Electromagnetic braking: A simple quantitative model

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A calculation is presented that quantitatively accounts for the terminal velocity of a cylindrical magnet falling through a long copper or aluminum pipe. The experiment and the theory are a dramatic illustration of Faraday's and Lenz's laws. © *2006 American Association of Physics Teachers.*
[DOI: 10.1119/1.2203645]

I. INTRODUCTION

Take a long metal pipe made of a nonferromagnetic material, such as copper or aluminum, hold it vertically with respect to the ground, and place a small magnet at its top opening. When the magnet is released, will it fall faster, slower, or at the same rate as a nonmagnetic object of the same mass and shape? The answer is a dramatic demonstration of Lenz's law, which amazes students and professors alike. The magnet takes much more time to reach the ground than a nonmagnetic object. For a copper pipe of length L =1.7 m, the magnet takes more than 20 s to fall through the pipe, while a nonmagnetic object covers the same distance in less than a second! When various magnets are stuck together and then dropped through the pipe, the time of passage varies nonmonotonically with the number of magnets. This behavior is contrary to the prediction of the point dipole approximation which is commonly used to justify the slow speed of the falling magnets.[1,2] The easy availability of powerful rare-earth magnets make this demonstration a must in any electricity and magnetism course.[1–4]

In this paper, we go beyond a qualitative discussion of the dynamics of the falling magnet and present a theory that quantitatively accounts for all the experimental observations. The theory is accessible to students with only an intermediate exposure to Maxwell's equations in their integral form.

II. THEORY

Consider a long vertical copper pipe of internal radius a and wall thickness w. A cylindrical magnet of cross-sectional radius r, height d, and mass m is held over its top aperture as shown in Fig. 1.

It is convenient to imagine that the pipe is uniformly subdivided into parallel rings of height ℓ. When the magnet is released, the magnetic flux in each of the rings begins to change. In accordance with Faraday's law, this flux change induces an electromotive force and an electric current inside the ring. The magnitude of the current depends on the distance of each ring from the falling magnet and on the magnet's speed. The law of Biot-Savart states that an electric current produces a magnetic field, which according to Lenz's law, opposes the action that induced it, that is, the motion of the magnet. Thus, if the magnet is moving away from a given ring, the induced field will attract it, while if it is moving toward a ring, the induced field will repel it. The net force on the magnet can be calculated by summing the magnetic interactions with all the rings. The electromagnetic force is an increasing function of the velocity and will decelerate the falling magnet. When the velocity reaches the value at which the magnetic force completely compensates gravity,

the acceleration will be zero and the magnet will fall at a constant terminal speed v. For a sufficiently strong magnet, the terminal speed is reached very quickly.

It is interesting to consider the motion of the magnet from the point of view of energy conservation. When an object falls freely in a gravitational field, its potential energy is converted into kinetic energy. For a falling magnet inside a copper pipe, the situation is quite different. Because the magnet moves at a constant velocity, its kinetic energy does not change and the gravitational potential energy must be transformed into something else, that is, the ohmic heating of the copper pipe. The gravitational energy is, therefore, dissipated by the eddy currents induced inside the pipe. In the steady state, the rate at which the magnet loses its gravitational energy is equal to the rate of the energy dissipation by the ohmic resistance,

$$mgv = \sum_z I(z)^2 R, \tag{1}$$

where v is the speed of the falling magnet, z is the coordinate along the pipe length, $I(z)$ is the current induced in the ring located at position z, and R is the resistance of the ring. Because the time scales associated with the speed of the falling magnet are much larger than the ones associated with the decay of eddy currents,[1,5] almost all the variation in the electric current through a given ring results from the changing flux due to the magnet's motion. The self-induction effects can thus be safely ignored.

Our goal is to calculate $I(z)$, the distribution of currents in each ring. To do so, we start by considering the rate of change of the magnetic flux through one ring as the magnet moves through the pipe. We first consider the functional form of the magnetic field produced by a stationary magnet. Because the magnetic permeability of copper and aluminum is very close to that of a vacuum, the magnetic field inside the pipe is almost identical to the one produced by the same magnet in vacuum. Usually, this field is approximated by that of a point dipole. This approximation is sufficient as long as we want to study the far field properties of the magnetic field. For a magnet confined to a pipe whose radius is comparable to its size, this approximation is no longer valid. Because a large portion of the energy dissipation occurs in the near field, we would have to sum all of the magnetic moments to correctly account for the field in the magnet's vicinity. This sum is impractical, and we shall adopt a different approach.

Let us suppose that the magnet has a uniform magnetization $\mathbf{M} = M\hat{\mathbf{z}}$. That is, the effective magnetic charge density $\rho_M = -\nabla \cdot \mathbf{M}$ inside the magnet is zero, and the effective magnetic surface charge density $\sigma_M = \mathbf{n} \cdot \mathbf{M}$ ($\mathbf{n}$ is the outward unit
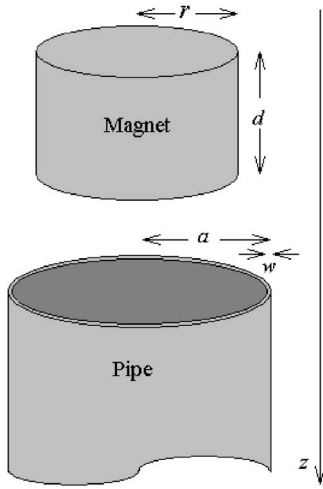
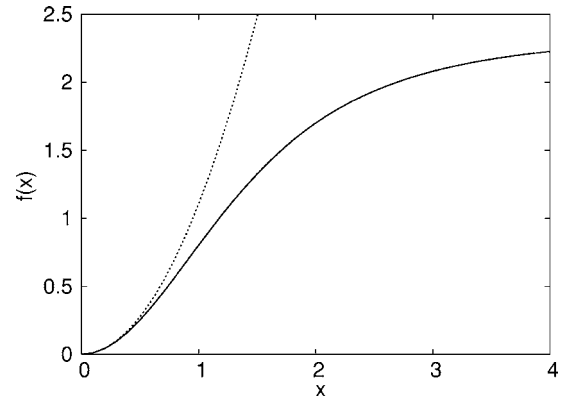Fig. 1. The magnet and pipe used in the experiment.

Fig. 2. The scaling function $f(x)$ (solid curve) and the limiting form, Eq. (9) (dotted curve). Note the strong deviation from the parabola (point dipole approximation) when $x > 1$.

normal vector) vanishes on the side of the cylinder and is $\pm M$ on the top and bottom, respectively. The flux produced by the magnet is, therefore, equivalent to the field of two uniformly charged disks, each of radius r, separated by a distance d. This geometry still does not lead to an easy calculation, because the magnetic field of a charged disk involves integration over Bessel functions. We shall, therefore, make a further approximation and replace the charged disks by point monopoles of the same net charge $q_m = \pi r^2 \sigma_M$. The flux through a ring produced by the two monopoles is given by

$$\Phi(z) = \frac{\mu_0 q_m}{2} \left[\frac{z+d}{\sqrt{(z+d)^2 + a^2}} - \frac{z}{\sqrt{z^2 + a^2}} \right], \quad (2)$$

where μ_0 is the permeability of vacuum and z is the distance from the nearest monopole, which we take to be the positively charged one, to the center of the ring. As the magnet falls, the flux through the ring changes, which results in an electromotive force given by Faraday's law,

$$\mathcal{E}(z) = -\frac{d\Phi(z)}{dt}, \quad (3)$$

and an electric current

$$I(z) = \frac{\mu_0 q_m a^2 v}{2R} \left[\frac{1}{(z^2 + a^2)^{3/2}} - \frac{1}{[(z+d)^2 + a^2]^{3/2}} \right]. \quad (4)$$

The rate of energy dissipation can be calculated by evaluating the sum on the right-hand side of Eq. (1). By taking the continuum limit, we find the power dissipated to be

$$P = \frac{\mu_0^2 q_m^2 a^4 v^2}{4R} \int_{-\infty}^{\infty} \frac{dz}{\ell} \left[\frac{1}{(z^2 + a^2)^{3/2}} - \frac{1}{[(z+d)^2 + a^2]^{3/2}} \right]^2. \quad (5)$$

Because most of the energy dissipation takes place near the magnet, we have explicitly extended the limits of integration to infinity. The resistance of each ring is $R = 2\pi a \rho/(w\ell)$, where ρ is the electrical resistivity. Equation (5) can now be rewritten as

$$P = \frac{\mu_0^2 q_m^2 v^2 w}{8 \pi \rho a^2} f\left(\frac{d}{a}\right), \quad (6)$$

where $f(x)$ is a scaling function defined as

$$f(x) = \int_{-\infty}^{\infty} dy \left[\frac{1}{(y^2+1)^{3/2}} - \frac{1}{[(y+x)^2 + 1]^{3/2}} \right]^2. \quad (7)$$

We substitute Eq. (6) into Eq. (1) and find that the terminal velocity of a falling magnet is

$$v = \frac{8 \pi m g \rho a^2}{\mu_0^2 q_m^2 w f\left(\frac{d}{a}\right)}. \quad (8)$$

The scaling function $f(x)$ is plotted in Fig. 2. For small x,

$$f(x) \approx \frac{45\pi}{128} x^2, \quad (9)$$

and the terminal velocity reduces to that of a point dipole[1,2] of moment $p = q_m d$,

$$v = \frac{1024}{45} \frac{m g \rho a^4}{\mu_0^2 p^2 w}. \quad (10)$$

From Fig. 2, we see that as soon as the length of the magnet becomes comparable to the radius of the pipe; the point dipole approximation fails. For the cylindrical magnets used in most demonstrations, the point dipole approximation limit is not applicable, and the full expression for $f(x)$ must be used in Eq. (8).

III. DEMONSTRATION AND DISCUSSION

In our demonstrations, we use a copper pipe (conductivity $\rho = 1.75 \times 10^{-8}$ Ω m) (Ref. 6) of length $L = 1.7$ m, radius $a = 7.85$ mm, and wall thickness $w = 1.9$ mm; three neodymium cylindrical magnets of mass 6 g each, radius $r = 6.35$ mm, and height $d = 6.35$ mm; a stop watch; and a teslameter. We start by dropping one magnet into the pipe and measure its time of passage. For one magnet $T = 22.9$ s. For two magnets stuck together, the time of passage increases to $T = 26.7$ s. If the point dipole approximation were valid, the time of passage would increase by a factor of 2, which is clearly not the case. (In the point dipole approximation, the time of passage

Table I. Experimental and theoretical (Eq. (13)) values of the terminal velocity.

n magnets	B (mT)	v_{exp} (10^{-2} m/s)	v_{theory} (10^{-2} m/s)
1	393	7.4	7.3
2	501	6.4	5.8
3	516	7.2	6.9

is proportional to p^2 and inversely proportional to the mass, see Eq. (10); sticking two magnets together increases both the dipole moment and the mass of the magnet by a factor of 2.) When three magnets are stuck together, the time of passage drops to $T=23.7$ s.

Because the terminal velocity is reached very quickly, a constant speed of fall is justified for the entire length of the pipe. In Table I, we present the values for the measured velocity $v_{exp}=L/T$. We next compare these measurements with the predictions of the theory. First, the value of q_m for the magnet must be obtained. To do so, we measure the magnetic field at the *center* of one of the flat surfaces of the magnet using a digital teslameter[7] (the probe of the teslameter is brought in direct contact with the surface of the magnet). Within our uniform magnetization approximation this field is produced by two parallel disks of radius r and effective magnetic surface charge density $\pm\sigma_M$, separated by distance d. On the axis of symmetry the field is easily calculated to be

$$B = \frac{\mu_0\sigma_m}{2}\frac{d}{\sqrt{d^2+r^2}}.$$ (11)

The effective magnetic charge is therefore

$$q_m = \frac{2\pi Br^2\sqrt{d^2+r^2}}{\mu_0 d}.$$ (12)

For n magnets stuck together $d \rightarrow h=nd$, and it can be checked using the values of the measured magnetic field presented in Table I, that q_m is independent of n to within experimental error, justifying our uniform magnetization approximation. We rewrite Eq. (8) in terms of the measured magnetic field for a combination of n magnets and obtain

$$v = \frac{2Mg\rho a^2 h^2}{\pi B^2 r^4 w(h^2+r^2)f\left(\dfrac{h}{a}\right)},$$ (13)

where $M=nm$. In Table I, we compare the values of the measured and the calculated terminal velocities.

Considering the complexity of the problem, the simple theory we have presented accounts reasonably well for the experimental results. In particular, the theory correctly predicts that two magnets stuck together fall slower than either one magnet or three magnets together. Therefore, for each pipe there is an optimum magnetic size that falls the slowest.

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