Sine-Gordon renormalization of the orientational roughening transition

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The Laplacian roughening model is believed to undergo two interfacial roughening transitions. Upon reducing the temperature, the symmetry of the Hamiltonian is broken in two stages, the first in which orientational order develops, and the second in which the interface becomes completely flat. Within the traditional Coulomb gas picture these two transitions correspond, respectively, to unbinding of the dipole and charge pairs. We show that the orientational transition can be described by a continuous interface Hamiltonian and studied in the context of the sine-Gordon type of renormalization group. The flow equations deduced by the present techniques are, to leading order, equivalent to those for a vector Coulomb gas.

I. INTRODUCTION

In some investigations of the two-dimensional melting problem[1] Nelson was led to study the discrete Laplacian roughening model on a triangular lattice[2]

$$H=\frac{-\kappa}{2}\sum_n [\Delta h(\mathbf{n})]^2 \;, \tag{1.1}$$

where $h(\mathbf{n})$ is the height of a solid-on-solid surface at lattice site $\mathbf{n}=(x,y)$, and Δ is a lattice approximation to $\nabla^2 h(\mathbf{x})$:

$$\Delta h(\mathbf{n})\equiv\tfrac{2}{3}\sum_{i=1}^{6} h(\mathbf{n}+\delta_i)-4h(\mathbf{n}) \;. \tag{1.2}$$

$\{\delta_i\}$ are the six interlattice unit vectors centered on site $\mathbf{n}$.

This work indicated that the Hamiltonian (1.1) should have two phase transitions, the first in which a completely disordered surface develops orientational order, the second being a conventional transition in which the interface becomes translationally and orientationally flat.[3] The orientational phase transition was, it was argued, equivalent to that of a vector Coulomb gas[4-6] and therefore should have the characteristics of a Kosterlitz-Thouless transition.[7] Some years earlier Ohta and Kawasaki[8] had considered a modified discrete Gaussian model for the description of simple interfacial transitions. Ohta and Jasnow[9] and Knops and den Ouden[10] then developed a momentum space renormalization group and Amit, Goldschmidt, and Grinstein[11] carried out a field-theoretic treatment of the equivalent sine-Gordon model. As was expected, the model had the usual scalar Coulomb-gas phase transition. In this paper we show that a similar transcription is possible for the orientational roughening problem.

The sine-Gordon renormalization method is developed in an entirely different manner from the Coulomb gas procedure and it is interesting to be able to study interfacial transitions in either formalism. Furthermore, the study of an interfacial Hamiltonian with both surface tension and curvature energy terms is more appropriate within this scheme.[12] Finally, the problem of roughening of many surfaces or domain walls requires one to include surface-surface interactions and then it is not possible to work within the Coulomb gas formalism, whereas the sine-Gordon interfacial Hamiltonian is readily generalizable. We now develop the sine-Gordon renormalization-group method for orientational roughening.

II. MODEL

The discrete Laplacian model for a triangular lattice defined by Eq. (1.1) possesses an orientational phase transition which may be described by the Hamiltonian (we set the lattice constant equal to 1)

$$H=-\tfrac{1}{2}\kappa\int d^2\mathbf{x}(\nabla^2\varphi)^2+2y\sum_{i=1}^{3}\int d^2\mathbf{x}\cos[\mathbf{e}_i\cdot\nabla\varphi(\mathbf{x})] \;, \tag{2.1}$$

where $\varphi(\mathbf{x})$ is the height of the interface at position $\mathbf{x}$ and

$$\mathbf{e}_i=(1,0),(-\tfrac{1}{2},\tfrac{1}{2}\sqrt{3}),(-\tfrac{1}{2},-\tfrac{1}{2}\sqrt{3}) \;. \tag{2.2}$$

In the Appendix we show that this Hamiltonian may be transcribed to a vector Coulomb gas. We now construct the flow equations for renormalization of the curvature κ and fugacity y.

We begin by defining $\phi(q)$ to be a cut-off function such that $\phi(q)\approx 0$ for $q>1$ and $\phi(q)\approx 1$ for $q<1$. The momentum shell propagator that will appear during the renormalization-group procedure is

$$G_{\mathbf{e}_i\mathbf{e}_j}(\mathbf{x})=-\frac{\epsilon}{\kappa}\int\frac{(\mathbf{e}_i\cdot\mathbf{q})(\mathbf{e}_j\cdot\mathbf{q})e^{i\mathbf{q}\cdot\mathbf{x}}}{q^4}\times\frac{d\phi}{dq}\frac{1}{(2\pi)^2}d^2\mathbf{q}\equiv\epsilon g_{\mathbf{e}_i\mathbf{e}_j}(\mathbf{x}) \;. \tag{2.3}$$

After integrating over high momentum fluctuations ($\varphi^>$) in the variable $\varphi(\mathbf{x})$, we find the Hamiltonian

$$H'=-\tfrac{1}{2}\kappa\int d^2\mathbf{x}\varphi^{<}(\mathbf{x})\nabla^4\varphi^{<}(\mathbf{x}) +2y\sum_{i=1}\int d^2\mathbf{x}\langle\cos[\mathbf{e}_i\cdot\nabla\varphi(\mathbf{x})]\rangle^{>} +\tfrac{1}{2}(2y)^2\sum_{\substack{i,j\\ i,j=1}}\int d^2\mathbf{x}\,d^2\mathbf{y}\langle\cos[\mathbf{e}_i\cdot\nabla\varphi(\mathbf{x})] \times\cos[\mathbf{e}_j\cdot\nabla\varphi(\mathbf{y})]\rangle_c^{>}\ , \tag{2.4}$$

which is correct to second order in fugacity.

The superscripts less than and greater than stand, respectively, for slow and fast modes. Now, the averages in Eq. (2.4) are

$$\langle\cos[\mathbf{e}_i\cdot\nabla\varphi(\mathbf{x})]\rangle^{>}=\cos[\mathbf{e}_i\cdot\nabla\varphi^{<}(\mathbf{x})]\exp[-\tfrac{1}{2}G_{\mathbf{e}_i\mathbf{e}_i}(0)] \tag{2.5}$$

and

$$\langle\cos[\mathbf{e}_i\cdot\nabla\varphi(\mathbf{x})]\cos[e_j\cdot\nabla\varphi(\mathbf{y})]\rangle_c^{>}=\tfrac{1}{2}\exp[-G_{\mathbf{e}_i\mathbf{e}_i}(0)](\{\exp[-G_{\mathbf{e}_i\mathbf{e}_j}(\mathbf{x}-\mathbf{y})]-1\}\cos[\mathbf{e}_i\cdot\nabla\varphi^{<}(\mathbf{x})+\mathbf{e}_j\cdot\nabla\varphi^{<}(\mathbf{y})] +\{\exp[G_{\mathbf{e}_i\mathbf{e}_j}(\mathbf{x}-\mathbf{y})]-1\}\cos[\mathbf{e}_i\cdot\nabla\varphi^{<}(\mathbf{x})-\mathbf{e}_j\cdot\nabla\varphi^{<}(\mathbf{y})])\ . \tag{2.6}$$

For a triangular lattice it is important to note that the basic interlattice vectors satisfy the relation

$$\mathbf{e}_1+\mathbf{e}_2=-\mathbf{e}_3\ . \tag{2.7}$$

From the Hamiltonian (2.4) we now seek to extract those terms that will renormalize the bare curvature and fugacity. Thus defining $\mathbf{r}=\mathbf{x}-\mathbf{y}$ and $\mathbf{u}=\frac{1}{2}(\mathbf{x}+\mathbf{y})$ the term of order y^2 in Eq. (2.4) will become

$$y^2\sum_{\substack{i,j\\ i,j=1}}\int\exp[-G_{e_ie_i}(0)](\{\exp[-G_{e_ie_j}(\mathbf{r})]-1\}\cos[\mathbf{e}_i\cdot\nabla\varphi^{<}(\mathbf{u}+\tfrac{1}{2}\mathbf{r})+\mathbf{e}_j\cdot\nabla\varphi^{<}(\mathbf{u}-\tfrac{1}{2}\mathbf{r})] +\{\exp[G_{\mathbf{e}_i\mathbf{e}_j}(\mathbf{r})]-1\}\cos[\mathbf{e}_i\nabla\varphi^{<}(\mathbf{u}+\tfrac{1}{2}\mathbf{r})-\mathbf{e}_j\cdot\nabla\varphi^{<}(\mathbf{u}-\tfrac{1}{2}\mathbf{r})])d\mathbf{u}\,d\mathbf{r}\ . \tag{2.8}$$

At this stage of the calculation one might use a gradient expansion to isolate those contributions that will renormalize κ and y. However, this procedure is incorrect since each operator that appears in the expansion will in general contain a relevant or marginal term and thus will contribute to renormalization.[10] Thus one must use Kadanoff's operator algebra formalism.[13] The higher harmonics in Eq. (2.8) can be shown to be less relevant.

There are two types of contribution from (2.8), one that will renormalize the fugacity and one that will renormalize the curvature energy. These are, respectively,

$$y^2\exp[-G_{\mathbf{e}_i\mathbf{e}_i}(0)]\sum_{\substack{i,j\\ i\neq j}}\int\{\exp[-G_{\mathbf{e}_i\mathbf{e}_j}(\mathbf{r})]-1\} \times\cos[\mathbf{e}_i\cdot\nabla\varphi^{<}(\mathbf{u}+\tfrac{1}{2}\mathbf{r}) +\mathbf{e}_j\cdot\nabla\varphi^{<}(\mathbf{u}-\tfrac{1}{2}\mathbf{r})]d\mathbf{u}\,d\mathbf{r} \tag{2.9}$$

and

$$y^2\exp[-G_{\mathbf{e}_1\mathbf{e}_1}(0)]\sum_{e_i}\int(\{\exp[G_{e_ie_i}(\mathbf{r})]-1\} \times\cos[\mathbf{e}_i\cdot\nabla\varphi^{<}(\mathbf{u}+\tfrac{1}{2}\mathbf{r}) -\mathbf{e}_i\cdot\nabla\varphi^{<}(\mathbf{u}-\tfrac{1}{2}\mathbf{r})])d\mathbf{u}\,d\mathbf{r}\ . \tag{2.10}$$

To proceed we consider the operator product expansion of

$$\cos[\mathbf{e}_1\cdot\nabla\varphi(\mathbf{x})+\mathbf{e}_2\cdot\nabla\varphi(\mathbf{y})] \underset{x\to y}{\sim}\tilde{O}(\mathbf{x})+a(\mathbf{x}-\mathbf{y})\cos[\mathbf{e}_3\cdot\nabla\varphi(\mathbf{x})]+b(\mathbf{x}-\mathbf{y})\ . \tag{2.11}$$

Here $\tilde{O}(\mathbf{x})$ is an irrelevant operator and $b(\mathbf{x}-\mathbf{y})$ is a constant that will contribute to renormalization of the free energy. To obtain $a(\mathbf{x}-\mathbf{y})$ we construct a correlation function,

$$g(R)=\langle\{\cos[\mathbf{e}_1\cdot\nabla\varphi(x)+\mathbf{e}_2\cdot\nabla\varphi(y)] -a(x-y)\cos[\mathbf{e}_3\cdot\nabla\varphi(x)]\} \times\cos[\mathbf{e}_3\cdot\nabla\varphi(x+R)]\rangle_0\ , \tag{2.12}$$

and require that in the limit $x\to y$ it satisfies the "orthogonality" property that it not contain any terms that decay algebraically in R. The expectation values are taken with respect to the quadratic part of the Hamiltonian (since all other contributions will lead to terms of higher order in y). One can then show that

$$a(\mathbf{x}-\mathbf{y})=\exp[-V_{\mathbf{e}_1\mathbf{e}_2}(\mathbf{x}-\mathbf{y})]\ , \tag{2.13}$$

where

$$V_{\mathbf{e}_i\mathbf{e}_j}(\mathbf{x})\frac{1}{\kappa}\int\frac{(\mathbf{e}_i\cdot\mathbf{q})(\mathbf{e}_j\cdot\mathbf{q})(e^{i\mathbf{q}\cdot(\mathbf{x}-\mathbf{y})}-1)}{q^4} \times\varphi(q)(2\pi)^2\frac{1}{(2\pi)^2}d^2\mathbf{q}\ . \tag{2.14}$$

Similarly,

$$\cos[\mathbf{e}_i\cdot\nabla\varphi(\mathbf{x})-\mathbf{e}_i\cdot\nabla\varphi(\mathbf{y})]=\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}O_i^{2n}\ , \tag{2.15}$$

where

$$O_i(\mathbf{x})=\mathbf{e}_i\cdot\nabla\varphi(\mathbf{x})-\mathbf{e}_i\cdot\nabla\varphi(\mathbf{y}) , \tag{2.16}$$

and the marginal operator will be $O_i^2(\mathbf{x})$. We can now write[10]

$$O_i^{2n}(\mathbf{x})=[\mathbf{e}_i\cdot\nabla\varphi(\mathbf{x})-\mathbf{e}_i\cdot\nabla\varphi(\mathbf{y})]^{2n} =\tilde{O}_{2n,i}(\mathbf{x})+a_{2n,i}(\mathbf{x}-\mathbf{y})O_i^2(\mathbf{x})+b_{2n,i}(\mathbf{x}-\mathbf{y}) , \tag{2.17}$$

Again $\tilde{O}_{2n,i}(\mathbf{x})$ are irrelevant operators and $b_{2n,i}(\mathbf{x}-\mathbf{y})$ are constants that will renormalize the free energy. The constant $a_{2n,i}(\mathbf{x}-\mathbf{y})$ is calculated using the operator algebra formalism of Kadanoff.[13] Thus we consider the correlation function

$$g_{2n,i}(R)=\langle[O_i^{2n}(\mathbf{x})-a_{2n}O_i^2(\mathbf{x})]O_i^2(\mathbf{x}+\mathbf{R})\rangle_0 . \tag{2.18}$$

We now require that $g_{2n,i}(R)$ satisfy the "orthogonality" property, namely that it does not contain any terms that fall off proportional to R^{-4} (as would be the case for a marginal operator). The average in (2.18) is taken with respect to the Gaussian Hamiltonian since all other contributions will lead to higher order terms in y. The $a_{2n,i}$ is given by all possible contractions that contain terms $\langle O_i(\mathbf{x})O_i(\mathbf{x}+\mathbf{R})\rangle_0^2$ since these decay as R^{-4}. It is then possible to show that

$$a_{2n,i}(\mathbf{x}-\mathbf{y})=\frac{(2n)!}{2^n(n-1)!}\langle O_i^2(\mathbf{x})\rangle_0^{n-1} , \tag{2.19}$$

where

$$\langle O_i^2(\mathbf{x})\rangle=-2V_{\mathbf{e}_i\mathbf{e}_i}(\mathbf{x}-\mathbf{y}) . \tag{2.20}$$

Thus Eq. (2.15) becomes

$$\cos[\mathbf{e}_i\cdot\nabla\varphi(\mathbf{x})-\mathbf{e}_i\cdot\nabla\varphi(\mathbf{y})] \underset{\mathbf{x}\to\mathbf{y}}{\sim} \tilde{O}_i(\mathbf{x})-\tfrac{1}{2}\exp[V_{\mathbf{e}_i\mathbf{e}_i}(\mathbf{x}-\mathbf{y})][\mathbf{e}_i\cdot\nabla\varphi(\mathbf{x})-\mathbf{e}_i\cdot\nabla\varphi(\mathbf{y})]^2 +B(\mathbf{x}-\mathbf{y}) . \tag{2.21}$$

Applying these results in expressions (2.9) and (2.10) one obtains that part of the second cumulant that renormalizes the fugacity to be

$$-2y^2\exp[-G_{\mathbf{e}_1\mathbf{e}_2}(0)]\epsilon\int g_{\mathbf{e}_1\mathbf{e}_2}(\mathbf{r})e^{-V_{\mathbf{e}_1\mathbf{e}_2}(\mathbf{r})}d^2\mathbf{r} \times\int\cos[\mathbf{e}_3\cdot\nabla\varphi(\mathbf{u})]d^2\mathbf{u} , \tag{2.22}$$

and the term that renormalizes the curvature is

$$-\tfrac{1}{2}y^2\exp[-G_{\mathbf{e}_1\mathbf{e}_1}(0)]\epsilon\sum_{e_i}g_{e_ie_i}(\mathbf{r})\exp[V_{\mathbf{e}_i\mathbf{e}_i}(\mathbf{r})] \times(\mathbf{e}_i\cdot\nabla)^2[\mathbf{r}\cdot\nabla\varphi^<(\mathbf{u})]^2d^2\mathbf{r}\,d^2\mathbf{u} . \tag{2.23}$$

Now, it is possible to show that

$$g_{\mathbf{e}_i\mathbf{e}_j}(\mathbf{r})=r\frac{dV_{\mathbf{e}_i\mathbf{e}_j}(\mathbf{r})}{dr}+\frac{\mathbf{e}_i\cdot\mathbf{e}_j}{4\pi\kappa} , \tag{2.24}$$

and performing the angular integration in Eq. (2.14), one finds that

$$V_{\mathbf{e}_i\mathbf{e}_j}(\mathbf{r})=\frac{\mathbf{e}_i\cdot\mathbf{e}_j}{4\pi\kappa}\int[J_0(qr)-1]\phi(q)\frac{1}{q}dq -\frac{(\mathbf{e}_i\cdot\mathbf{r})(\mathbf{e}_j\cdot\mathbf{r})}{2\pi\kappa r^2}\int J_2(qr)\phi(q)\frac{1}{q}dq +\frac{\mathbf{e}_i\cdot\mathbf{e}_j}{4\pi\kappa}\int J_2(qr)\phi(q)\frac{1}{q}dq . \tag{2.25}$$

After rescaling $(k\to k/\delta, r\to r\delta, \varphi\to\varphi\delta$, where $\delta=1+\epsilon)$, we find the recurrence relations

$$y(\epsilon)=y\exp[-\tfrac{1}{2}G_{\mathbf{e}_1\mathbf{e}_1}(0)]\delta^2 +y^2\exp[-G_{\mathbf{e}_1\mathbf{e}_2}(0)]\epsilon\delta^2\int d\mathbf{r}\,r^{1+\mathbf{e}_1\cdot\mathbf{e}_2/4\pi\kappa}\frac{d}{dr}\left[\exp\left[-V_{\mathbf{e}_1\mathbf{e}_2}(r)-\frac{\mathbf{e}_1\cdot\mathbf{e}_2}{4\pi\kappa}\ln r\right]\right] , \tag{2.26}$$

$$\kappa(\epsilon)=\kappa+y^2\exp[-G_{\mathbf{e}_1\mathbf{e}_1}(0)]\epsilon\sum_{\{e_i\}}\int d\mathbf{r}\,r^{3-1/4\pi\kappa}\frac{d}{dr}\left[\exp\left[V_{\mathbf{e}_i\mathbf{e}_i}(\mathbf{r})+\frac{1}{4\pi\kappa}\ln r\right](\mathbf{e}_i\cdot\nabla)^2(\hat{\mathbf{r}}\cdot\hat{\mathbf{n}})^2\right] , \tag{2.27}$$

$$\hat{\mathbf{n}}=\frac{\nabla\varphi(\mathbf{u})}{|\nabla\varphi|} . \tag{2.28}$$

Upon performing the angular integrations these equations may be written in the differential form

$$\frac{dy}{d\epsilon}=\left[2-\frac{c(\phi)}{2}\right]y+2\pi y^2\int dr\,r^{2-1/8\pi\kappa}\frac{d}{dr}\left[\exp\left[\tfrac{1}{2}A(\phi;r)+\frac{1}{8\pi\kappa}\ln r\right]I_0(B(\phi;r))\right] , \tag{2.29}$$

$$\frac{d\kappa}{d\epsilon}=\tfrac{3}{2}\pi y^2\int dr\,r^{4-1/4\pi\kappa}\frac{d}{dr}\left[\exp\left[A(\phi;r)+\frac{1}{4\pi\kappa}\ln r\right][I_0(B(\phi;r))-\tfrac{1}{2}I_1(B(\phi;r))]\right] , \tag{2.30}$$

where

$$c(\phi)=\frac{-1}{4\pi\kappa}\int dq\frac{1}{q}\frac{d\phi}{dq}\ , \tag{2.31}$$

$$A(\phi;r)=\frac{1}{4\pi\kappa}\int[J_0(qr)-1]\phi(q)\frac{1}{q}dq\ , \tag{2.32}$$

$$B(\phi;r)=\frac{1}{4\pi\kappa}\int J_2(qr)\phi(q)\frac{1}{q}dq \tag{2.33}$$

For the special case of a sharp cutoff at the transition temperature ($\kappa=1/16\pi$) we obtain the usual flow equations for the vector Coloumb gas

$$\frac{dy}{d\epsilon}=\left[2-\frac{1}{8\pi\kappa}\right]y+2\pi y^2 I_0(2)\ , \tag{2.34}$$

$$\frac{d\kappa}{d\epsilon}=\tfrac{3}{2}y^2[I_0(2)-\tfrac{1}{2}I_1(2)]\ . \tag{2.35}$$

These equations have been studied by a number of authors[5,6] and may be shown to predict an interfacial transition.

We have thus shown that the interfacial Hamiltonian (2.1) does indeed possess an orientational phase transition, and the two types of renormalization produce the same flow equations to leading order in fugacity. It is interesting to note that this Hamiltonian can be generalized to describe surfaces with interfacial tension and curvature energies.

III. DISCUSSION

In this paper we have focused mainly on the technical aspects of the sine-Gordon renormalization formulation of the interfacial orientational phase transition. It is, however, worth discussing the physical interpretation of the Hamiltonian (2.1). We emphasize that $\varphi(\mathbf{x})$ is the height of the column arising from a coarse graining of the discrete Gaussian model (1.1). Evidently, the first term in (2.1) is a curvature energy originating in long-wavelength deformations of a tensionless interface. The second term arises from coarse graining of the microscopic Hamiltonian for fixed average orientation $\nabla\varphi(\mathbf{x})$.[14]. We also point out that this term is the most relevant perturbation for the Gaussian model of a tensionless interface. These may be checked by calculating the second cumulant of various interaction terms. In the absence of a tension the simple cosine of the height is irrelevant, while higher harmonics of cosine of the gradient are less relevant then the first.

Besides the mathematical connection to a vector Coulomb gas (Appendix), one might seek to interpret the orientational phase transition in terms of the interface Hamiltonian. Thus, for the simple roughening transition, the Coulomb gas picture indicates that the unbinding of charges is the effect that drives the transition. In the sine-Gordon formulation this unbinding is accompanied by the coefficient of the potential term becoming finite and the height fluctuations of the interface are therefore suppressed. Similarly, for orientational roughening the unbinding of dipoles in the Coulomb gas formulation means that the coefficient of the potential term in Eq. (2.1) becomes finite. Here, however, we find that the normals to the surface (being conjugate to the dipoles) are restrained. This observation is quite general. When a complexion of charges unbinds, the fluctuations of the conjugate surface order parameter are suppressed by an appropriate potential term. It is straightforward to derive the appropriate Hamiltonian by examination of the basic Coulomb gas transcription.

In a future report we will discuss various phenomena suggested by the present formulation. These include orientational preroughening [which arises from considerations of higher harmonics[15] in the Hamiltonian (2.1)], and the progression of the Laplacian roughening model of a tensionless interface to one where the tension is finite. Finally, we seek to understand the collective roughening of weakly coupled interfaces.

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APPENDIX: VECTOR COULOMB GAS FORMULATI0N OF THE GRADIENT SINE-GORDON MODEL

As we pointed out in Sec. II, we expect that the field-theoretic model considered above should give us the same equations as those that arise for the vector Coulomb gas renormalization, given that there exists a transformation between the two formulations. In this appendix we consider a mapping which takes one model into the other.

Thus we consider the following partition function:

$$Z=\sum_{\mathbf{b}(\mathbf{x})}\int\mathcal{D}\varphi(\mathbf{x})\exp\Big[-\tfrac{1}{2}\kappa\int d^2\mathbf{x}(\nabla^2\varphi)^2 +2\pi i\int d^2\mathbf{x}\,\mathbf{b}(\mathbf{x})\cdot\nabla\varphi(\mathbf{x}) +\ln y\int d^2\mathbf{x}\,\mathbf{b}^2(\mathbf{x})\Big]\ . \tag{A1}$$

Performing the integration over the φ we obtain

$$Z=Z_0\sum_{\mathbf{b}(\mathbf{x})}\exp\Big[-2\pi^2\int d^2\mathbf{x}\,d^2\mathbf{y}\,b_\alpha(\mathbf{x})b_\beta(\mathbf{y}) \times\langle\partial_\alpha\varphi(\mathbf{x})\partial_\beta(\mathbf{y})\rangle_0+\ln y\int d^2\mathbf{x}\,b^2(\mathbf{x})\Big]\ , \tag{A2}$$

where Z_0 is a Gaussian partition function. We define

$$G_{\alpha\beta}(\mathbf{x}-\mathbf{y})\equiv\langle\partial_\alpha\varphi(\mathbf{x})\partial_\beta\varphi(\mathbf{y})\rangle_0\ , \tag{A3}$$

and one can show that

$$G_{\alpha\beta}(\mathbf{x}-\mathbf{y})=\int q_\alpha q_\beta\frac{e^{i\mathbf{q}\cdot(\mathbf{x}-\mathbf{y})}}{q^4}\frac{1}{\kappa(2\pi)^2}d^2\mathbf{q}\ . \tag{A4}$$

Using this in Eq. (A2) we obtain a divergent result unless

$$\int d^2\mathbf{x}\,\mathbf{b}(\mathbf{x})=0\ . \tag{A5}$$

Thus restricting the allowed configurations to satisfy this constraint we can replace $G_{\alpha\beta}(\mathbf{r})$ by

$$V_{\alpha\beta}(\mathbf{x}-\mathbf{y})=G_{\alpha\beta}(\mathbf{x}-\mathbf{y})-G_{\alpha\beta}(0)\ . \tag{A6}$$

If we now use a sharp cutoff we find that for large separations

$$V_{\alpha\beta}(\mathbf{r})=-\frac{1}{4\pi\kappa}\ln r\,\delta_{\alpha\beta}-\frac{1}{4\pi\kappa}\frac{r_\alpha r_\beta}{r^2}+\frac{1}{8\pi\kappa}\delta_{\alpha\beta}\ . \tag{A7}$$

Substituting this result into Eq. (A2) we finally obtain[16]

$$H_b=\frac{\pi}{2\kappa}\int_{|\mathbf{x}-\mathbf{y}|>1} d^2\mathbf{x}\,d^2y\left[\ln r\,\mathbf{b}(\mathbf{x})\cdot\mathbf{b}(\mathbf{y})+\frac{[\mathbf{b}(\mathbf{x})\cdot\mathbf{r}][\mathbf{b}(\mathbf{y})\cdot\mathbf{r}]}{r^2}-\tfrac{1}{2}\mathbf{b}(\mathbf{x})\cdot\mathbf{b}(\mathbf{y})\right]+\ln y\int d^2\mathbf{x}\,\mathbf{b}^2(\mathbf{x})\ . \tag{A8}$$

On the other hand if, in Eq. (A1), we perform the trace over $\mathbf{b}(\mathbf{x})$, including only vectors of length 0 and ±1, the leading contribution to the resulting Hamiltonian is

$$H_\varphi=-\tfrac{1}{2}\kappa\int d^2\mathbf{x}(\nabla^2\varphi)^2+2y\sum_{i(=1)}\int d^2\mathbf{x}\cos[\mathbf{e}_i\cdot\nabla\varphi(\mathbf{x})]\ , \tag{A9}$$

which is the Hamiltonian (2.1).

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[16]The sign of the angular term in Eq. (A8) is different from the usual vector Coulomb gas, but as was pointed out by A. P. Young (Ref. 5), this will not effect the flow equations.