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Generalized Laplacian roughening model on a triangular lattice

Y. Levin
Department of Physics, University of California, Berkeley, Berkeley, California 94720
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We consider a generalized Laplacian roughening model with isotropic interactions. It is shown that depending on the range of interaction, the model can have a variety of phases, all of which can be classified by the Z_6 group. Critical exponents are calculated for an arbitrary range of interactions.

I. INTRODUCTION

During the last decade there has been considerable interest in the study of melting in two dimensions.[1] In his investigation of this problem, Nelson[2] was led to study the discrete Laplacian roughening model on the triangular lattice, using the Hamiltonian

$$H=\frac{-\kappa}{2}\sum_{n}[\Delta h(\mathbf{n})]^2 , \qquad (1.1)$$

where $h(\mathbf{n})$, is the height of a solid-on-solid (SOS) surface at lattice site $\mathbf{n}=(x,y)$. The quantity $\Delta h(\mathbf{n})$ is a lattice approximation to $\nabla^2 h(\mathbf{x})$, defined as

$$\Delta h(\mathbf{n})\equiv\tfrac{2}{3}\sum_{i=1}^{6} h(\mathbf{n}+\delta_i)-4h(\mathbf{n}) , \qquad (1.2)$$

where the $\{\delta_i\}$, are the six interlattice unit vectors centered on site $\mathbf{n}$.

Nelson was able to perform a duality transformation that mapped the above model onto a gas of charges interacting by a $r^2\ln r$ potential. As is usual with duality transformations, the low temperatures of the original lattice model get mapped onto high temperatures of the dual model. Nelson then argued that the model will have two phase transitions: the first in which the completely disordered surface develops an orientational order, and the second in which the surface becomes translationally and orientationally flat.[3] In the dual model of an interacting charged gas, the lowest temperature phase, corresponding to a completely disordered interface, consists of the charges grouped in quartets with no net charge and zero dipole moment. The orientational roughening transition in this language is described by the unbinding of the dipoles comprising each quartet, while the usual roughening transition corresponds to the unbinding of the charges making up each dipole.[4–6]

Here we will consider a more general Laplacian roughening model of order p given by

$$H=\frac{-\kappa}{2}\sum_{n}[\Delta^p h(\mathbf{n})]^2 . \qquad (1.3)$$

One might suspect that this model would have $2p$ phase transitions, corresponding to the unbinding of the higher moments of the charge distribution. However, one can show that the structure of higher multipole moments in two dimensions, consisting of the nearest neighbors on a triangular lattice, is not rich enough to allow this. Thus, for example, the only dipole-neutral quadrupoles will have diagonal quadrupole tensors. So, if one wants to study a class of models described by Eq. (1.3), the Coulomb-gas transcription does not seem to be the most general approach.

In a previous report,[7] we have shown that the dipole unbinding transition in model (1.1) can be mapped onto a Landau-Ginzburg-Wilson (LGW) Hamiltonian of the form

$$H=-\tfrac{1}{2}\kappa\int d^2\mathbf{x}(\nabla^2\varphi)^2+2y\sum_{i=1}^{3}\int d^2\mathbf{x}\cos[e_i\cdot\nabla\varphi(\mathbf{x})] , \qquad (1.4)$$

where

$$e_i=(1,0),\ \left(-\frac{1}{2},\frac{\sqrt{3}}{2}\right),\ \left(-\frac{1}{2}-\frac{\sqrt{3}}{2}\right) , \qquad (1.5)$$

and $\nabla\varphi$ plays the role of an order parameter. The charge unbinding transition can then be described by the usual sine-Gordon model.[8–10]

We can now take a different approach to this problem. Instead of looking for an explicit transformation that will map the model (1.1) onto the model (1.4), we can postulate that the effective LGW Hamiltonian must have the same low-temperature symmetry as an underlying microscopic Hamiltonian (1.1). If we adopt this point of view, we must then study the symmetries of the microscopic Hamiltonian (1.1). It is easy to see that this Hamiltonian is invariant under the *discrete* transformation

$$h(\mathbf{n})\to h(\mathbf{n})+m+\frac{1}{2\pi}\mathbf{G}\cdot\mathbf{n} , \qquad (1.6)$$

where $m\in\mathbb{Z}$ and $\{\mathbf{G}\}$ are the reciprocal lattice vectors. At high temperatures we expect that the symmetry will be restored and the effective LGW Hamiltonian will be invariant under the *continuous* transformation

$$\varphi(\mathbf{x})\to\varphi(\mathbf{x})+a+\mathbf{b}\cdot\mathbf{x} . \qquad (1.7)$$

where $a \in \mathbb{R}$ and $\mathbf{b} \in \mathbb{R}^2$. We expect that this restoration of symmetry will occur in two stages. First, the translational symmetry will be restored as $T \to {}^{-}T_1$ and then the orientational symmetry as $T \to {}^{-}T_2$.

The effective Hamiltonian that describes restoration of translational symmetry must then be invariant under the transformation $\varphi(\mathbf{x}) \to \varphi(\mathbf{x}) + 2\pi m$ in the low temperature phase and $\varphi(\mathbf{x}) \to \varphi(\mathbf{x}) + a$ in the high-temperature phase. This is exactly the situation with the sine-Gordon model given by[8-10]

$$H = -\tfrac{1}{2}\kappa \int d^2\mathbf{x} (\nabla\varphi)^2 + 2y \int d^2\mathbf{x} \cos[\varphi(\mathbf{x})] \ . \tag{1.8}$$

In the low-temperature phase y is large and the Hamiltonian is invariant under the *discrete* transformation

$$\varphi(\mathbf{x}) \to \varphi(\mathbf{x}) + 2\pi m \ , \tag{1.9}$$

signifying the flatness of the interface, while in the high-temperature phase y renormalizes to zero, the interface is rough, and the Hamiltonian is invariant under the *continuous* transformation

$$\varphi(\mathbf{x}) \to \varphi(\mathbf{x}) + a \ . \tag{1.10}$$

As the temperature is raised farther, $T \to {}^{-}T_2$, the second-phase transition occurs. In the low-temperature phase of this transition, $T < T_2$, the LGW Hamiltonian must be invariant under the *discrete* transformation

$$\varphi(\mathbf{x}) \to \varphi(\mathbf{x}) + \mathbf{G}\cdot\mathbf{x} \ , \tag{1.11}$$

implying the existence of orientational order. This order is no longer present in the high-temperature phase, $T > T_2$, where the LGW Hamiltonian is invariant under the *continuous* transformation

$$\varphi(\mathbf{x}) \to \varphi(\mathbf{x}) + \mathbf{b}\cdot\mathbf{x} \ . \tag{1.12}$$

This is exactly the case for the Hamiltonian (1.4), where in the low-temperature phase y is large, and the symmetry is discrete given by (1.11), while in the high-temperature y renormalizes to zero, and the continuous symmetry (1.12) is restored. It should be emphasized that the Hamiltonian (1.4) does not describe the usual roughening transition because both its high- and low-temperature phases are invariant under the continuous transformation (1.10) implying that the interface has been roughened previously.

Based solely on the considerations of symmetry, we have been able to reduce the study of the discrete Hamiltonian (1.1) to the study of two field-theoretic Hamiltonians, (1.4) and (1.8), completely bypassing a transcription onto a Coulomb gas.

II. THE GENERALIZED LAPLACIAN ROUGENING MODEL

We now apply the analysis of the preceding section to the Hamiltonian (1.3). It is easy to see that this Hamiltonian has the symmetry

$$h(\mathbf{n}) \to h(\mathbf{n}) + m + \tilde{T}^{1}_{\alpha} r_a + \cdots + \tilde{T}^{2p-1}_{\alpha\beta\cdots\gamma} r_\alpha r_\beta \cdots r_\gamma \ , \tag{2.1}$$

where $m \in \mathbb{Z}$ and $\{\tilde{T}^{n}_{\alpha\beta\cdots\gamma}\}$ are the "reciprocal-lattice tensors," which can be constructed by the procedure described below. For our purposes it is important only to note that this is a discrete symmetry, similar to (1.6). We can then expect that the symmetry is going to be restored in steps and the Laplacian roughening model of order p will undergo $2p$ phase transitions. The appropriate LGW Hamiltonians needed to study these phase transitions will be of the form

$$H_n = -\tfrac{1}{2}\kappa_n \int d^2\mathbf{x} (\nabla^{n+1}\varphi)^2 + 2y_n \sum_{i=1}^{3} \int d^2\mathbf{x} \cos[{}^{i}T^{n}_{\alpha\beta\cdots\gamma} \nabla_\alpha \nabla_\beta \cdots \nabla_\gamma \varphi(\mathbf{x})] \ , \tag{2.2}$$

where n ranges from 0 to $2p-1$ and there are n gradients inside the cosine term. It is easy to see that the interaction of the form chosen above is a relevant perturbation. $\varphi(\mathbf{x})$ is a scalar field, which could be thought of as the height of the coarse grained interface (repeated Greek indices are summed). The $\nabla_\alpha \nabla_\beta \cdots \nabla_\gamma \varphi(\mathbf{x})$ will play the role of the order parameter and ${}^{i}T^{n}_{\alpha\beta\cdots\gamma}$ are the irreducible lattice tensors that can be determined from the symmetry of the lattice. To do this we must consider representations of the group $\mathbb{Z}_6$, since it is the group corresponding to the structure of triangular lattice. The one-dimensional representations of the group are then given by the set

$$\{e^{i2\pi m/6}; m = 0, 1, \ldots, 5\} \ . \tag{2.3}$$

We expect that ${}^{i}T^{n}_{\alpha\beta\cdots\gamma}$ will transform under irreducible representations of $\mathbb{Z}_6$. To actually construct the ${}^{i}T^{n}_{\alpha\beta\cdots\gamma}$ we use the following recipe. We find the basis $E_\pm$, which under rotations by $2\pi/6$ transform into $e^{\pm(2\pi i/6)}E_\pm$. We then express the lattice vectors in terms of this new basis. This will correspond to the spin-1 representation of rotations by $2\pi/6$. To construct the higher-rank tensors all one needs to do is make a substitution

$$E_+ \to E_+ \otimes E_+ \otimes \cdots \otimes E_+ \equiv [E_+ \otimes]^n$$

and

$$E_- \to E_- \otimes E_- \otimes \cdots \otimes E_- \equiv [E_- \otimes]^n \ ,$$

where there are n tensor products. These objects will, under rotations by $2\pi/6$, transform into themselves acquiring a phase factor of $e^{\pm(2\pi n/6)}$. We can then write

$${}^{1}T^{n} = 2^{(n-2)/2}[(E_+ \otimes)^n + (E_- \otimes)^n] \ , \tag{2.4}$$

$${}^{2}T^{n} = 2^{(n-2)/2}\left[-\frac{1}{2}[(E_+ \otimes)^n + (E_- \otimes)^n] - \frac{\sqrt{3}}{2} i [(E_+ \otimes)^n - (E_- \otimes)^n]\right] \ , \tag{2.5}$$

$${}^{3}T^{n} = 2^{(n-2)/2}\left[-\frac{1}{2}[(E_+ \otimes)^n + (E_- \otimes)^n] + \frac{\sqrt{3}}{2} i [(E_+ \otimes)^n - (E_- \otimes)^n]\right] \ . \tag{2.6}$$

$$ {}^4T^n = -{}^1T^n, \quad {}^5T^n = -{}^2T^n, \quad {}^6T^n = -{}^3T^n . \tag{2.7} $$

The powers of two are included for normalization. The tensors ${}^iT^n$ transform under the irreducible representations of the group $\mathbb{Z}_6$. For the case $n=1$ all six tensors will belong to the same group and transform into each other by rotations of $2\pi/6$. For $n=2$, there will be two groups of three tensors each, $({}^1T^n, {}^2T^n, {}^3T^n)$ and $(-{}^1T^n, -{}^2T^n, -{}^3T^n)$, closed under the rotation of $2\pi/6$, and for $n=3$, there will be three groups of two tensors each, $({}^1T^n, -{}^1T^n)$, $({}^2T^n, -{}^2T^n)$, and $({}^3T^n, -{}^3T^n)$. The case of $n=4$ is similar to $n=2$; that is there will once again be two groups of three objects each, not connected by rotations of $2\pi/6$. $n=5$ will correspond to $n=1$, and finally $n=0$ will split the set of tensors into six groups of one element each, which are not connected by rotations. For $n>5$ this structure will repeat itself modulus 6. Thus it will be convenient to characterize different types of Hamiltonians by the spin of ${}^iT^n$, where spin s is defined by

$$ s = n(\text{mod}6) . \tag{2.8} $$

The tensor we have constructed possess the interesting property

$$ {}^1T^m_{\alpha\beta\cdots\gamma} n_\alpha n_\beta \cdots n_\gamma = \cos(m\theta) , \tag{2.9} $$

$$ {}^2T^m_{\alpha\beta\cdots\gamma} n_\alpha n_\beta \cdots n_\gamma = -\cos(m\theta + 2\pi/6) , \tag{2.10} $$

$$ {}^3T^m_{\alpha\beta\cdots\gamma} n_\alpha n_\beta \cdots n_\gamma = -\cos(m\theta - 2\pi/6) , \tag{2.11} $$

where $\mathbf{n}$ is a unit vector and θ is the angle that $\mathbf{n}$ makes with the x axis.

From this point onward the discussion very closely follows that in Ref. 7. We begin by defining $\phi(q)$ to be a cutoff function such that $\phi(q) \approx 0$ for $q>1$ and $\phi(q) \approx 1$ for $q<1$. The momentum shell propagator that will appear during the renormalization-group procedure is

$$ G^n_{ij}(\mathbf{x}) = -\frac{\varepsilon}{\kappa} \int \frac{{}^iT^n_{\alpha\beta\cdots\gamma}\, {}^jT^n_{\delta\nu\cdots\sigma} q_\alpha q_\beta \cdots q_\gamma q_\delta q_\nu \cdots q_\sigma e^{i\mathbf{q}\cdot\mathbf{x}}}{q^{2n+1}} \frac{d\phi}{dq} \frac{d^2\mathbf{q}}{(2\pi)^2} \equiv \varepsilon g^n_{ij}(\mathbf{x}) . \tag{2.12} $$

After integrating over high-momentum fluctuations ($\varphi^>$) in the variable $\varphi(\mathbf{x})$, we find the Hamiltonian,

$$ \begin{aligned} H' = & -\tfrac{1}{2}\kappa_n \int d^2\mathbf{x}\, \varphi^<(\mathbf{x}) \nabla^{2(n+1)} \varphi^<(\mathbf{x}) + 2y_n \sum_{i=1} \int d^2\mathbf{x} \langle \cos[{}^iT^n_{\alpha\beta\cdots\gamma} \nabla_\alpha \nabla_\beta \ldots \nabla_\gamma \varphi(\mathbf{x})] \rangle^> \\ & + \tfrac{1}{2}(2y_n)^2 \sum_{i=1} \sum_{j=1} \int d^2\mathbf{x}\, d^2\mathbf{y} \langle \cos[{}^iT^n_{\alpha\beta\cdots\gamma} \nabla_\alpha \nabla_\beta \ldots \nabla_\gamma \varphi(\mathbf{x})] \cos[{}^jT^n_{\alpha\beta\cdots\gamma} \nabla_\alpha \nabla_\beta \ldots \nabla_\gamma \varphi(\mathbf{y})] \rangle^>_c \end{aligned} \tag{2.13} $$

which is correct to second order in the fugacity (y).

The superscript symbols $^<$ and $^>$ stand, respectively, for slow and fast modes. The averages in Eq. (2.13) are

$$ \langle \cos[{}^iT^n_{\alpha\beta\cdots\gamma} \nabla_\alpha \nabla_\beta \ldots \nabla_\gamma \varphi(\mathbf{x})] \rangle^> = \cos[{}^iT^n_{\alpha\beta\cdots\gamma} \nabla_\alpha \nabla_\beta \ldots \nabla_\gamma \varphi^<(\mathbf{x})] e^{-(1/2)G^n_{ii}(0)} , \tag{2.14} $$

$$ \begin{aligned} & \langle \cos[{}^iT^n_{\alpha\beta\cdots\gamma} \nabla_\alpha \nabla_\beta \ldots \nabla_\gamma \varphi(\mathbf{x})] \cos[{}^iT^n_{\alpha\beta\cdots\gamma} \nabla_\alpha \nabla_\beta \ldots \nabla_\gamma \varphi(\mathbf{y})] \rangle^>_c \\ & \quad = \frac{1}{2} e^{-G^n_{ii}(0)} \left\{ \begin{aligned} & (e^{-G^n_{ij}(\mathbf{x}-\mathbf{y})} - 1) \cos[{}^iT^n_{\alpha\beta\cdots\gamma} \nabla_\alpha \nabla_\beta \ldots \nabla_\gamma \varphi^<(\mathbf{x}) + {}^jT^n_{\alpha\beta\cdots\gamma} \nabla_\alpha \nabla_\beta \ldots \nabla_\gamma \varphi^<(\mathbf{y})] \\ & + (e^{G^n_{ij}(\mathbf{x}-\mathbf{y})} - 1) \cos[{}^iT^n_{\alpha\beta\cdots\gamma} \nabla_\alpha \nabla_\beta \ldots \nabla_\gamma \varphi^<(\mathbf{x}) - {}^jT^n_{\alpha\beta\cdots\gamma} \nabla_\alpha \nabla_\beta \ldots \nabla_\gamma \varphi^<(\mathbf{y})] \end{aligned} \right\} . \end{aligned} \tag{2.15} $$

Note that the tensors ${}^1T^n, {}^2T^n, {}^3T^n$ satisfy the relation,

$$ {}^1T^n + {}^2T^n + {}^3T^n = 0 . \tag{2.16} $$

We also note that

$$ G^n_{11}(0) = G^n_{22}(0) = G^n_{33}(0) = \frac{\varepsilon}{4\pi\kappa} \tag{2.17} $$

and are thus independent of n.

From the Hamiltonian (2.13) we now seek to extract those terms that will renormalize the bare curvature and fugacity (κ, y). Thus, defining $\mathbf{r} = \mathbf{x} - \mathbf{y}$ and $\mathbf{u} = \frac{1}{2}(\mathbf{x} + \mathbf{y})$ the term of order y^2 in Eq. (2.13) will become

$$ y_n^2 \sum_{i,j=1} \int e^{-G^a_{ii}(0)} \left\{ \begin{aligned} & (e^{-G^n_{ij}(\mathbf{r})} - 1) \cos[{}^iT^n_{\alpha\beta\cdots\gamma} \nabla_\alpha \nabla_\beta \ldots \nabla_\gamma \varphi^<(\mathbf{u} + \tfrac{1}{2}\mathbf{r}) + {}^jT^n_{\alpha\beta\cdots\gamma} \nabla_\alpha \nabla_\beta \ldots \nabla_\gamma \varphi^<(\mathbf{u} - \tfrac{1}{2}\mathbf{r})] \\ & + (e^{G^n_{ij}(\mathbf{r})} - 1) \cos[{}^iT^n_{\alpha\beta\cdots\gamma} \nabla_\alpha \nabla_\beta \ldots \nabla_\gamma \varphi^<(\mathbf{u} + \tfrac{1}{2}\mathbf{r}) - {}^jT^n_{\alpha\beta\cdots\gamma} \nabla_\alpha \nabla_\beta \ldots \nabla_\gamma \varphi^<(\mathbf{u} - \tfrac{1}{2}\mathbf{r})] \end{aligned} \right\} d\mathbf{u}\, d\mathbf{r} . \tag{2.18} $$

There are two contributions from (2.18), one that will renormalize the fugacity and one that will renormalize the curvature energy. These are, respectively,

$$ y_n^2 e^{-G^n_{ii}(0)} \sum_{i \neq j} \int (e^{-G^n_{ii}(\mathbf{r})} - 1) \cos[{}^iT^n_{\alpha\beta\cdots\gamma} \nabla_\alpha \nabla_\beta \ldots \nabla_\gamma \varphi^<(\mathbf{u} + \tfrac{1}{2}\mathbf{r}) + {}^jT^n_{\alpha\beta\cdots\gamma} \nabla_\alpha \nabla_\beta \ldots \nabla_\gamma \varphi^<(\mathbf{u} - \tfrac{1}{2}\mathbf{r})] d\mathbf{u}\, d\mathbf{r} \tag{2.19} $$

and

$$y_n^2 e^{-G_{ii}^n(0)} \sum_i \int \{(e^{G_{ii}^n(\mathbf{r})}-1)\cos[{}^iT^n_{\alpha\beta\cdots\gamma}\nabla_\alpha\nabla_\beta\cdots\nabla_\gamma\varphi^<(\mathbf{u}+\tfrac{1}{2}\mathbf{r})-{}^iT^n_{\alpha\beta\cdots\gamma}\nabla_\alpha\nabla_\beta\cdots\nabla_\gamma\varphi^<(\mathbf{u}-\tfrac{1}{2}\mathbf{r})]\}d\mathbf{u}\,d\mathbf{r}\ . \tag{2.20}$$

Here we have neglected terms that will renormalize higher harmonics and can be shown to be less relevant. To proceed we consider the operator product expansion of

$$\cos[{}^1T^n_{\alpha\beta\cdots\gamma}\nabla_\alpha\nabla_\beta\cdots\nabla_\gamma\varphi(\mathbf{x})+{}^2T^n_{\alpha\beta\cdots\gamma}\nabla_\alpha\nabla_\beta\cdots\nabla_\gamma\varphi(\mathbf{y})]_{\mathbf{x}\to\mathbf{y}}=\tilde{O}(\mathbf{x})+a_n(\mathbf{x}-\mathbf{y})\cos[{}^3T^n_{\alpha\beta\cdots\gamma}\nabla_\alpha\nabla_\beta\cdots\nabla_\gamma\varphi(\mathbf{x})]\ . \tag{2.21}$$

Here $\tilde{O}(\mathbf{x})$ is an irrelevant operator. The constant $a_n(\mathbf{x}-\mathbf{y})$ is calculated using the operator product formalism of Kadanoff.[11] We construct a correlation function

$$g_n(R)=\left\langle \begin{bmatrix}\cos[{}^1T^n_{\alpha\beta\cdots\gamma}\nabla_\alpha\nabla_\beta\cdots\nabla_\gamma\varphi(\mathbf{x})+{}^2T^n_{\alpha\beta\cdots\gamma}\nabla_\alpha\nabla_\beta\cdots\nabla_\gamma\varphi(\mathbf{y})]\\ -a_n(\mathbf{x}-\mathbf{y})\cos[{}^3T^n_{\alpha\beta\cdots\gamma}\nabla_\alpha\nabla_\beta\cdots\nabla_\gamma\varphi(\mathbf{x})]\end{bmatrix}\cos[{}^3T^n_{\alpha\beta\cdots\gamma}\nabla_\alpha\nabla_\beta\cdots\nabla_\gamma\varphi(\mathbf{x}+\mathbf{R})]\right\rangle_0\ , \tag{2.22}$$

and require that in the limit $\mathbf{x}\to\mathbf{y}$ it satisfies an "orthogonality" property that for any given n it does not contain any terms that decay algebraically in R. The expectation values are taken with respect to the quadratic part of the Hamiltonian, since all other contributions will lead to terms of higher order in y. One can then show that

$$a_n(\mathbf{x}-\mathbf{y})=e^{-V_{12}^n(\mathbf{x}-\mathbf{y})}\ , \tag{2.23}$$

where

$$V_{ij}^n(\mathbf{x})=\frac{1}{\kappa}\int\frac{({}^iT^n_{\alpha\beta\cdots\gamma}q_\alpha q_\beta\cdots q_\gamma)({}^iT^n_{\alpha\beta\cdots\gamma}q_\alpha q_\beta\cdots q_\gamma)(e^{i\mathbf{q}\cdot(\mathbf{x}-\mathbf{y})}-1)}{q^{2(n+1)}}\varphi(q)\frac{d^2\mathbf{q}}{(2\pi)^2}\ . \tag{2.24}$$

Similarly,

$$\cos[{}^iT^n_{\alpha\beta\cdots\gamma}\nabla_\alpha\nabla_\beta\cdots\nabla_\gamma\varphi(\mathbf{x})-{}^iT^n_{\alpha\beta\cdots\gamma}\nabla_\alpha\nabla_\beta\cdots\nabla_\gamma\varphi(\mathbf{y})]=\sum_{m=0}^{\infty}\frac{(-1)^m}{(2m)!}O_i^{2m}\ , \tag{2.25}$$

where

$$O_i(\mathbf{x})\equiv{}^iT^n_{\alpha\beta\cdots\gamma}\nabla_\alpha\nabla_\beta\cdots\nabla_\gamma\varphi(\mathbf{x})-{}^iT^n_{\alpha\beta\cdots\gamma}\nabla_\alpha\nabla_\beta\cdots\nabla_\gamma\varphi(\mathbf{y})\ . \tag{2.26}$$

For notational simplicity we suppress index n on O_i. Each O_i^{2m} will contain some $O_i^2(\mathbf{x})$. We can then write[8]

$$O_i^{2m}(\mathbf{x})=[{}^iT^n_{\alpha\beta\cdots\gamma}\nabla_\alpha\nabla_\beta\cdots\nabla_\gamma\varphi(\mathbf{x})-{}^iT^n_{\alpha\beta\cdots\gamma}\nabla_\alpha\nabla_\beta\cdots\nabla_\gamma\varphi(\mathbf{y})]^{2m}=\tilde{O}_{2m,i}(\mathbf{x})+a_{2m,i}(\mathbf{x}-\mathbf{y})O_i^2(\mathbf{x})+b_{2m,i}(\mathbf{x}-\mathbf{y})\ . \tag{2.27}$$

Again $\tilde{O}_{2m,i}(\mathbf{x})$ are irrelevant operators and $b_{2m,i}(\mathbf{x}-\mathbf{y})$ are constants that will renormalize the free energy. To project out $O_i^2(\mathbf{x})$, we consider the correlation function

$$g_{2m,i}(R)=\langle[O_i^{2m}(\mathbf{x})-a_{2m,i}O_i^2(\mathbf{x})]O_i^2(\mathbf{x}+\mathbf{R})\rangle_0\ . \tag{2.28}$$

We now require that $g_{2n,i}(R)$ satisfy the orthogonality property of not containing any terms that fall off proportional to $R^{-2(n+1)}$ [as would be the case for the operator $O_i^2(\mathbf{x})$]. The $a_{2m,i}$ are given by all possible contractions that contain terms $\langle O_i(\mathbf{x})O_i(\mathbf{x}+\mathbf{R})\rangle_0^2$, since these decays as $R^{-2(n+1)}$. It is then possible to show that

$$a_{2m,i}(\mathbf{x}-\mathbf{y})=\frac{(2m)!}{2^m(m-1)!}\langle O_i^2(\mathbf{x})\rangle_0^{m-1}\ , \tag{2.29}$$

where

$$\langle O_i^2(\mathbf{x})\rangle=-2V_{ii}^n(\mathbf{x}-\mathbf{y})\ . \tag{2.30}$$

Thus, Eq. (2.25) becomes

$$\begin{aligned}&\cos[{}^iT^n_{\alpha\beta\cdots\gamma}\nabla_\alpha\nabla_\beta\cdots\nabla_\gamma\varphi(\mathbf{x})-{}^iT^n_{\alpha\beta\cdots\gamma}\nabla_\alpha\nabla_\beta\cdots\nabla_\gamma\varphi(\mathbf{y})]_{\mathbf{x}\to\mathbf{y}}\\ &\qquad=\tilde{O}_i(\mathbf{x})-\tfrac{1}{2}e^{V_{ii}(\mathbf{x}-\mathbf{y})}[{}^iT^n_{\alpha\beta\cdots\gamma}\nabla_\alpha\nabla_\beta\cdots\nabla_\gamma\varphi(\mathbf{x})-{}^iT^n_{\alpha\beta\cdots\gamma}\nabla_\alpha\nabla_\beta\cdots\nabla_\gamma\varphi(\mathbf{y})]^2+B(\mathbf{x}-\mathbf{y})\ .\end{aligned} \tag{2.31}$$

Applying these results in expressions (2.19) and (2.20), one obtains the part of the second cumulant that renormalizes the fugacity to be

$$-2y_n^2 e^{-G_{12}(0)}\varepsilon\int g_{12}(\mathbf{r})e^{-V_{12}^n(\mathbf{r})}d^2\mathbf{r}\int\cos[{}^3T^n_{\alpha\beta\cdots\gamma}\nabla_\alpha\nabla_\beta\cdots\nabla_\gamma\varphi^<(\mathbf{u})]d^2\mathbf{u}\ . \tag{2.32}$$

Similarly, the term that renormalizes the curvature is,

$$-\tfrac{1}{2}y_n^2 e^{-G_{11}(0)}\varepsilon \sum_i \int g_{ii}^n(\mathbf{r}) e^{V_{ii}(\mathbf{r})}({}^iT^n_{\alpha\beta\cdots\gamma}\nabla_\alpha\nabla_\beta\cdots\nabla_\gamma)^2[\mathbf{r}\cdot\nabla\varphi^<(\mathbf{u})]^2 d^2\mathbf{r}\, d^2\mathbf{u} . \tag{2.33}$$

One can show that

$$g_{ij}^n(\mathbf{r}) = r\frac{dV_{ij}^n(\mathbf{r})}{dr} + \frac{e_i\cdot e_j}{4\pi\kappa} , \tag{2.34}$$

where the e_i are given by Eq. (1.5).

After rescaling ($k \to k/\delta$, $r \to r\delta$, $\varphi \to \varphi\delta^n$, where $\delta = 1+\varepsilon$) and changing to momentum space, we find the recurrence relations

$$y_n(\varepsilon) = y_n e^{-(1/2)G_{11}(0)}\delta^2 + y_n^2 e^{-G_{12}(0)}\varepsilon\delta^2 \int d\mathbf{r}\, r^{1+(e_1\cdot e_2/4\pi\kappa)}\frac{d}{dr}(e^{-V_{12}^n(\mathbf{r})-(e_1\cdot e_2/4\pi\kappa)\ln r}) , \tag{2.35}$$

$$\kappa_n(\varepsilon) = \kappa_n + y_n^2 e^{-G_{11}(0)}\varepsilon \sum_i \int d\mathbf{r}\, r^{3-(1/4\pi\kappa)}\frac{d}{dr}[e^{V_{ii}^n(\mathbf{r})+(1/4\pi\kappa)\ln r}({}^iT^n_{\alpha\beta\cdots\gamma}\hat{\mathbf{q}}_\alpha\hat{\mathbf{q}}_\beta\cdots\hat{\mathbf{q}}_\gamma)^2(\hat{\mathbf{r}}\cdot\hat{\mathbf{q}})^2] . \tag{2.36}$$

Using Eqs. (2.9)–(2.11) it is easy to perform the angular integration in the equations above. For the case of $n=0$ we get the usual sine-Gordon model.[8] For $n=1$ the equations have already been given in Ref. 7, for $n>1$ we have

$$\frac{dy_n}{d\varepsilon} = \left[1-\frac{1}{8\pi\kappa}\right]y + 2\pi y_n^2 \int dr\, r^{2-(1/8\pi\kappa)}\frac{d}{dr}[e^{(1/2)A(\phi;r)+(1/8\pi\kappa)\ln r}I_0(B_n(\phi;r))] \tag{2.37}$$

and

$$\frac{d\kappa_n}{d\varepsilon} = \frac{3}{2}\pi y_n^2 \int dr\, r^{4-(1/4\pi\kappa)}\frac{d}{dr}[e^{A(\phi;r)+(1/4\pi\kappa)\ln r}I_0(B_n(\phi;r))] , \tag{2.38}$$

where

$$A(\phi;r) = \frac{1}{4\pi\kappa}\int [J_0(qr)-1]\phi(q)\frac{dq}{q} \tag{2.39}$$

and

$$B_n(\phi;r) = \frac{1}{4\pi\kappa}\int J_{2n}(qr)\phi(q)\frac{dq}{q} . \tag{2.40}$$

Using the Bessel function integral identity,

$$\int_0^\infty \frac{J_n(x)}{x}dx = \frac{1}{n} , \tag{2.41}$$

for the special case of a sharp cutoff, near the transition temperature, ($\kappa = 1/16\pi$), we obtain

$$\frac{dy_n}{d\varepsilon} = \left[2-\frac{1}{8\pi\kappa}\right]y_n + 2\pi y^2 I_0\left[\frac{2}{n}\right] \tag{2.42}$$

and

$$\frac{d\kappa_n}{d\varepsilon} = \frac{3}{2}y_n^2 I_0\left[\frac{2}{n}\right] . \tag{2.43}$$

The procedure used to obtain critical exponents from these types of equations has been given in Refs. 1 and 5. We do not repeat it here but simply state the results. Defining the quantity

$$\alpha_n^2 = \frac{I_0(2/n)}{48} , \tag{2.44}$$

the critical exponents ν_n are then given by the formula

$$\nu_n = 1-\tfrac{1}{2}[1+\alpha_n^2-\alpha_n(1+\alpha_n^2)^{1/2}]^{-1} . \tag{2.45}$$

Note that as $n\to\infty$ the critical exponent $\nu_n \to \frac{3}{7}$.

III. DISCUSSION

In this paper we have considered an isotropic discrete Laplacian roughening model with an arbitrary range of interaction. By considering the symmetry of the microscopic Hamiltonian (1.3) we have constructed a coarse-grained LGW Hamiltonian (2.2), which we believe summarizes the main aspects of the roughening on a triangular lattice. Thus we have shown that a Laplacian roughening model of order p (p is the range of interaction given by Eqs. (1.3) will possess $2p$ phase transitions, each of a modified Kosterlitz-Thouless type, given by Eqs. (2.42) and (2.43). Using the $\mathbb{Z}_6$ group we are able to characterize all possible orientational symmetries of different phases that develop by the spin of the tensors conjugate to the order parameter. There will be only six different orientational symmetries for the surface corresponding to six possible values of spin. However, each new phase that develops will have a different radial structure. The situation here is similar to that of quantum-mechanical hydrogen atom. There, the different orbital wave functions can have the same angular behavior, characterized by the angular-momentum quantum number, while possessing distinct radial structure. To study the symmetry of these phases we must construct appropriate correlation functions that will reflect this symmetry. Thus, for example, to study the symmetry of the interface with orientational structure corresponding to $s=1$, Nelson considered correlations around a hexagon.[2] It is not yet clear what correlation functions are needed to study the interfaces with higher-spin-symmetry values.

The structure of the phase transitions for the Laplacian roughening model can then be viewed as a $2p$ by $2p+1$ array of the form in Fig. 1. F and R stand for the flat and rough structures respectively, while the number corresponds to the rank of the tensor conjugate to the order parameter. Transition between $F0$ and $R0$ is described by the usual sine-Gordon model, which is known to be in the Kosterlitz-Thouless universality class.[8–10] It was shown by Ohta and Kowasaki[12] that below the roughening temperature the width of the interface, given by $L^2 \approx g_0(r) = \langle (h(\mathbf{r}) - h(\mathbf{0}))^2 \rangle$, is a constant proportional to $\ln\xi_0$ for large r. As the roughening temperature is approached ξ_0 will diverge as $\xi_0 = e^{ct^{-\nu_0}}$, signifying the divergence of the interfacial width. In the rough phase $g_0(r) \approx \ln r$. The transition between $F1$ and $R1$ (corresponding to Nelson's orientational roughening) was shown to be in the vector Coulomb gas universality class.[1,7] Nelson suggested that the appropriate correlation function needed to study this transition is

$$g_1(r) = \sum_{i=1}^{6} \langle (h(\mathbf{r}) - h(\mathbf{r}+\delta_i) - h(\mathbf{0}) + h(\delta_i))^2 \rangle , \quad (3.1)$$

where $\{\delta_i\}$ are the six interlattice unit vectors. This function measures the correlations between the tangents to the interface. Once again below the orientational roughening temperature $T_1, g_1(r) \approx \ln\xi_1$, where $\xi_1 = e^{ct^{-\nu_1}}$. Here c is some nonuniversal constant and the critical exponent is the $n=1$ entry in the array above. In the orientationally rough phase $g_1(r) \approx \ln r$. It is easy to see that the appropriate generalizations of the correlation functions given above will be

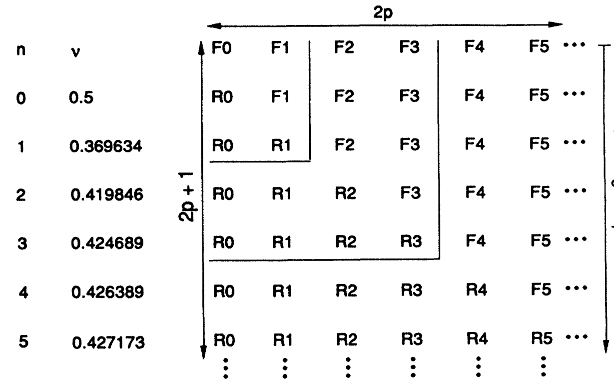

FIG. 1. The phase diagram of a sequence of roughening transitions for the generalized Laplacian roughening model of order p in the form of $2p$ by $2p+1$ array.

$$g_n(r) = \langle (\Delta^{n/2}h(\mathbf{r}) - \Delta^{n/2}h(\mathbf{0}))^2 \rangle , \quad (3.2)$$

$$g_n(r) = \sum_{i=1}^{6} \langle [\Delta^{(n-1)/2}h(\mathbf{r}) - \Delta^{(n-1)/2}h(\mathbf{r}+\delta_i) - \Delta^{(n-1)/2}h(\mathbf{0}) + \Delta^{(n-1)/2}h(\delta_i)]^2 \rangle , \quad (3.3)$$

for n even and odd, respectively. Below the transition temperature $T_n, g_n(r) \approx \ln\xi_n$, where $\xi_n = e^{ct^{-\nu_n}}$ with the critical exponent given by the n's entry of the array above. Above the transition temperature $g_n(r) \approx \ln r$. These higher-order phase transitions correspond to the interface losing more and more of its structure as the temperature is increased.

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[1]D. Nelson, Phys. Rev. B **18**, 2318 (1978).
[2]D. Nelson, Phys. Rev. B **26**, 269 (1982).
[3]S. T. Chui and J. D. Weeks, Phys. Rev. B **14**, 4978 (1976).
[4]J. M. Kosterlitz and D. J. Thouless, J. Phys. C **6**, 1181 (1973); J. V. José, L. P. Kadanoff, S. Kirkpatrick, and D. R. Nelson, Phys. Rev. B **16**, 1217 (1977).
[5]A. P. Young, Phys. Rev. B **19**, 1855 (1979).
[6]D. R. Nelson and B. I. Halperin, Phys. Rev. B **19**, 2457 (1979).
[7]Y. Levin and K. Dawson, Phys. Rev. A **42**, 3507 (1990).
[8]H. J. F. Knops and L. W. J. den Ouden, Physica **103A**, 597 (1980).
[9]T. Ohta, Prog. Theor. Phys. **60**, 968 (1978); T. Ohta and D. Jasnow, Phys. Rev. B **20**, 139 (1979).
[10]D. J. Amit, Y. Y. Goldschmidt, and G. Grinstein, J. Phys. A **13**, 585 (1980).
[11]L. P. Kadanoff, Ann. Phys. **120**, 39 (1978).
[12]T. Ohta and K. Kawasaki, Prog. Theor. Phys. **60**(2), 365 (1978).