the breakdown of the SR expansion.

The results reported here are in contrast to recent work by Yamazaki[6] who suggested a double expansion in ϵ and ϵ', with either $\epsilon\geq\epsilon'$ or $\epsilon\leq\epsilon'$, that leads to multiple expressions for the critical exponents between which a choice has to be made. There is a recent rigorous proof by van Enter[7] that a system with SR interactions and broken rotational symmetry (the XY model) has a phase transition that is unstable to weak LR perturbations in dimension $d\gtrsim 3$.

The bare effective Hamiltonian for the rotationally invariant n-vector model in momentum space may be written as

$$H_0=\frac{1}{2}\int_{\vec{k}}[G^0(k,m_0)]^{-1}\vec{\phi}_{\vec{k}}\cdot\vec{\phi}_{-\vec{k}}+\frac{\tilde{u}_0}{4!}\int_{\vec{k}}\int_{\vec{p}}\int_{\vec{q}}(\vec{\phi}_{\vec{k}}\cdot\vec{\phi}_{\vec{p}})(\vec{\phi}_{\vec{q}}\cdot\vec{\phi}_{-\vec{k}-\vec{p}-\vec{q}})\ , \tag{2}$$

and with this we do renormalized perturbation theory with dimensional regularization.[8] As usual, with the renormalization at the critical theory, one can leave aside the mass term m_0 in the free-field propagator, and we assume either of two forms:

$$G^0_{\mathrm{SR}}(k)=(k^2+\tilde{v}_0k^\sigma)^{-1} \tag{3}$$

for a LR perturbation in a SR system (case A), or

$$G^0_{\mathrm{LR}}(k)=(k^\sigma+\tilde{w}_0k^2)^{-1} \tag{4}$$

for a SR perturbation in a LR system (case B), in which $\tilde{v}_0$ and $\tilde{w}_0$ denote bare dimensional coupling parameters that will be assumed to be of the same order of magnitude as $\tilde{u}_0$. The one-particle irreducible two- and four-point vertex functions $\Gamma^{(2)}(k)$ and $\Gamma^{(4)}(\{\vec{k}_i\})$ are constructed to $O(\tilde{u}_0^3)$, the same order as in the work of previous authors.

The expansion parameter depends on the propagator to be used. For case A, $\tilde{u}_0=\kappa^{4-d}u_0$, whereas for case B, $\tilde{u}_0=\kappa^{2\sigma-d}u_0$, in which κ is an arbitrary momentum-scale parameter and u_0 is the dimensionless quartic coupling. This leads to the expansions in *either* ϵ *or* ϵ', respectively. Renormalization of the vertex functions consists in the removal of dimensional poles in Γ functions either in ϵ or in ϵ'.

In calculating the first-order contribution of $\tilde{v}_0$ to the one-loop integral in $\Gamma^{(4)}(\{k_i\})$, for case A, one finds that this is given by

$$I_1^{(1)}(\{k_i\}) = -\frac{2}{\sigma}\frac{\Gamma(1-\sigma/2)}{\Gamma(2-\sigma/2)}\tilde{v}_0[|\vec{k}_1+\vec{k}_2|^{-(2-\sigma)-\epsilon}+\cdots] \quad (5)$$

to leading order in ϵ, where the ellipsis includes also two other terms generated by permutations. In the free-field limit, where $\epsilon=0$, this leads to the infrared divergence for any $\sigma<2$, and a similar behavior is found in higher powers of $\tilde{v}_0$. This suggests that there is no region of competing SR and LR interactions where the LR part is irrelevant.

Turning next to case B, the bare vertex functions are, to first order in $\tilde{w}_0$,

$$\kappa^{-\sigma}\Gamma^{(2)}(k) = \left(\frac{k}{\kappa}\right)^{\sigma}(1-B_2^{(0)}u_0^2+B_3^{(0)}u_0^3) + \left(\frac{k}{\kappa}\right)^2 w_0(1-B_2^{(1)}u_0^2) \; , \quad (6)$$

$$\kappa^{-\epsilon'}\Gamma^{(4)}(\{\vec{k}_i\}) = u_0(1-A_1^{(0)}u_0-A_1^{(1)}u_0w_0+A_2^{(0)}u_0^2) \; , \quad (7)$$

in terms of the dimensionless $w_0=\kappa^{2-\sigma}\tilde{w}_0$, where $A_L^{(j)}$ and $B_L^{(j)}$ are the contributions from the L-loop diagrams of jth order in w_0. Renormalized couplings u and w are introduced through

$$u_0=u(1+a_1^{(0)}u+a_1^{(1)}uw+a_2^{(0)}u^2) \; , \quad (8)$$

$$w_0=w(1+b_2^{(1)}u^2) \; . \quad (9)$$

Noting that

$$A_1^{(1)} = -\frac{n+8}{3\sigma}\frac{(\sigma-1)}{\Gamma(\sigma/2)} \times\Gamma\left(-1+\frac{\sigma}{2}+\frac{\epsilon'}{2}\right)(|\vec{k}_1+\vec{k}_2|^{(2-\sigma)-\epsilon'}+\cdots) \quad (10)$$

(where the ellipsis includes two other terms generated by permutations) does not have a pole for any $\sigma<2$, one finds that $a_1^{(1)}=0$ and Eq. (8) is the same as for pure LR interactions. This implies that the fixed-point coupling u^*, which is a zero of the Wilson β function, $\beta_u\equiv\kappa\partial u/\partial\kappa$ at fixed $\tilde{u}_0$ and $\tilde{w}_0$, is that for LR interactions given here by

$$u^*=\frac{6}{n+8}\epsilon'\left[1+\frac{2(5n+22)}{(n+8)^2}S(\sigma)\epsilon'\right]+O(\epsilon'^3) \; , \quad (11)$$

in which a factor $K_d=[2^{d-1}\pi^{d/2}\Gamma(d/2)]^{-1}$ is, as usual, absorbed in u, $S(\sigma)\equiv\psi(1)-2\psi(\sigma/2)+\psi(\sigma)$, where $\psi(z)$ is the logarithmic derivative of the Γ function. Calculation of $\Gamma^{(2)}(k)$ shows that there is no need for wave-function renormalization, provided that $b_2^{(1)}$ in Eq. (9) is chosen as

$$b_2^{(1)}=\left(1-\frac{1}{\sigma}\right)\frac{n+2}{24\epsilon'} \quad (12)$$

to cancel the dimensional pole in $B_2^{(1)}$. The renormalized vertex functions are

$$\Gamma_R^{(N)}(\{\vec{k}_i\};u,w)=Z_\phi^{N/2}\Gamma^{(N)}(\{\vec{k}_i\};u_0,w_0) \; ,$$

with $Z_\phi=1$, and this yields the known $\eta=2-\sigma$, for any $\sigma<2$.

Consider next the terms $B_2^{(0)}u_0^2$ and $B_3^{(0)}u_0^3$ in Eq. (6). The parts that are relevant near $\sigma\cong 2$ in the limit $\epsilon'\to 0$ are given by

$$B_2^{(0)}=\frac{n+2}{72}\frac{\Gamma^2(\sigma)}{\Gamma(3\sigma/2)}\Gamma\left(-\frac{\sigma}{2}+\epsilon'\right)k^{-2\epsilon'} \quad (13)$$

and

$$B_3^{(0)}=\frac{(n+2)(n+8)}{432\epsilon'}\frac{\Gamma^2(\sigma)}{\Gamma(3\sigma/2)}\Gamma\left(-\frac{\sigma}{2}+3\frac{\epsilon'}{2}\right)k^{-3\epsilon'} \; . \quad (14)$$

Taking the limit $\epsilon'\to 0$ for fixed σ will lead to coefficients in the terms $B_2^{(0)}u^2$ and $(B_3^{(0)}-2a_1^{(0)}B_2^{(0)})u^3$ in the renormalized two-point vertex which develop poles in the Γ functions only when σ becomes 2, and it is then that the LR expansion breaks down. It is interesting to note here that if one sets $\sigma=2$ only in the integrals before taking the limit $\epsilon'\to 0$, one needs wave-function renormalization which yields the erroneous result

$$\eta=2-\sigma+\gamma_\phi^* \quad (15)$$

by means of the usual scaling argument,[8,9] in which $\gamma_\phi^*\equiv\kappa\partial\ln Z_\phi/\partial\kappa$ at the fixed point, where $\gamma_\phi^*=O(\epsilon'^2)$.[10] Then η would reach the classical value $2-\sigma$ only asymptotically as $2-\sigma >> O(\epsilon^2)$. This seems to be in clear contradiction with the early result of Fisher, Ma, and Nickel.[1] The reason why a γ_ϕ^* that is different from zero in Eq. (15) is incorrect is because in deriving it one fails to recognize that $2-\sigma$ is *not* an expansion parameter. We come back to this point later.

The fixed point w^* follows as the zero of Wilson's β function for w,

$$\beta_w\equiv\left(\kappa\frac{\partial w}{\partial\kappa}\right)_{\tilde{u}_0,\tilde{w}_0} = -\left[\left(\kappa\frac{\partial\tilde{w}_0}{\partial\kappa}\right)_{u,w}+\beta_u\left(\frac{\partial\tilde{w}_0}{\partial u}\right)_{\kappa,w}\right]\left[\left(\frac{\partial\tilde{w}_0}{\partial w}\right)_{\kappa,u}\right]^{-1} = w[(2-\sigma)-2\beta_u b_2^{(1)}u(1-b_2^{(1)}u^2)] \; . \quad (16)$$

Clearly, $w^*=0$, and since $\partial\beta_w/\partial w=2-\sigma>0$ at the fixed point, where $\beta_u=0$, the LR fixed point is stable to SR perturbations for any $\sigma<2$. This, together with Eq. (11), is the basis for our calculation that yields LR critical exponents for any $\sigma\leq 2^-$. So far, we referred only to η, but calculation of $\gamma_{\phi^2}^*=\kappa\partial\ln Z_{\phi^2}/\partial\kappa$, at the fixed point, where Z_{ϕ^2} is the renormalization constant for a ϕ^2 insertion into the N-point vertex function, yields the known values of the critical exponents[1,2] through $\nu^{-1}=2-\eta-\gamma_{\phi^2}^*$ and $\gamma=(2-\eta)\nu$.

The main consequence of the fact that $2-\sigma$ is not an expansion parameter is that if one sets $\sigma=2$ in the coefficients of a given vertex function one has to everywhere else replace σ by 2. When one sets $\sigma=2$, however, ϵ has to coincide with ϵ' in accordance with Eq. (1). Thus ϵ becomes the SR expansion parameter taking the place of ϵ'. At the same time there is a new term appearing in the coefficient a_2 in Eq. (8) which now requires wave-function renormalization. Then it follows that γ_ϕ^* takes its SR value $\gamma_\phi^*(\mathrm{SR})$ and Eq. (15) yields $\eta=\gamma_\phi^*(\mathrm{SR})=\eta_{\mathrm{SR}}$.

When one does not ignore that $2-\sigma$ is not an expansion parameter, calculation of the exact RG recursion relations[11,12] yields

$$a'=s^{-\eta}\left[a-A_n\frac{u^2}{b^4}\left(2-\frac{2}{\sigma}\right)\left(3a\ \ln s-R(\sigma)b\frac{s^{2-\sigma}}{2-\sigma}\right)\right] , \quad (17)$$

$$b'=s^{2-\sigma-\eta}b , \quad (18)$$

for fixed $\sigma<2$ and first order in a/b when the free-field propagator is written as $G^0(k)=(ak^2+bk^\sigma)^{-1}$, and momentum-shell integrations are done over $s^{-1}<|\vec{k}|<1$, while A_n is a constant that depends only on n and $R(\sigma)$ tends to 1 as $\sigma\to 2$. The fixed point comes from the last term in Eq. (17) which, together with Eq. (18), yields

$$a^*/b^*=R(\sigma)\eta_{SR}(\epsilon')/(2-\sigma) ,$$

in which $b^*=\text{const}\neq 0$ and at the same time $\eta=2-\sigma$, while

$$\eta_{SR}(\epsilon')=\epsilon'^2(n+2)/2(n+8)$$

follows from the fixed-point value $u^*=2\pi^2\epsilon' b^{*2}/(n+8)$, to the order needed here. A nonzero a^* shows that iteration of the recursion relations generates SR interactions for nonzero b even if the initial $a=0$, in accordance with current expectations. However, as long as a^*/b^* remains finite, as in our case, a nonzero a^* (in contrast to $w^*=0$) does not change the critical behavior for fixed $\sigma<2$.

Since we cannot resum the terms in the recursion relation for a to all orders in a/b without setting $\sigma=2$, our result cannot be directly compared with Sak's Eq. (10),[4] which may be written as

$$a'=a-\eta a\ \ln s+\eta_{SR}(a+b)\ln s . \quad (19)$$

Nevertheless, it is interesting to note that Eq. (19) is obtained assuming that $\eta=2-\sigma$ is small, so that $s^{-\eta}$ is expandable in powers of η, with $\sigma=2$ in the integral that goes into the last term and ϵ' being replaced by ϵ. Accordingly, Eq. (18) yields $b'=b$—the second of Sak's recursion relations—and with this, $a^*=\eta_{SR}b^*/(\eta-\eta_{SR})$, which lead Sak to conclude that the LR expansion breaks down when $\eta<\eta_{SR}$. We argue that if there is a breakdown of the latter, this should be detected within the ϵ' expansion. In contrast, the logarithmic terms in Eq. (19), which appear only in expansion in powers of $(2-\sigma)$, already assume that ϵ is vanishingly small. It seems, therefore, that Eq. (19) is valid only in the limit $\sigma\to 2$, where $b^*=0$ from Eq. (18) and $\eta=\eta_{SR}$ with $a^*=\text{const}\neq 0$.

We conclude that there is a discontinuity of critical exponents at $\sigma=2$. Although the discussion presented here is in dimension $d=2\sigma-\epsilon'$, we expect to reach the same conclusion through the low-temperature RG in $d=\sigma+\epsilon'$, as well as for other models with LR interactions decaying as $1/|\vec{r}|^{d+\sigma}$, basically because what should be common to all of them is that $2-\sigma$ is not an expansion parameter, which is the crucial point of our argument.

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[1]M. E. Fisher, S.-K. Ma, and B. G. Nickel, Phys. Rev. Lett. 29, 917 (1972).
[2]M. Suzuki, Y. Yamazaki, and G. Igarashi, Phys. Lett. 42A, 313 (1972); Y. Yamazaki and M. Suzuki, Prog. Theor. Phys. 57, 1886 (1977).
[3]G. S. Joyce, Phys. Rev. 146, 349 (1966).
[4]J. Sak, Phys. Rev. B 8, 281 (1973).
[5]J. Sak, Phys. Rev. B 15, 4344 (1977).
[6]Y. Yamazaki, Nuovo Cimento A55, 59 (1980).
[7]A. C. D. van Enter, Phys. Rev. B 26, 1336 (1982).
[8]D. J. Amit, *Field Theory, the Renormalization Group and Critical Phenomena* (McGraw-Hill, New York, 1978).
[9]E. Brézin, J. C. Le Guillou, and J. Zinn-Justin, in *Phase Transitions and Critical Phenomena,* Vol. 6, edited by C. Domb and M. S. Green (Academic, New York, 1976).
[10]Some of the results of Yamazaki, Ref. 6, are precisely of this form.
[11]K. G. Wilson and M. E. Fisher, Phys. Rev. Lett. 28, 240 (1972); K. G. Wilson, *ibid.* 28, 548 (1972); K. G. Wilson and J. B. Kogut, Phys. Rep. 12C, 75 (1974).
[12]For the application to LR interactions in which σ is not replaced by 2, see A. Theumann, J. Phys. A 14, 2759 (1981).