# Image effects on the transport of intense nonaxisymmetric charged beams

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The effect of conducting pipes on the equilibrium of intense nonaxisymmetric continuous beams of charged particles is investigated. For a cylindrical pipe and an elliptical beam, we obtain an exact closed form analytical expression for the electrostatic potential. Using a variational principle, we then explore the distortions that equilibrium beams suffer due to the conducting channel. Finally, we present an exact proof that despite the nonlinear forces acting on beams of arbitrary cross section inside conducting pipes of arbitrary shape, the density of these beams remains homogeneous and their cross sectional area remains the same as the one in free space. © *2007 American Institute of Physics*. [DOI: 10.1063/1.2827580]

In many applications where intense charged particle beams are employed, it is desirable to have a nonaxisymmetric beam distribution. This is particularly the case for vacuum electronic devices for which the use of ribbon electron beams allows for the transport of a large amount of current at low space-charge forces.[1–5] This is accomplished by distributing the current along a *large* beam width, while keeping its height small enough to go across the small aperture sizes required by the modern high-frequency devices. Although the focusing field configuration necessary to transport nonaxisymmetric beams is naturally more complex than that of axisymmetric beams, a number of different schemes has been proposed in the past few decades.[1–4] They are all based on anisotropic periodic focusing fields which generate stronger focusing in one transverse direction than in another. Both stability studies[3,4] and experiments with low-intensity beams[2] show the viability of the transport of such beams. More recently, Zhou *et al.*[5] demonstrated the existence of a class of equilibrium solutions for the transport of intense, cold, nonaxisymmetric beams with variable aspect ratios through periodic magnetic focusing fields in free space. In equilibrium, the beam is uniformly distributed along an ellipse whose angle and semiaxis radii undergo some small amplitude rapid oscillations around stationary, average, values. A very important question then arises: how will the equilibrium beam configuration be affected by the presence of a conducting wall of a vacuum chamber? While round beams will not be disturbed by the coaxial circular conducting pipe—such pipe is an equipotential surface—elliptical beams may be strongly modified in such environment. In particular, one might expect that the charge induced on a grounded conducting wall will strongly attract the beam particles, causing a significant modification of the beam equilibrium shape and its homogeneity.

In this paper, we study in detail the effects of a conducting pipe on the equilibrium of intense nonaxisymmetric beams. We start by analyzing the image effects of a cylindrical conducting pipe on a continuous beam of an elliptical form. In contrast to previous studies, which employed multipole expansions,[6,7] we derive an explicit analytical expression for the self-field potential of the beam inside the pipe. By means of a variational method, we then calculate the relaxed equilibrium shape after a free space beam enters into a conducting pipe. Finally, we prove that despite the nonlinear forces produced by the conducting walls of *arbitrary cross section*, intense beams preserve their homogeneity and conserve the cross sectional area.

We consider an intense, cold, unbunched beam propagating with an axial velocity $v_z$ through a magnetic focusing channel enclosed by a conducting pipe, both aligned with the $z$ axis. The focusing force is assumed to be linear and anisotropic along the transverse directions. In the smooth-beam approximation, where the fast oscillations due to the periodic focusing field are averaged out, the dynamics of any beam particle is governed by[3,8]

$$\mathbf{r}'' + \nabla_\perp U_B + \nabla_\perp \psi = 0, \tag{1}$$

where $\mathbf{r}=x\hat{\mathbf{e}}_x+y\hat{\mathbf{e}}_y$, $r=(x^2+y^2)^{1/2}$ is the radial distance from the $z$ axis, the prime denotes derivative with respect to $z$, $\nabla_\perp \equiv (\partial/\partial x)\hat{\mathbf{e}}_x+(\partial/\partial y)\hat{\mathbf{e}}_y$, $U_B=k_x x^2/2+k_y y^2/2$ is the effective confining potential due to the external field, $k_i=\xi_i^2 q^2\overline{B(z)^2}/2\gamma_b^2\beta_b^2 m^2 c^4$, $i=x,y$, $B(z)$ is the magnetic field along the $z$ axis, the bar represents average over one focusing period, $\xi_i$ are the form factors which satisfy $\xi_x+\xi_y=1$, $\beta_b=v_z/c$, $\gamma_b=(1-\beta_b^2)^{-1/2}$, $q$ and $m$ are the mass and charge of the beam particles, respectively, and $c$ is the speed of light *in vacuo*. In Eq. (1), $\psi$ is a normalized potential that incorporates both the self-electric and the self-magnetic fields and is also affected by the presence of a conducting wall. It is related to the self-scalar and self-vector potentials by $\phi^s=\beta_b^{-1}A_z^s=\gamma_b^3 m\beta_b^2 c^2\psi(\mathbf{r},s)/q$ and satisfies the Poisson equation

$$\nabla_\perp^2 \psi = -\frac{2\pi K}{N_b} n_b(\mathbf{r},z), \tag{2}$$

subjected to the boundary condition $\psi=0$ at the grounded conducting wall. Here, $n_b(\mathbf{r},z)$ is the beam density profile, $N_b=$const is the number of particles per unit axial length, and $K=2q^2N_b/\gamma_b^3\beta_b^2 mc^2$ is the so-called beam perveance, which can be interpreted as a measure of the total two-dimensional beam charge.

We begin by studying how to include the effects of cylindrical conducting pipe of radius $r_w$ in the self-field potential. Let us consider an arbitrary transverse particle distribution of total charge $K$ contained in the region $r<r_w$. In the

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free space, the electrostatic potential satisfies the Laplace equation $\nabla_\perp^2 \psi^{\text{free}}=0$ for $r>r_w$. We assume that $\psi^{\text{free}}$ is known. One can then verify that the function $\psi^{\text{free}}(r_w^2/r,\theta)$ also satisfies the Laplace equation. This function, however, has a singular point of charge $K$ at $r=0$, as well as other singularities—corresponding to the image charges—located at $r>r_w$. We then note that the combination

$$\psi(\mathbf{r}) = \psi^{\text{free}}(r,\theta) - \psi^{\text{free}}(r_w^2/r,\theta) - K\log(r/r_w) \tag{3}$$

satisfies the Poisson equation with the original charge density and vanishes at $r=r_w$. This means that $r_w$ corresponds to the location of a grounded conducting wall. Note that although we are only interested in the electrostatic potential $\psi(\mathbf{r})$ inside the pipe, to obtain it, it is necessary to know $\psi^{\text{free}}(\mathbf{r})$ over *the whole* space.

We employ the above result to investigate the transport of an elliptically symmetric inhomogeneous beam propagating inside a conducting pipe of radius $r_w$. We assume a parabolic density profile of the form

$$n(x,y) = \frac{N_b}{\pi ab}\left[1 + \chi - 2\chi\left(\frac{x^2}{a^2} + \frac{y^2}{b^2}\right)\right] \tag{4}$$

inside the beam core, $(x/a)^2+(y/b)^2 \leq 1$, where $\chi$ is the inhomogeneity parameter $(-1 \leq \chi \leq 1)$, and $a$ and $b$ are the elliptical semiaxis radii. In the absence of a conducting wall, a free space analytic solution to the Poisson equation $\psi^{\text{free}}(\mathbf{r})$ is known inside the beam core.[8,9] Solving the Laplace equation in elliptical coordinates, imposing the continuity of the electric field at the beam boundary, and taking advantage of complex variable properties, we can also obtain a closed-form expression for the self-field outside the beam core

$$\psi^{\text{free}}(\mathbf{r}) = Re\left[\frac{3+\chi}{6}\mathcal{H} + \frac{\chi}{12}\mathcal{H}^2 - \text{arccosh}\left(\frac{\zeta}{c}\right)\right], \tag{5}$$

where $\zeta=x+iy$, $\mathcal{H}=1-2(\zeta/c)^2[1-\sqrt{1-(c/\zeta)^2}]$, and $c=\sqrt{a^2-b^2}$. In the limit of $c\to 0$, $\psi^{\text{free}}\to\log(2r/c)$, as expected. Substituting $\psi^{\text{free}}(\mathbf{r})$ in Eq. (3), we find an exact analytic expression for the electrostatic potential of an inhomogeneous elliptic beam inside a grounded conducting pipe. The electric field generated by such a confined beam can be then calculated and used to study, for instance, the dynamics of test particles.[10] Here, we will use the analytical expressions and a variational calculation to investigate the effects of conductors on the equilibrium beam profile.

For a given set of transport channel parameters $k_x$, $k_y$, and $r_w$, we compute the total beam energy per particle

$$\mathcal{E}_T = \frac{1}{N_b}\int\left[\frac{\psi(\mathbf{r})}{2} + U_B(\mathbf{r})\right]n(\mathbf{r})d^2r, \tag{6}$$

as a function of the beam parameters $a$, $b$, and $\chi$. By minimizing $\mathcal{E}_T(a,b,\chi)$ with respect to these parameters, we then determine the equilibrium beam shape inside the pipe. Note that in this variational calculation, the beam shape is constrained to be elliptical, while its inhomogeneity is allowed to vary parabolically. For $r_w\to\infty$, the minimization can be performed explicitly to find that the equilibrium corresponds to a uniform beam $(\chi=0)$ with $a=a_0\equiv\sqrt{2Kk_y/[k_x(k_x+k_y)]}$ and $b=b_0\equiv\sqrt{2Kk_x/[k_y(k_x+k_y)]}$. The free space radii $a_0$ and $b_0$ can then be used to characterize the focusing field intensities $k_x$ and $k_y$. In Fig. 1, we show the results obtained for $r_w/a_0=1.2$ and varying values of $b_0/a_0$. For later comparison with the full $N$-particle simulations, we present in panel (a) the equilibrium *effective* semiaxis $a_{\text{rms}}\equiv 2\langle x^2\rangle^{1/2}=a(1-\chi/3)^{1/2}$ and $b_{\text{rms}}\equiv 2\langle y^2\rangle^{1/2}=b(1-\chi/3)^{1/2}$, where $\langle\cdots\rangle$ stands for the average over the beam distribution. The figure confirms that for nearly axisymmetric beams with $b_0/a_0\approx 1$, wall effects are negligible and $a_{\text{rms}}/a_0$ and $b_{\text{rms}}/b_0$ are close to unity. As the focusing channel becomes more anisotropic with $b_0/a_0<1$, wall effects become important, always acting to further intensify the beam anisotropy. The figure also reveals that the dependence of the equilibrium beam sizes on the focusing field anisotropy $b_0/a_0$ is nonmonotonic, being more pronounced for aspect ratios close to $b_0/a_0=0.5$. This feature was verified for different wall positions $r_w$. In all the cases, the inhomogeneity parameter $\chi$ was found to be small, on the order of a few percent. Full $N$-particle simulations were performed to compare with the results of the variational calculations. In the simulations, a large number $N=20\,000$ of macroparticles evolved according to Eq. (1). The influence of a grounded conducting wall was taken into account using the image charges.[11] The particles were launched in an arbitrary configuration and attained equilibrium state through a slow damping in their dynamics. The simulation results obtained for $a_{\text{rms}}$ and $b_{\text{rms}}$ are represented by the symbols in Fig. 1(a), showing a very good agreement with the predictions of the variational calculation. An intriguing property of the variational calculation is that the effective area occupied by the beam in the presence of a wall is *exactly* the same as that of the free beam, i.e., $a_{\text{rms}}b_{\text{rms}}=a_0b_0$ [see Fig. 1(a)]. This feature in a variational calculation suggests a hidden symmetry which, however, cannot be seen at the level of variational equations because of their complexity.

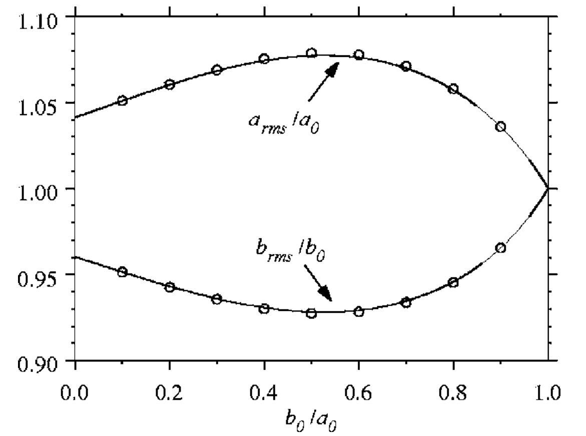


FIG. 1. Equilibrium beam dimensions as a function of the aspect ratio $b_0/a_0$ for $r_w/a_0=1.2$. The curves correspond to the results of the variational calculation, whereas the symbols correspond to those obtained from the full $N$-particle simulations.

To uncover the hidden symmetry, we now appeal to the following mathematical adiabatic construction. Consider an equilibrium beam in the absence of a wall. Because each particle is in equilibrium, the force balance Eq. (1) requires that

$$\nabla_\perp U_B + \nabla_\perp \psi = 0. \tag{7}$$

If we picture the beam as a continuous charge distribution, this equation holds at *all* points inside the the beam distribution. We now suppose that a conducting boundary of an arbitrary shape initially at infinity is adiabatically approaching the beam (shrinking toward it). The equilibrium beam charge distribution deforms but since the wall motion is adiabatic, the equilibrium condition Eq. (7) remains unaltered. Operating with $\nabla_\perp$ on Eq. (7), we obtain

$$\nabla_\perp^2 U_B(\mathbf{r}) - \frac{2\pi K}{N_b} n_b(\mathbf{r}, r_w) = 0, \tag{8}$$

where $r_w$ now stands for a typical distance from the focusing channel axis to the conductor and use has been made of the Poisson equation [Eq. (2)]. Note that while $n_b$ is a function of $r_w$, $U_B(\mathbf{r})$ only depends on the external focusing field. As the wall undergoes a small displacement $\delta r_w$ with $r_w \to r_w + \delta r_w$, the beam particle positions will be modified, $\mathbf{r} \to \mathbf{r} + \delta\mathbf{r}$. Using this in Eq. (8) and expanding to linear order in $\delta r_w$ and $\delta\mathbf{r}$, we obtain

$$\left[\frac{\partial}{\partial r_w} + \mathbf{v} \cdot \nabla_\perp\right] n_b(\mathbf{r}, r_w) = \frac{N_b}{2\pi K} \mathbf{v} \cdot \nabla_\perp (\nabla_\perp^2 U_B), \tag{9}$$

where $\mathbf{v} \equiv \delta\mathbf{r} / \delta r_w$. Thus, as long as $U_B$ is a quadratic function of $\mathbf{r}$—the focusing force is linear—the right-hand side of Eq. (8) vanishes and the total (convective) derivative of $n_b(\mathbf{r})$ with respect to variations in $r_w$ is zero. Since in the absence of the wall the density is uniform, vanishing of the convective derivative implies that it remains so for any $r_w$, preserving the beam cross sectional area. Therefore, the beam density and cross-sectional area are adiabatic invariants. This places a stringent constraint on the effects that conductors can exert on beam equilibria. To illustrate this, in Fig. 2, we show the large distortion of the equilibrium charge distribution caused by a nearby conducting plate. Nevertheless, in agreement with Eq. (9), the particle density inside the beam remains uniform and constant and the cross sectional area of the beam is unchanged.

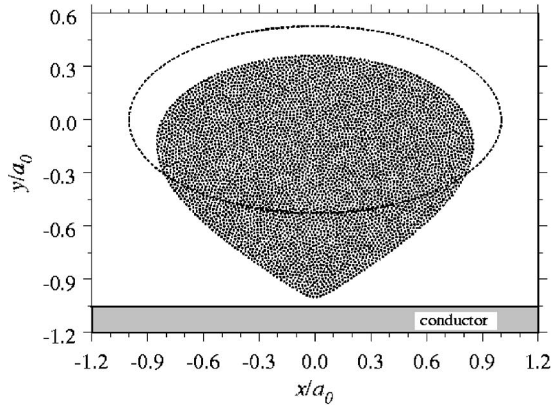


FIG. 2. Equilibrium distribution in a focusing field with $b_0/a_0 = 1/1.9$ near a conducting plate obtained using a full $N$-particle simulation. The dashed ellipse shows the equilibrium beam border in the absence of the conducting plate.

To conclude, we have investigated the effects of a conducting pipe on the equilibrium of intense nonaxisymmetric beams. First, we analyzed the image effects of a cylindrical conducting pipe on a continuous beam with elliptical symmetry and derived an exact expression for the electrostatic potential. Using a variational method, we then calculated the equilibrium beam shape and its charge distribution. It was found that the presence of a pipe does not alter the effective beam cross sectional area. This suggested that the variational equations possess an underlying hidden symmetry. Using an adiabatic construction, we were able to prove that despite the nonlinear forces exerted by the induced charges, the intense particle beams preserve a uniform equilibrium density, as long as the focusing forces are linear. Furthermore, the cross sectional area of the beam remains the same as in the absence of a conductor. These findings should have important practical implications for the design of intense beam transport channels.

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