FIP10601 − Text 8

Applications of Boltzmann equation

Charge and energy (heat) transport

As a first example of using Boltzmann's equation in the relaxation-time approximation, we will obtain transport coefficients for a metal simultaneously subjected to a static electric field $\mathbf{E}$ and to a temperature gradient $\boldsymbol{\nabla} T$. In fact, we will see that the presence of one implies the presence of the other.

Since the temperature varies with position, we replace the usual Fermi function by a *local-equilibrium distribution function*, which is written as

$$f^0(\mathbf{k}, \mathbf{r}) = \frac{1}{e^{[\varepsilon(\mathbf{k}) - \mu(\mathbf{r})]/k_B T(\mathbf{r})} + 1} \ . \tag{1}$$

Note that the temperature variation induces position dependence of the chemical potential,

$$\mu(\mathbf{r}) = \mu[T(\mathbf{r})] \ . \tag{2}$$

As seen in Text 7, the steady-state Boltzmann equation in the relaxation-time approximation is

$$\frac{\partial f}{\partial \mathbf{r}} \cdot \dot{\mathbf{r}} + \frac{\partial f}{\partial \mathbf{k}} \cdot \dot{\mathbf{k}} = -\frac{\delta f}{\tau} \ . \tag{3}$$

The dependence of f on $\mathbf{r}$ is due to T and μ, so that

$$\frac{\partial f}{\partial \mathbf{r}} = \frac{\partial f}{\partial T} \boldsymbol{\nabla} T + \frac{\partial f}{\partial \mu} \boldsymbol{\nabla} \mu \ . \tag{4}$$

We will consider a constant $\boldsymbol{\nabla} T$, which implies a constant $\boldsymbol{\nabla} \mu$. It usually suffices to take the linear-response limit, which allows the replacements

$$
\begin{aligned}
\frac{\partial f}{\partial T} &\longrightarrow \frac{\partial f^0}{\partial T} \ , \\
\frac{\partial f}{\partial \mu} &\longrightarrow \frac{\partial f^0}{\partial \mu} \ .
\end{aligned} \tag{5}
$$

Now the temperature T and the chemical potential μ are to be seen as average values over the entire system, since inclusion of their spatial variations would lead to higher-order contributions on $\boldsymbol{\nabla} T$. So, f^0 now denotes the "normal" Fermi function.

From the explicit form of f^0, it is easy to see that

$$
\begin{aligned}
\frac{\partial f^0}{\partial \mu} &= -\frac{\partial f^0}{\partial \varepsilon_{\mathbf{k}}} \,, \\
\frac{\partial f^0}{\partial T} &= \frac{\varepsilon_{\mathbf{k}} - \mu}{T} \left(-\frac{\partial f^0}{\partial \varepsilon_{\mathbf{k}}} \right) \,.
\end{aligned}
\tag{6}
$$

As $\dot{\mathbf{k}} = -e\mathbf{E}/\hbar$, being therefore linear in $\mathbf{E}$, we may perform the replacement

$$
\frac{\partial f}{\partial \mathbf{k}} \longrightarrow \frac{\partial f^0}{\partial \mathbf{k}} = \frac{\partial f^0}{\partial \varepsilon_{\mathbf{k}}} \hbar \mathbf{v}_{\mathbf{k}}
\tag{7}
$$

in Eq. (3). Furthermore, using the fact that $\dot{\mathbf{r}} = \mathbf{v}_{\mathbf{k}}$, the solution of that equation is

$$
\delta f(\mathbf{k}) = \left(-\frac{\partial f^0}{\partial \varepsilon_{\mathbf{k}}} \right) \tau \mathbf{v}_{\mathbf{k}} \cdot \left[-e\mathbf{E} + \frac{\varepsilon_{\mathbf{k}} - \mu}{T} (-\boldsymbol{\nabla} T) \right] \,.
\tag{8}
$$

Here we explicitly see that δf depends only on $\mathbf{k}$ and on the "external fields" $\mathbf{E}$ and $(-\boldsymbol{\nabla} T)$. For this, $\boldsymbol{\nabla} \mu$ has been absorbed on a redefinition of the electric field $\mathbf{E}$, which is actually an effective electric field, given by the gradient of the electrochemical potential $(\phi - \mu/e)$. In practice, the difference between this effective $\mathbf{E}$ and the applied electric field tends to be negligible due to the weak temperature dependence of the chemical potential in metals at normal experimental conditions.

Electric and thermal currents

One of the most important characteristics of metals is that they transport electric current. The corresponding intrinsic quantity is the current density, i.e., the total current per unit volume. It is often written as $\mathbf{j} = -ne\mathbf{v}$, in terms of the global electronic density and of some average (drift) velocity of the ensemble of conduction electrons. Therefore it encompasses contributions from all the electrons in their respective states, and can be written as

$$
\mathbf{j} = \frac{(-e)}{4\pi^3} \int \mathrm{d}^3 k \, \mathbf{v}_{\mathbf{k}} \, f(\mathbf{k}) \,.
\tag{9}
$$

Using the previously defined non-equilibrium part of the distribution function, $\delta f(\mathbf{k}) \equiv f(\mathbf{k}) - f^0(\mathbf{k})$, and taking into account that the equilibrium current is zero, we have

$$
\mathbf{j} = \frac{(-e)}{4\pi^3} \int \mathrm{d}^3 k \, \mathbf{v}_{\mathbf{k}} \, \delta f(\mathbf{k}) \,.
\tag{10}
$$

Similarly, we can define a *thermal-current density*

$$
\mathbf{u} = \frac{1}{4\pi^3} \int \mathrm{d}^3 k \, (\varepsilon_{\mathbf{k}} - \mu) \, \mathbf{v}_{\mathbf{k}} \, \delta f(\mathbf{k}) \,.
\tag{11}
$$

which is effectively an energy-current density. Note that the extra energy carried by an electron, (i.e., its excitation energy) is $\varepsilon_\mathbf{k} - \mu$.

Using Eq. (8), these two current densities can be written as

$$\mathbf{j} \;=\; e^2\,\mathbf{K}_0 \cdot \mathbf{E} - \frac{e}{T}\,\mathbf{K}_1 \cdot (-\boldsymbol{\nabla}T) \,, \tag{12}$$

$$\mathbf{u} \;=\; -e\,\mathbf{K}_1 \cdot \mathbf{E} + \frac{1}{T}\,\mathbf{K}_2 \cdot (-\boldsymbol{\nabla}T) \,, \tag{13}$$

where we defined the tensors

$$\mathbf{K}_n \equiv \frac{\tau}{4\pi^3\hbar} \int \mathrm{d}\varepsilon \left(-\frac{\partial f^0}{\partial \varepsilon} \right) (\varepsilon - \mu)^n \int_{S_\varepsilon} \frac{\mathrm{d}S_\mathbf{k}}{|\mathbf{v_k}|} \mathbf{v_k}\mathbf{v_k} \,. \tag{14}$$

Low temperatures

If we take the zero temperature limit, in which

$$\mu = \varepsilon_F \,, \qquad \left(-\frac{\partial f^0}{\partial \varepsilon} \right) = \delta(\varepsilon - \varepsilon_F) \,, \tag{15}$$

only $\mathbf{K}_0$ is non null. We must therefore utilize the low-temperature expansion developed in Text 5. Defining

$$\mathbf{K}(\varepsilon) \equiv \frac{\tau}{4\pi^3\hbar} \int_{S_\varepsilon} \frac{\mathrm{d}S_\mathbf{k}}{|\mathbf{v_k}|} \mathbf{v_k}\mathbf{v_k} \,, \tag{16}$$

Eq. (14) assumes the form

$$\mathbf{K}_n = \int \mathrm{d}\varepsilon \left(-\frac{\partial f^0}{\partial \varepsilon} \right) (\varepsilon - \mu)^n \mathbf{K}(\varepsilon) \,. \tag{17}$$

In Text 5 we obtained

$$\int \mathrm{d}\varepsilon \left(-\frac{\partial f^0}{\partial \varepsilon} \right) G(\varepsilon) = G(\mu) + \frac{\pi^2}{6}(k_BT)^2 \left[\frac{\partial^2 G}{\partial \varepsilon^2} \right]_{\varepsilon=\mu} + O(T^4) \,. \tag{18}$$

The following correspondences apply in calculating $\mathbf{K}_n$ from Eq. (17):

$$\begin{aligned}
\mathbf{K}_0 &\longrightarrow & G(\varepsilon) &= \mathbf{K}(\varepsilon) \,, \\
\mathbf{K}_1 &\longrightarrow & G(\varepsilon) &= (\varepsilon - \mu)\,\mathbf{K}(\varepsilon) \,, \\
\mathbf{K}_2 &\longrightarrow & G(\varepsilon) &= (\varepsilon - \mu)^2\,\mathbf{K}(\varepsilon) \,.
\end{aligned} \tag{19}$$

Then, taking the most important term in each case, and considering that μ can be replaced by ε_F at the same level of approximation, we have

$$\begin{aligned}
\mathbf{K}_0 &= \mathbf{K}(\varepsilon_F) \,, \\
\mathbf{K}_1 &= \frac{\pi^2}{3}(k_BT)^2 \left[\frac{\partial \mathbf{K}(\varepsilon)}{\partial \varepsilon} \right]_{\varepsilon=\varepsilon_F} \,, \\
\mathbf{K}_2 &= \frac{\pi^2}{3}(k_BT)^2\,\mathbf{K}(\varepsilon_F) = \frac{\pi^2}{3}(k_BT)^2\,\mathbf{K}_0 \,.
\end{aligned} \tag{20}$$

Note that $\mathbf{K}_0$ does not depend explicitly on temperature, while $\mathbf{K}_1$ and $\mathbf{K}_2$ are both of order T^2.

Electrical conductivity

The electrical conductivity $\boldsymbol{\sigma}$ is, by definition, the coefficient of proportionality between electric field and electric-current density. More generally, this coefficient is a tensor. Therefore, from Eq. (12),

$$\boldsymbol{\sigma} = e^2 \, \mathbf{K}_0 \; . \tag{21}$$

Depending on the specific lattice symmetries, the conductivity tensor may assume simpler forms. For example, for a cubic lattice

$$\sigma_{xx} = \sigma_{yy} = \sigma_{zz} \equiv \sigma \; , \tag{22}$$

and all the other components are **null**. In this case, if $\mathbf{E} \parallel x$,

$$(\mathbf{vv}) \cdot \mathbf{E} = v_x^2 E = \frac{1}{3} v^2 E \; . \tag{23}$$

Then, using the definition of $\mathbf{K}(\varepsilon)$, Eq. (16), we obtain

$$\sigma = \frac{e^2 \tau}{12\pi^3 \hbar} \int_{S_F} v(\mathbf{k}) \mathrm{d}S_{\mathbf{k}} \; . \tag{24}$$

Further simplification can be achieved by focusing on a case of Fermi level near a band minimum (or maximum), so that $\varepsilon(\mathbf{k})$ can be taken as approximately parabolic, and the electrons (or holes) have a scalar effective mass m^*. In this case, the Fermi surface is approximately spherical. Hence, the velocity has uniform magnitude over the entire surface, being given by

$$v_F = \frac{\hbar \bar{k}_F}{m^*} \; , \tag{25}$$

where $\bar{k}_F$ is the Fermi-surface radius, measured from the location of the band minimum (or maximum). Taking v_F out of the integral in Eq, (24), it reduces to

$$\int_{S_F} \mathrm{d}S_{\mathbf{k}} = 4\pi \bar{k}_F^2 \; , \tag{26}$$

and we obtain

$$\sigma = \frac{e^2 \bar{k}_F^3 \tau}{3\pi^2 m^*} \; . \tag{27}$$

If there are n electrons (or holes) per unit volume in the band, it is easy to see that $\bar{k}_F^3 = 3\pi^2 n$, so that the conductivity can be written as

$$\sigma = \frac{n e^2 \tau}{m^*} \; . \tag{28}$$

This is the same as for free electrons, except that it applies also to a hole band, and lattice effects are taken into account through the effective mass.

Thermal conductivity

The thermal conductivity $\boldsymbol{\kappa}$ is a transport coefficient which relates thermal-current density with temperature drop (negative of the gradient). So, by Eq. (13),

$$\boldsymbol{\kappa} = \frac{1}{T}\mathbf{K}_2 \ . \tag{29}$$

In practice, the thermal conductivity is measured in open circuit, that is, $\mathbf{j} = 0$. Using this condition in Eq. (12), one notices that there is a nonzero electric field which is **induced** by the temperature gradient (we will come back to this in more detail later). Then, Eq. (13) yields the combined coefficient of $-\boldsymbol{\nabla}T$ as

$$\boldsymbol{\kappa} = \frac{1}{T}(\mathbf{K}_2 - \mathbf{K}_1 \cdot \mathbf{K}_0^{-1} \cdot \mathbf{K}_1) \ . \tag{30}$$

But the correction term is of order T^4 while $\mathbf{K}_2$ is of order T^2.

Wiedmann-Franz law

Equations (21) and (29) for $\boldsymbol{\sigma}$ and $\boldsymbol{\kappa}$, together with the relationship between $\mathbf{K}_2$ and $\mathbf{K}_0$ in the last line of Eqs. (20), allow to write

$$\frac{\boldsymbol{\kappa}}{T} = \frac{\pi^2 k_B^2}{3e^2}\,\boldsymbol{\sigma} \ . \tag{31}$$

This expression (or, originally, its scalar form) is known as *Wiedmann-Franz law*. It serves as a test for the relaxation-time approximation, since the coefficient of proportionality between $\boldsymbol{\kappa}/T$ and $\boldsymbol{\sigma}$ does not depend on τ. This coefficient is known as *Lorenz number*. Its value from the above expression is 2.44×10^{-8} W$\,\Omega$/K^2. Experimental results for a large number of metals (such as Au, Ag, Cu, Be, Mg, Zn, Al, Sn, . . .) yield values ranging from 2.0–2.5 $\times 10^{-8}$ W$\,\Omega$/K^2, and weakly dependent on temperature, in good agreement with the Wiedmann-Franz law. However, there is also a large number of other metals for which deviations of the Wiedmann-Franz law are quite significant. For these the validity conditions of the relaxation-time approximation are not fulfilled.

Thermoelectric power

As mentioned above, an electric field appears due to the presence of a temperature gradient, which characterizes a thermoelectric effect. Using the open-circuit condition $\mathbf{j} = 0$ in Eq. (12), we obtain

$$\mathbf{E} = -\frac{1}{eT}\mathbf{K}_0^{-1} \cdot \mathbf{K}_1 \cdot \boldsymbol{\nabla}T \ . \tag{32}$$

This equation can be written as

$$\mathbf{E} = \mathbf{Q} \cdot \boldsymbol{\nabla}T \ , \tag{33}$$

where

$$\mathbf{Q} = -\frac{1}{eT}\,\mathbf{K}_0^{-1}\cdot\mathbf{K}_1 \tag{34}$$

is known as *thermoelectric power* (also called *thermopower*), and relates the induced electric field to the applied temperature gradient.

From the experimental point of view, thermoelectric effects are observed in two basic forms:

- *Seebeck Effect* – Two wires made of different metals are connected at one extremity, and stretched side by side but not connect at the opposite end. Such a set is usually called a *thermocouple.* If the connected extremities are kept at a different temperature with respect to the non-connected ones, a voltage difference appears between the latter. This happens because both metals are subjected to the same temperature gradient, but the electric fields induced in each of them are not the same due to their distinct thermoelectric powers. The thermopower Q (scalar in this setup) is also called *Seebeck coefficient* (denoted by S).

- *Peltier effect* – In this case, two wires of the same metal are connected to terminals of a voltage source, while their other extremities are connected through a wire made of a different metal. A stationary electric current will flow around the circuit, but heat will be radiated at one junction and absorbed at the other. This occurs because the different thermoelectric powers of the two metals cause different thermal currents to flow in each of them. The thermal and charge current-density intensities may be related as $u = \Pi\,j$, through the *Peltier coefficient* ($\Pi = T\,Q$).

We must remark that additional temperature effects on the transport coefficients may appear through the relaxation time, which takes part in the $\mathbf{K}_n$ factors through $\mathbf{K}(\varepsilon)$, Eq. (16), and here was taken as constant. To obtain temperature-dependent relaxation times it is necessary to analyze in detail the relevant scattering mechanisms. As previously mentioned in Text 7, the two dominant mechanisms in (non-magnetic) metals are scattering by impurities and by phonons. It is quite obvious that scattering by impurities is essentially independent of temperature, since atomic (electronic) excitation energies are large. However, electron-phonon interactions yield temperature-dependent relaxation times which agree well with experimental observations of the behavior of electrical conductivity with temperature, as we will see in Unit 3.

Transport in the presence of electric and magnetic fields

Let us consider a system subjected to an electric field $\mathbf{E}$ and a magnetic field corresponding to a magnetic induction $\mathbf{B}$, both static and uniform. In order to focus on the interplay between these two fields, we will neglect thermal effects. Then the distribution function is spatially uniform, and Boltzmann's equation in the relaxation time approximation takes the simple form

$$\frac{\partial f}{\partial \mathbf{k}}\cdot\dot{\mathbf{k}} = -\frac{\delta f}{\tau}\ . \tag{35}$$

Using the semiclassical equation of motion for $\dot{\mathbf{k}}$ obtained in Text 7, we have

$$\delta f(\mathbf{k}) = -\frac{e\tau}{\hbar}\left(-\frac{\partial f}{\partial \mathbf{k}}\right) \cdot [\mathbf{E} + \mathbf{v_k} \times \mathbf{B}] \ . \tag{36}$$

Note that we cannot make the substitution (7) on the magnetic-field term because this would involve $\mathbf{v_k} \cdot (\mathbf{v_k} \times \mathbf{B}) = 0$. It is therefore necessary to include $\nabla_{\mathbf{k}}\delta f(\mathbf{k})$, neglected before. Thus, the equation determining $\delta f(\mathbf{k})$ is

$$\left[1 - \frac{e\tau}{\hbar}(\mathbf{v_k} \times \mathbf{B}) \cdot \nabla_{\mathbf{k}}\right]\delta f(\mathbf{k}) = -e\tau\left(-\frac{\partial f^0}{\partial \varepsilon_{\mathbf{k}}}\right)\mathbf{v_k} \cdot \mathbf{E} \ . \tag{37}$$

Comparing with the case without magnetic field, Eq. (8), we suppose a solution of the type

$$\delta f(\mathbf{k}) = -e\tau\left(-\frac{\partial f^0}{\partial \varepsilon_{\mathbf{k}}}\right)\mathbf{v}(\mathbf{k}) \cdot \mathbf{\Lambda} \ , \tag{38}$$

where $\mathbf{\Lambda}$ is a vector to be determined. Restricting ourselves to the validity conditions of the relaxation time approximation, which justify the hypothesis of a scalar effective mass, it is easy to show that the equation determining $\mathbf{\Lambda}$ from the applied fields can be written as

$$\mathbf{\Lambda} - \frac{e\tau}{m^*}\mathbf{B} \times \mathbf{\Lambda} = \mathbf{E} \ , \tag{39}$$

and that the current density is given by

$$\mathbf{j} = \sigma_0\mathbf{\Lambda} \ , \tag{40}$$

where $\sigma_0 = ne^2\tau/m^*$, the electrical conductivity in the absence of magnetic field.

As an application of the above equations, let us consider a standard geometry of the *Hall effect*, with $\mathbf{B} = (0, 0, B)$, $\mathbf{j} = (j_x, 0, 0)$, and $E_x \neq 0$ due to an applied voltage along x. The steady state current is restricted to the x direction, implying that only the Λ_x component is non null. Equation (39) admits the solution $\mathbf{\Lambda} = (E_x, 0, 0)$, with the electric field $\mathbf{E} = (E_x, E_y, 0)$, if the E_y component satisfies the condition

$$E_y = -\frac{e\tau E_x B}{m^*} = -\frac{\sigma_0 E_x B}{ne} \ . \tag{41}$$

The *Hall coefficient* is defined as

$$R_H = \frac{E_y}{j_x B} \ . \tag{42}$$

Considering that $\mathbf{j} = (\sigma_0 E_x, 0, 0)$, we obtain

$$R_H = -\frac{1}{ne} \ . \tag{43}$$

Because only the E_x component is due to the applied voltage, the E_y component reveals that a transverse voltage is **induced** in the sample by the presence of the magnetic field, which characterizes the *Hall effect*. This can be understood as the result of the transversal deviation of the electrons under the magnetic field, yielding an accumulation of negative charge on one side of the sample, with a corresponding non-compensated positive charge on the opposite side.

The negative sign of the Hall coefficient in Eq. (43) comes from the assumption that we have conduction **electrons**. Applying the same formalism to a *hole-like band*, we obtain the same form but with opposite sign. Therefore, a Hall-effect experiment yields information on which type of charge carrier is present in the material.

It is worth remarking that the current density calculated here (i.e., for a single conduction band, and in the Hall-effect geometry) does not show any *magnetoresistance*, i.e., magnetic-field dependence of the electrical conductivity. Magnetoresistance appears when one deals with less trivial band structures, involving the presence of more than one band at the Fermi level, or the contribution of "open orbits" (very far from spherical Fermi surfaces). Other observations of magnetoresistance occur in much different contexts, like systems presenting strong electronic correlations, spin-orbit interaction, or magnetic order.

Transport in confined systems

Up to now we have focused on *bulk* electronic properties of three-dimensional crystalline solids. Even though we considered, for convenience, a finite volume V, we used periodic boundary conditions in all directions, which makes sense because because the thermodynamic limit can be implied, expressing the relevant quantities in terms of appropriate densities. In the presence of static external fields, stationary electric an thermal currents where seen to be well described using the Boltzmann equation in the *relaxation-time approximation*. This is no longer true when the system size is sufficiently small in one or more directions. The first step is to quantify how small.

We will restrict our analysis to charge transport, i.e., electric current. Our previous description for bulk systems in the linear response regime related the current density **j** to the applied electric field **E** through the electrical conductivity $\boldsymbol{\sigma}$. For an isotropic system, and under simplifying assumptions, we obtained a scalar conductivity

$$\sigma = \frac{n\,e^2\,\tau_{\text{sc}}}{m^*}\;. \tag{44}$$

Here we made a change of notation from Eq. (28), denoting the relaxation time as τ_{sc}, instead of just τ, in order to explicitly relate it to scattering processes, which are *intrinsic* to the material, and not related to specific features of a sample. We already know that such processes (basically due to lattice defects or phonons) are essentially elastic, and only involve electrons near the Fermi level. This allows to define a **mean free path** $\ell_{\text{sc}} = v_F \tau_{\text{sc}}$, where v_F is the velocity of an electron with energy close to ε_F.

Supposing a macroscopic sample of a solid, with length L and cross-section area A, its electric **resistance** is given by

$$R = \frac{L}{\sigma A} \; . \tag{45}$$

Alternatively, we can write its **conductance**, which is the inverse of the resistance, as

$$G = \frac{A}{L} \sigma \; . \tag{46}$$

The well-known relation $V = R\,I$, between the applied voltage and the resulting current, may be written in terms of the conductance as $I = G\,V$.

In macroscopic systems it is convenient to work with the **intrinsic** quantity σ, or its inverse, the resistivity ρ, which are characteristics of the material, not the sample. However, the dependence of σ on τ_{sc} and the dependence of the resistance or the conductance on the sample's dimensions, as seen in Eqs. (45) and (46), allow us to infer that these equations should cease to be valid when at least one of the system's dimensions is smaller than the mean free path ℓ_{sc}.

We can also infer that transport properties are changed if there is *quantum confinement*, which occur when one or more of the system dimensions is smaller than some (system dependent) confinement length, ℓ_{C}. We can loosely define this length as being of the order of the electron's de Broglie wavelength, λ_{dB}. In bulk metals, since the relevant electrons are those near the Fermi surface, we should have $\lambda_{\mathrm{dB}} \simeq 2\pi/k_F$, which is of the order of the lattice parameters, i.e., less than 1 nm. On the other hand, in doped semiconductors, which have much fewer electrons in the conduction band, k_F is reduced by one or two orders of magnitude. Consequently, λ_{dB} grows in the same proportion, and ℓ_{C} assumes typical values between 10 and 100 nm. For this reason, confined systems are usually engineered from semiconductors.

A common construction has a thin layer of a given semiconductor "sandwiched" between two layers of another semiconductor, with a larger gap. Upon doping the side layers, the added charge carriers (electrons or holes) "fall" into the central well, that has levels of lower energy. The wavefunctions of particles in this quantum well are confined in the direction perpendicular to the layers, having discrete quantized energies associated to this degree of freedom. On the other hand, they are free to move in the two directions parallel to the layers, having free-particle energies associated to these two degrees of freedom. The result is that for each quantized energy due the transversal confinement there is a *two-dimensional electron gas* (2DEG). Now, it is easy to infer that confinement in two directions will yield a one-dimensional electron gas (1DEG) for each *channel* defined by the discrete energies resulting from quantization in the other two dimensions. This latter system is called a *quantum wire.* Finally, confinement in all three dimensions results in a totally discrete energy spectrum. Such a system, being small in all directions, is called a *quantum dot.*

Obviously, we can only talk about *transport* when the system is not quantum confined in the direction of flow of an electric current. This means that the system size L in this

direction is larger than the confinement length ($L > \ell_C$), but there can be confinement in the transversal directions.

The most confined system for which one can still have *transport* is a quantum wire. Supposing that its two extremities are connected to bulk conductors, there is no confinement along its length L. If the wire is very "clean" (few defects), and the temperature is very low (no phonons), we may have $L < \ell_{\rm sc}$, resulting in *ballistic transport.*

What is the conductance of such a wire? Our experience with bulk metals leads us to think that the conductance would be infinite, since the electrons travel without scattering. But the right answer must be obtained from the analysis of an appropriate experimental situation. The metallic contacts at both ends of the wire may be viewed as electron "reservoirs", which are connected to a voltage source (a battery, for example), so that a voltage difference V is established between them, as schematically depicted in Fig. 1.

In one dimension, the current I is the same as the current density j, and can be evaluated by the expression

$$I = -e\,\frac{2}{L}\sum_{\mathbf{k}} v_{\mathbf{k}} = -2\,e \int \frac{dk}{2\pi}\, v_{\mathbf{k}} \;, \tag{47}$$

where the factor 2 comes from spin degeneracy. It is convenient to rewrite this equation in terms of an energy integral,

$$I = -e \int_{\mu_1}^{\mu_2} d\varepsilon\, \tfrac{1}{2}\mathcal{D}(\varepsilon)\, v(\varepsilon) \;. \tag{48}$$

The integration limits take into account that only *non-compensated* electrons, i.e., those in states in between the two chemical potentials (Fermi levels), contribute to the current. The factor $\tfrac{1}{2}$ is necessary because these electrons are moving from left to wright while the DOS includes the states with opposite wavevectors.
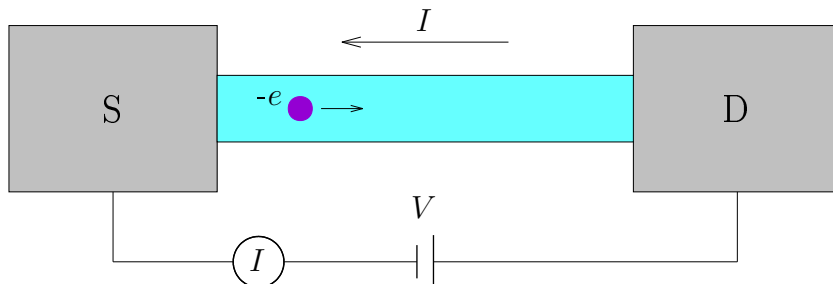
Figure 1: Schematic representation of a quantum wire connected to two metallic reservoirs subjected to a voltage difference. The letters S e D stand for *source* and *drain* with respect to the electrons.

The relevant DOS is that of a one-dimensional electron gas. From the general equations seen in Text 3, it is easy to verify (**EXERCISE**) that it can be written as

$$\mathcal{D}(\varepsilon) = \left(\frac{m^*}{2\pi^2\hbar^2}\right)^{1/2} \varepsilon^{-1/2} . \tag{49}$$

The complete specification of the energies involves a label referring to the confined part. We can write the energies as $\varepsilon_\alpha(k) = \epsilon_\alpha + \varepsilon(k)$, where the subscript α represents a pair of quantum numbers associated to the quantization due to confinement in the transversal directions. Each α defines a *conduction channel*, since it refers to a one-dimensional band of energies $\varepsilon(k)$. Eq. (48) gives the contribution of a single channel.

From the above form of the one-dimensional DOS, and taking into account that $v = \hbar\, k/m^*$ for the corresponding parabolic band, it is easy to verify that

$$\tfrac{1}{2}\mathcal{D}(\varepsilon) = \frac{2}{h\, v(\varepsilon)} . \tag{50}$$

Inserting into Eq. (48), we obtain the simple relationship

$$I = -2\frac{e}{h}\Delta\mu . \tag{51}$$

On the other hand, a difference between the chemical potentials is due to the applied potential, i.e., $\Delta\mu = -eV$. So,

$$I = \frac{2e^2}{h}\, V . \tag{52}$$

Since we should have $I = GV$, we obtain the *quantum conductance* (or *conductance quantum*)

$$G_0 = \frac{2e^2}{h} . \tag{53}$$

The conductance of a given quantum wire is, then, quantized, being the product of G_0 by the number of channels in the energy window of width $\Delta\mu$.

It is interesting to observe that the conductance quantum is *universal*, since it does not depend on specific characteristics of the material from which the wire is made. Furthermore, the value of G_0 is far from infinity, in contrast to our first conjecture. In fact, the corresponding resistance is approximately $12.9\,\mathrm{k\Omega}$, which is quite large in comparison with typical metallic resistors. For instance, since the resistivity of copper at room temperature is about $1.6\,\mu\Omega\,\mathrm{cm}$, the resistance of a copper wire with a diameter of 6 mm and a length of 5 cm is approximately $28\,\mu\Omega$.