Relaxation and emittance growth of a thermal charged-particle beam

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We present a theory that allows us to accurately calculate the distribution functions and the emittance growth of a thermal charged-particle beam after it relaxes to equilibrium. The theory can be used to obtain the fraction of particles, which will evaporate from the beam to form a halo. The calculated emittance growth is found to be in excellent agreement with the simulations. © *2009 American Institute of Physics*. [doi:10.1063/1.3254245]

The understanding of physics involved in the transport of high-intensity charged-particle beams is of fundamental importance in the development of a new generation of accelerators and electromagnetic wave generators to be used in applications such as heavy ion fusion, high-energy physics, communication, materials processing, and cancer therapy. A very detrimental effect that may seriously influence the efficiency of such devices is a halo formation and emittance growth of the beam.[1–5] These not only cause degradation of the beam quality but may also be responsible for the activation of accelerator channel wall and pulse shortening in microwave devices. Emittance growth is generally associated with the relaxation of initially nonstationary beam toward a more stable stationary configuration. The emittance growth can be calculated if the final stationary distribution is known.[3,6,7] However, the determination of this distribution is not an easy task[6,8,9] because particles in an intense beam interact through long-range forces, which prevent the system from relaxing to the true thermodynamic equilibrium.[10–12] Instead these systems get trapped in metastable states, the lifetime of which diverges with the number of particles. To understand the properties of these states, one cannot use the standard statistical mechanics, and new nonequilibrium theories must be developed.[13]

In this letter, we will present a theoretical framework that will allow us to accurately calculate the density and the velocity distributions of particles in the final stationary state achieved by a space-charge dominated beam focused by a uniform external magnetic field. Our approach is based on the theory of violent relaxation in gravitational systems,[14] modified so as to explicitly account for the effects of single particle resonances[1] responsible for the halo formation.[13] The theory is applicable to arbitrary initial conditions. In this letter we will show how the theory can be used to accurately calculate the density and the velocity distributions as well as to account for the emittance growth of a charged-particle beam launched with a thermal (Maxwell) velocity distribution. The predictions of the theory will be tested against the molecular dynamics simulations.

The physical system considered here is an intense charged-particle beam of perveance $K=2q^2N_b/\gamma_b^3 v_z^2 m$—where c is the speed of light *in vacuo*, and q, m, and $\gamma_b=[1-(v_z/c)^2]^{-1/2}$ are the charge, mass, and the relativistic factor of the beam particles, respectively—propagating with an axial velocity $v_z\hat{\mathbf{e}}_z$ through a magnetic focusing channel enclosed by a cylindrical conducting wall located at $r=r_w$.[8,15] The external focusing magnetic field is given by $\mathbf{B}=B_0\hat{\mathbf{e}}_z$. It is convenient to work in the Larmor frame, which rotates with respect to the laboratory frame with angular velocity $\Omega_L=qB_0/2\gamma_b v_z mc$, normalized to v_z. In the Larmor frame, the external magnetic field produces a parabolic confining potential $U(r)=\kappa_z r^2/2$, with the focusing field parameter $\kappa_z=\Omega_L^2/c^2$. The effective electromagnetic scalar potential between the particles ψ incorporates both the self-electric and the self-magnetic fields, $\mathbf{E}^s$ and $\mathbf{B}^s$. This potential satisfies the Poisson equation with the boundary condition $\psi(r_w)=0$,

$$\nabla^2\psi=-(2\pi K/N_b)n_b(\mathbf{r},s), \tag{1}$$

where N_b is the number of particles per unit axial length, $\mathbf{r}$ is the position vector in the transverse plane, and $n_b(\mathbf{r},s)=N_b\int f d^2\mathbf{v}$ is the transverse beam density profile, given in terms of the one particle distribution function $f(\mathbf{r},\mathbf{v};s)$. In the Larmor frame, the dynamics of the beam reduces to that of a two dimensional one component plasma with logarithmic interaction between the particles, confined by a parabolic potential $U(r)$. The axial coordinate $s=z=v_z t$ plays the role of time for this two dimensional system.

We will suppose that the initial (transverse) distribution of the beam is Gaussian in velocity space and is uniform in cross section,

$$f_0(\mathbf{r},\mathbf{v})=\frac{1}{2\pi^2\sigma^2 r_m^2}\Theta(r_m-r)e^{-(v^2/2\sigma^2)}, \tag{2}$$

where σ^2 is the initial mean square transverse velocity and r_m is the beam radius. The quality of the beam is inversely proportional to the emittance, defined as $\varepsilon^2=4\langle r^2\rangle\langle v^2\rangle$, for a *stationary* beam. For the distribution (2), the emittance is $\varepsilon_0=2\sigma r_m$.

It will be convenient to discretize Eq. (2) into a p-level distribution

$$f_p^0(\mathbf{r},\mathbf{v})=\sum_{j=1}^{p}\eta_j\rho_j^0(\mathbf{r},\mathbf{v}), \tag{3}$$

where $\rho_j^0(\mathbf{r},\mathbf{v})\equiv\Theta(v-v_{j-1})\Theta(v_j-v)\Theta(r_m-r)$ and v_j and η_j are the maximum velocity and the amplitude of the level j, respectively, with $v_0=0$. For a perfect description of Eq. (2), an infinite number of levels ($p\to\infty$) in Eq. (3) will be necessary. In practice, however, we find that a small number of levels are already sufficient to provide a very accurate ap-

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proximation for the beam dynamics. For a given value of p, the optimal values of η_j and v_j can be obtained by minimizing the functional $\mathcal{F}=\int(f_0-f_p^0)^2 d^2\mathbf{r}d^2\mathbf{v}$, with the constraints on the kinetic energy and normalization,

$$\delta\left\{\mathcal{F}+\lambda_1\left(\int \frac{v^2}{2}f_p^0 d^2\mathbf{r}d^2\mathbf{v}-\sigma^2\right)+\lambda_2\left(\int f_p^0 d^2\mathbf{r}d^2\mathbf{v}-1\right)\right\} = 0, \tag{4}$$

where λ_1 and λ_2 are the two Lagrange multipliers. Minimization of Eq. (4) yields the optimal parameters $\{\eta_j\}$ and $\{v_j\}$. The many-body dynamics of systems with unscreened long-range interaction is governed by the collisionless Boltzmann (Vlasov) equation. The distribution functions that satisfy the Vlasov equation evolve in time as the density of an incompressible fluid. In particular this means that the p hypervolumes, $\gamma(\eta_j)=\int\delta(f_p^0(\mathbf{r},\mathbf{v})-\eta_j)d^d\mathbf{r}d^d\mathbf{v}$, of the distribution (3) will be preserved by the Vlasov flow.[14]

In Ref. 14 it was argued that the stationary solution of the Vlasov equation could be obtained by maximizing the coarse grained entropy, with the constraints imposed by the conservation of energy and the hypervolumes of the p levels of the initial distribution function. For matched beams and water-bag initial conditions, the resulting distribution was shown to be in excellent agreement with the molecular dynamics simulations.[13] However, for mismatched beams, the plasma oscillations result in parametric resonances, which lead to a significant particle evaporation. After the relaxation process is complete, the stationary beam phase separates into a cold core, surrounded by a halo of highly energetic particles. For a water-bag initial condition ($p=1$), it was shown that the core was very well described by a cold Fermi–Dirac distribution with the temperature $T\approx T_F/40$, where T_F is the "Fermi temperature" of the beam. The halo was reasonably approximated by a step function with energy range of one particle resonance. The full distribution function had the form of

$$f(\mathbf{r},\mathbf{v})=f_c(\mathbf{r},\mathbf{v})+f_h(\mathbf{r},\mathbf{v}). \tag{5}$$

For a p-level system, which is used to approximate the thermal distribution given by Eq. (2), a similar phase separation will occur. The form of the core distribution function can be obtained, once again, by maximizing the coarse grained entropy to yield

$$f_c(\mathbf{r},\mathbf{v})=\sum_{j=1}^{p}(\eta_j-\chi)\rho_j(\mathbf{r},\mathbf{v}), \tag{6}$$

with

$$\rho_j(\mathbf{r},\mathbf{v})=\frac{e^{-\beta\eta_j\epsilon(\mathbf{r},\mathbf{v})+\alpha_j}}{\sum_{i=1}^{p}e^{-\beta\eta_i\epsilon(\mathbf{r},\mathbf{v})+\alpha_i}+1}, \tag{7}$$

where the mean particle energy is $\epsilon(\mathbf{r},\mathbf{v})=v^2/2+U(r)+\psi(r)$ and β and $\{\alpha_j\}$ are the Lagrange multipliers for the energy and the hypervolumes conservation. The oscillations of the mismatched beam excite the parametric resonances, resulting in a halo formation.[13] The parameter χ determines the fraction of the particles, which will evaporate to form the halo of the beam. The coarse grained distribution can no longer preserve all the hypervolumes of the original fine-grained distribution function so that only the lower energy hypervolumes will be conserved, while the particles from the higher energy states will evaporate to form a halo. We find that the halo can be modeled accurately by the distribution

$$f_h(\mathbf{r},\mathbf{v})=\chi\Theta[\epsilon_\zeta-\epsilon(\mathbf{r},\mathbf{v})]+\chi\Theta[\epsilon(\mathbf{r},\mathbf{v})-\epsilon_\zeta]\Theta[\epsilon_R-\epsilon(\mathbf{r},\mathbf{v})]e^{-\gamma(\epsilon-\epsilon_\zeta)}. \tag{8}$$

The extent of the halo[1] is up to one particle resonance energy ϵ_R. The low energy part of the halo distribution is flat, while for energies $\epsilon>\epsilon_\zeta=\epsilon_R/2$, it decays exponentially with exponent $\gamma\approx 8$. We can now, in principle, numerically solve Eqs. (1), (5), (7), and (8) to calculate the stationary distribution function $f(\mathbf{r},\mathbf{v})$ of the relaxed beam. There is, however, one problem. Equations (7) and (8) contain $p+2$ parameter: β, $\{\alpha_j\}$, and χ. The conservation of energy, norm, and lower energy hypervolumes gives us $p+1$ additional equations,

$$\int d^2\mathbf{r}d^2\mathbf{v}\epsilon(\mathbf{r},\mathbf{v})f(\mathbf{r},\mathbf{v})=\epsilon_0,$$

$$\int d^2\mathbf{r}d^2\mathbf{v}f(\mathbf{r},\mathbf{v})=1,$$

$$\int d^2\mathbf{r}d^2\mathbf{v}\rho_j(\mathbf{r},\mathbf{v})=\int d^2\mathbf{r}d^2\mathbf{v}\rho_j^0(\mathbf{r},\mathbf{v}), \tag{9}$$

where $1\le j\le p-1$ and ϵ_0 is the average energy per particle of the initial thermal distribution,

$$\epsilon_0=\sigma^2+\kappa_z\frac{r_m^2}{4}+K\left[\frac{1}{8}-\frac{1}{2}\ln\left(\frac{r_m}{r_w}\right)\right]. \tag{10}$$

There, however, still remains one missing condition necessary to uniquely determine the distribution function. For water-bag distributions, this condition was provided by the requirement that in the relaxed state, the core temperature is very low, $T\approx T_F/40$. It is difficult, however, to numerically implement this condition for p-level distributions. On the other hand, if we discretize the original thermal distribution into only one level ($p=1$), the condition $T\approx T_F/40$ is easily implemented and allows us to uniquely close all the equations and calculate the relaxed distribution function.[13] We find that although the core distribution is not well described by a $p=1$ system, the halo part of the distribution is found to be quite accurate. This allows us to fix the value of χ. Using this χ, we can now improve the description of the core region by including additional levels into the discretization procedure. To compare the predictions of the theory with the simulations, we calculated the number of particles in the interval $[r,r+dr]$, $N(r)dr=2\pi N_b r dr\int d^2\mathbf{v}f(\mathbf{r},\mathbf{v})$, and the number of particles with velocities between v and $v+dv$, $N(v)dv=2\pi N_b v dv\int d^2\mathbf{r}f(\mathbf{r},\mathbf{v})$, for various initial conditions. The simulations are based on the Vlasov dynamics in which particles interact with the mean-field potential. This avoids the collisional effects present in finite size systems but which must vanish for one component plasmas in thermodynamic limit. The simulation code uses the Gauss law to calculate the mean electric field felt by each particle.[7] For axisymmetric beams studied in this work, this proves to be very efficient since the electric field at a radial coordinate r is determined simply by counting the total number of particles with coordinates smaller than r. Simulations were performed with 20 000 particles. As can be seen from the Figs. 1 and 2, the agreement between the theory and the simulations is excel-

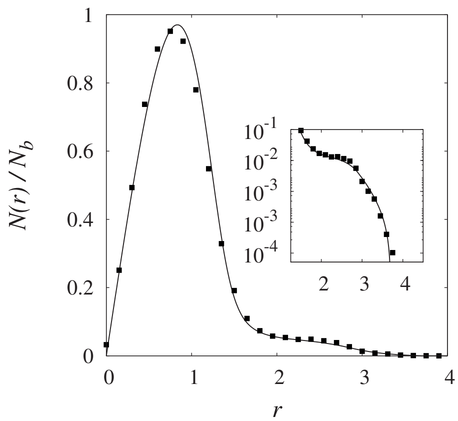

FIG. 1. The relaxed particle density of an initially thermal beam with scaled perveance $K^{*}=1$ and mismatch of 75% ($\mu=1.75$). The points are the results of the simulations, and the solid line is the prediction of the theory. Inset shows the exponential decay of the halo close to one particle resonance energy.

lent. In the figures, the distances are measured in units of $\sqrt{\varepsilon_0/\Omega_L}$, and the velocities are in units of $\sqrt{\varepsilon_0\Omega_L}$. We have also defined a scaled perveance $K^{*}\equiv K/\Omega_L\varepsilon_0$ and the mismatch parameter, $\mu\equiv r_m/r_0$, which measures the deviation of the initial beam radius from the corresponding virial value $r_0=\sqrt{K+4\sigma^2}/\Omega_L$, for which the oscillations of the beam envelope are very small. In particular, we find that the discretization of the Gaussian by only four levels already provides us with an almost perfect description of the core region.

As a direct application of the theory developed above, we calculate the emittance growth of an originally thermal beam. This quantity is of fundamental importance for the design and development of high-intensity space-charge dominated beams.[16] The calculations are performed for beams of varying scaled perveance K^{*} and mismatch parameter μ. The results are compared with the molecular dynamics simulations. Once again, an excellent agreement is found between the theory and the simulations (Fig. 3).

To conclude, we have presented a theory that allows us to calculate the density and the velocity distributions of an initially thermal beam after it relaxes to the final stationary state. Comparing to the simulations, the theory is found to be extremely accurate without any adjustable parameters. In particular, it can be used to calculate the emittance growth and the fraction of particles, which will evaporate as the beam evolves to its final stationary state.

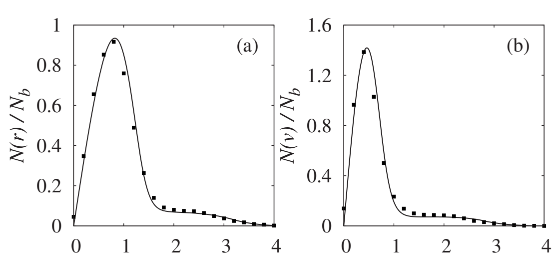

FIG. 2. The density and velocity distributions: solid line is the prediction of the theory, and points are the results of the molecular dynamics simulation. The scaled perveance is $K^{*}=1$ and mismatch is 50% ($\mu=1.50$).

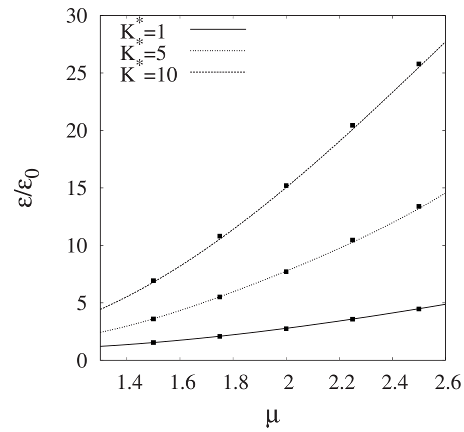

FIG. 3. The final emittance of an initially thermal beam. The points are the result of molecular dynamics simulation, and the lines are the predictions of the theory.

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