Relaxation and emittance growth of a thermal charged-particle beam

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We present a theory that allows us to accurately calculate the distribution functions and the emittance growth of a thermal charged-particle beam after it relaxes to equilibrium. The theory can be used to calculate the density and the velocity distributions as well as the emittance growth of the beam. The theory is applicable to arbitrary initial conditions. In this letter, we will present a theoretical framework that will allow us to accurately calculate the density and the velocity distributions of particles in the final stationary state achieved by a space-charge dominated beam focused by a uniform external magnetic field. Our approach is based on the theory of violent relaxation in gravitational systems, modified so as to explicitly account for the effects of single particle resonances responsible for the halo formation. The theory is applicable to arbitrary initial conditions. In this letter we will show how the theory can be used to accurately calculate the density and the velocity distributions as well as to account for the emittance growth of a charged-particle beam launched with a thermal (Maxwell) velocity distribution. The predictions of the theory will be tested against the molecular dynamics simulations.

The physical system considered here is an intense charged-particle beam of perveance $K=2q^2N_b/\gamma_p^2m^2$—where $c$ is the speed of light in vacuo, and $q$, $m$, and $\gamma_p=[1-(v/c)^2]^{-1/2}$ are the charge, mass, and the relativistic factor of the beam particles, respectively—propagating with an axial velocity $v_{\text{ax}}$ through a magnetic focusing channel enclosed by a cylindrical conducting wall located at $r=r_w$. The external focusing magnetic field is given by $B=B_0\hat{z}$. It is convenient to work in the Larmor frame, which rotates with respect to the laboratory frame with angular velocity $\Omega_L=gB_0/2y_0v_{\text{ax}}mc$, normalized to $v_c$. In the Larmor frame, the external magnetic field produces a parabolic confining potential $U(r) = k_v r^2/2$, with the focusing field parameter $k_v = \Omega_L^2/2$. The effective electromagnetic scalar potential between the particles incorporates both the self-electric and the self-magnetic fields, $E^*$ and $B^*$. This potential satisfies the Poisson equation with the boundary condition $\psi(r_0)=0$,

$$\nabla^2 \psi = -(2\pi K/N_b) n_b(r, s),$$

(1)

where $N_b$ is the number of particles per unit axial length, $r$ is the position vector in the transverse plane, and $n_b(r, s) = N_b \int d\mathbf{v}$ is the transverse beam density profile, given in terms of the one particle distribution function $f(r, \mathbf{v}, s)$. In the Larmor frame, the dynamics of the beam reduces to that of a two-dimensional one component plasma with logarithmic interaction between the particles, confined by a parabolic potential $U(r)$. The axial coordinate $s = z = v_c t$ plays the role of time for this two dimensional system.

We will suppose that the initial (transverse) distribution of the beam is Gaussian in velocity space and is uniform in cross section,

$$f_0(r, \mathbf{v}) = \frac{1}{2\pi \sigma^2 r_m^2} \Theta(r_m - r) e^{-\sigma^2/2r_m^2},$$

(2)

where $\sigma^2$ is the initial mean square transverse velocity and $r_m$ is the beam radius. The quality of the beam is inversely proportional to the emittance, defined as $\varepsilon^2 = 4\langle r^2 \rangle /r_m^2$, for a stationary beam. For the distribution (2), the emittance is $\varepsilon_0 = 2\sigma r_m$.

It will be convenient to discretize Eq. (2) into a $p$-level distribution

$$f_p(r, \mathbf{v}) = \sum_{j=1}^{p} \eta_j \rho_j^0(r, \mathbf{v}),$$

(3)

where $\rho_j^0(r, \mathbf{v}) = \Theta(v - v_{j-1})\Theta(v_j - v)\Theta(r_m - r)$ and $v_j$ and $\eta_j$ are the maximum velocity and the amplitude of the level $j$, respectively, with $v_0=0$. For a perfect description of Eq. (2), an infinite number of levels ($p \rightarrow \infty$) in Eq. (3) will be necessary. In practice, however, we find that a small number of levels are already sufficient to provide a very accurate ap-

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proximation for the beam dynamics. For a given value of $p$, the optimal values of $\eta_j$ and $v_j$ can be obtained by minimizing the functional $\mathcal{F}=\int (f_0^j - f_j^0)^2 d^3r d^3v$, with the constraints on the kinetic energy and normalization,

$$\delta \mathcal{F} + \lambda_1 \left( \int \frac{v^2}{2} f_j^0 d^3r d^3v - \sigma^2 \right) + \lambda_2 \left( \int f_j^0 d^3r d^3v - 1 \right) = 0, \quad (4)$$

where $\lambda_1$ and $\lambda_2$ are the two Lagrange multipliers. Minimization of Eq. (4) yields the optimal parameters $\{\eta_j\}$ and $\{v_j\}$. The many-body dynamics of systems with unscreened long-range interaction is governed by the collisionless Boltzmann (Vlasov) equation. The distribution functions that satisfy the Vlasov equation evolve in time as the density of an incompressible fluid. In particular this means that the $V$-incompressible equation will be conserved, while the particles from the higher energy states will evaporate to form a halo. We find that the halo can be modeled accurately by the distribution

$$f_h(r, v) = \chi \Theta[\epsilon_r - \epsilon(r, v)] + \epsilon \Theta[\epsilon(r, v) - \epsilon_r] \Theta[\epsilon_r - \epsilon(r, v)] e^{-\frac{\sigma^2}{2 \epsilon_r}}. \quad (8)$$

The extent of the halo is up to one particle resonance energy $\epsilon_r$. The low energy part of the halo distribution is flat, while for energies $\epsilon > \epsilon_r = \epsilon_r/2$, it decays exponentially with exponent $\gamma = 8$. We can now, in principle, numerically solve Eqs. (1), (5), (7), and (8) to calculate the stationary distribution function $f(r, v)$ of the relaxed beam. There is, however, one problem. Equations (7) and (8) contain $p+2$ parameter: $\beta$, $\{\alpha_j\}$, and $\chi$. The conservation of energy, norm, and lower energy hypervolumes gives us $p+1$ additional equations,

$$\int d^3r d^3v \epsilon(r, v) f(r, v) = \epsilon_0,$$

$$\int d^3r d^3v f(r, v) = 1,$$

$$\int d^3r d^3v \rho_j(r, v) = \int d^3r d^3v \rho_j^0(r, v), \quad (9)$$

where $1 \leq j \leq p-1$ and $\epsilon_0$ is the average energy per particle of the initial thermal distribution,

$$\epsilon_0 = \sigma^2 + \kappa \frac{r_m^2}{4} + K \left[ \frac{1}{8} - \frac{1}{2} \ln \left( \frac{r_m}{r_w} \right) \right]. \quad (10)$$

There, however, still remains one missing condition necessary to uniquely determine the distribution function. For water-bag distributions, this condition was provided by the requirement that in the relaxed state, the core temperature is very low, $T = T_f/40$. It is difficult, however, to numerically implement this condition for $p$-level distributions. On the other hand, if we discretize the original thermal distribution into only one level $(p=1)$, the condition $T = T_f/40$ is easily implemented and allows us to uniquely close all the equations and calculate the relaxed distribution function.\(^{13}\) We find that although the core distribution is not well described by a $p=1$ system, the halo part of the distribution is found to be quite accurate. This allows us to fix the value of $\chi$. Using this $\chi$, we can now improve the description of the core region by including additional levels into the discretization procedure. To compare the predictions of the theory with the simulations, we calculated the number of particles in the interval $[r, r + dr]$, $N(r) dr = 2 \pi N_p dr \int d^3v f(r, v)$, and the number of particles with velocities between $v$ and $v + dv$, $N(v) dv = 2 \pi N_p v dv \int d^3v f(r, v)$, for various initial conditions. The simulations are based on the Vlasov dynamics in which particles interact with the mean-field potential. This avoids the collisional effects present in finite size systems but which must vanish for one component plasmas in thermodynamic limit. The simulation code uses the Gauss law to calculate the mean electric field felt by each particle.\(^7\) For axisymmetric beams studied in this work, this proves to be very efficient since the electric field at a radial coordinate $r$ is determined simply by counting the total number of particles with coordinates smaller than $r$. Simulations were performed with 20,000 particles. As can be seen from the Figs. 1 and 2, the agreement between the theory and the simulations is excel-
also defined a scaled perveance \( K^* = K/\Omega L \varepsilon_0 \) and the mismatch parameter, \( \mu = r_m/r_0 \), which measures the deviation of the initial beam radius from the corresponding virial value \( r_0 = \sqrt{K + 4\sigma^2/\Omega L} \), for which the oscillations of the beam envelope are very small. In particular, we find that the discretization of the Gaussian by only four levels already provides us with an almost perfect description of the core region.

As a direct application of the theory developed above, we calculate the emittance growth of an originally thermal beam. This quantity is of fundamental importance for the design and development of high-intensity space-charge dominated beams.\(^\text{16}\) The calculations are performed for beams of varying scaled perveance \( K^* \) and mismatch parameter \( \mu \). The results are compared with the molecular dynamics simulations. Once again, an excellent agreement is found between the theory and the simulations (Fig. 3).

To conclude, we have presented a theory that allows us to calculate the density and the velocity distributions of an initially thermal beam after it relaxes to the final stationary state. Comparing to the simulations, the theory is found to be extremely accurate without any adjustable parameters. In particular, it can be used to calculate the emittance growth and the fraction of particles, which will evaporate as the beam evolves to its final stationary state.

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