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Wealth redistribution with conservative exchanges

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Abstract

We present a simplified model for the exploitation of resources by interacting agents, where each agent receives a random fraction of the available resources. An extremal dynamics ensures that the poorest agent has a chance to change its economic welfare. After a long transient; the system self-organizes into a critical state that maximizes the average performance of each participant. Our model exhibits a new kind of wealth condensation, where very few extremely rich agents are stable in time and the rest stays in the middle class.

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1. Introduction

The concepts and techniques of statistical mechanics have been applied for years to a wide variety of complex extended systems, physical and otherwise, in an effort to understand the emergent properties appearing in them. In general, extended systems

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showing critical behavior do not need any fine-tuning of a parameter to be in a critical state. In an attempt to explain this behavior, Bak, Tang and Wiesenfeld introduced the concept of self-organized criticality (SOC) [1]. In the self-organized critical state, there are long-range interactions, by which each part of the system is under the influence of all the others. More precisely, this means that many of the relevant observables in the system follow a power-law or Pareto–Lévy distribution with a non-trivial exponent. In economics, Pareto distributions are known to characterize, for instance, the income distribution in the high income strata of rich societies [2].

Economics is, by far, one of the more complex extended systems. Economic development has always been considered the driving force in determining the relationships inside a society. Similar to what happens in paleontology [3], it follows a *punctuated* pattern: Wars, famines, crashes and revolutions are the most evident and extreme illustrations of these bursts of historical activity. It is then natural, if nothing else by the force of mere analogy, to look for evidences of critical behavior in economic systems. In recent years a great deal of effort has been devoted to the analysis of economic data. From stock-exchange fluctuations [4], models of production [5], size distribution of companies [6], to the appearance of money [7], and the effects of controls on the market [8], it has finally been shown that market economy exhibits properties characteristic of a critical system [9].

In this paper we focus on one particular aspect of economic processes: We aim at modelling the competition among different agents (countries, enterprises, etc.) acting in an environment where all exchanges of resources between agents take place in a conservative manner, i.e., a conservative exchange market model (CEMM) [10]. This restriction has several motivations: On the one hand, it can be argued that resources are material objects, and consequently they cannot be created or destroyed by means of exchanging them. Even though there are resources that can be created, for instance knowledge, or destroyed, say fuel in an automobile, we assume that if this situation happens it is at the expense of others, thus making the totality of available resources at that particular instant constant. On the other hand, the use of the CEMM implies that our economy is a zero-sum game, something that may seem at odds with usual economic orthodoxy. At this stage we would like to stress that we pursue a reference behavior, i.e., behavior against which a non-zero sum system could (and should) be compared; any study of wealth increase among a population of agents requires understanding the behavior of the reference (conservative) system. However, our results also hold for systems in which the total amount of wealth increases uniformly and smoothly in time (no jumps); we will come back to this point later. Last but not least, our model is parameter free.

Within this context, conservative exchange mechanisms lead to an exponential wealth distribution, where very few extremely rich agents are stable in time and the rest is just above threshold. This is so even in the absence of a lower bound for the wealth parameter [11]. Since neither cost of debt nor returns or tax on wealth are explicitly included in this model, the role played by geography is reinforced. Indeed, an a-geographic mean field solution generates a completely different wealth distribution. This globally coupled solution can be compared with the results obtained in Refs. [12,13] for stochastic multiplicative market models, and reinforces the conclusions presented in Ref. [14] concerning wealth condensation with a finite number of agents. [1]

2. Conservative exchange market model

We will model our *economy* as a one-dimensional lattice, every site of which represents an agent. Agents with closer ties to each other (geographical or otherwise) will be neighbors on the lattice. For simplicity sake we assume periodic boundary conditions. Each agent will be characterized by some wealth-parameter that represents its welfare. The exact choice of this parameter is not straightforward. For instance, when thinking of countries in the world economy, the GDP, GNP or some function of macroeconomic indicators could be a reasonable choice. In the case of companies, equity, share price or some suitable combination of them with outstanding debt are reasonable candidates. To level the field and make sure that no agent has an unfair advantage, we choose an initial configuration where the wealth is distributed randomly among agents, the wealth of each agent being between 0 and 1.

In the marketplace, all agents strive to improve their situation. In particular the poorest agent is the one feeling the strongest pressure to move up the ladder. Thus, we model this process by an extremal dynamics. At each time step, the poorest agent, i.e., the one with the minimum wealth, will take some action trying to improve its economic state. That is, it will change its production methods, borrow money, increase the percentage of sown fields or take some other measure aiming at increasing its wealth. Since the outcome of any such measure is uncertain, this outcome is simulated as a random change in the wealth parameter of this agent; for simplicity, we take the new wealth parameter to be equal to a random number extracted from a uniform distribution (between 0 and 1). From a theoretical point of view, the randomness in the outcome may be understood as the combined effect of several things, such as changing market conditions, poor understanding of these conditions by the agent, the agent not having the necessary skills to make the right changes, etc. Whatever wealth is gained (lost) by the poorest agent will be at the expense of its neighbors and we assume it to be equally divided among its two nearest neighbors. We would like to remark that, apart from conservation, we do not impose any limit on the wealth evolution, so any negative value is in principle possible, corresponding to an agent having debt rather than wealth. Since a site with negative wealth will most probably be the minimum in the near future, we expect such a site to linger only a few steps *in red*. In this simplified version of the model, default is not taken into account, that is, any agent may stay for ever in debt, albeit with a very low probability.

[1] In this paper we are defining wealth as the total amount of resources available to an agent. It is a simplification, since for example in any commercial transaction wealth is, in a sense, created.

3. Results

When performing numerical simulations we observe that, after a relatively long transient, the system arrives at a self organized critical (SOC) state with a stationary wealth distribution; typical wealth's landscapes are shown in Fig. 1.

As in other extremal dynamics models the system self-organizes into a state in which almost all agents are beyond a certain threshold, $\eta_T \approx 0.4$. Above threshold, the distribution of agents is exponential, i.e., there are exponentially few rich agents while the mass of them remain in what we call a *middle class*. Wealth redistribution is then evident. This exponential distribution is consistent with the Boltzmann–Gibbs distribution discussed in Ref. [11] for the restricted case of bounded wealth parameter and time reversal symmetry although these two restrictions are not present in our model.

Turning our attention back to Fig. 1, we also show in it the globally coupled (*mean field*) case. This corresponds to a situation in which the agents with which the exchange takes place are chosen at random and not based on geographical proximity. This mean field solution exhibits a lower threshold and, more strikingly, an almost linear behavior beyond threshold. This departs from standard extremal dynamics models where both distributions are rather uniform. Furthermore, the distribution of avalanches follows a power law with the same exponent as the Bak–Sneppen universality class.

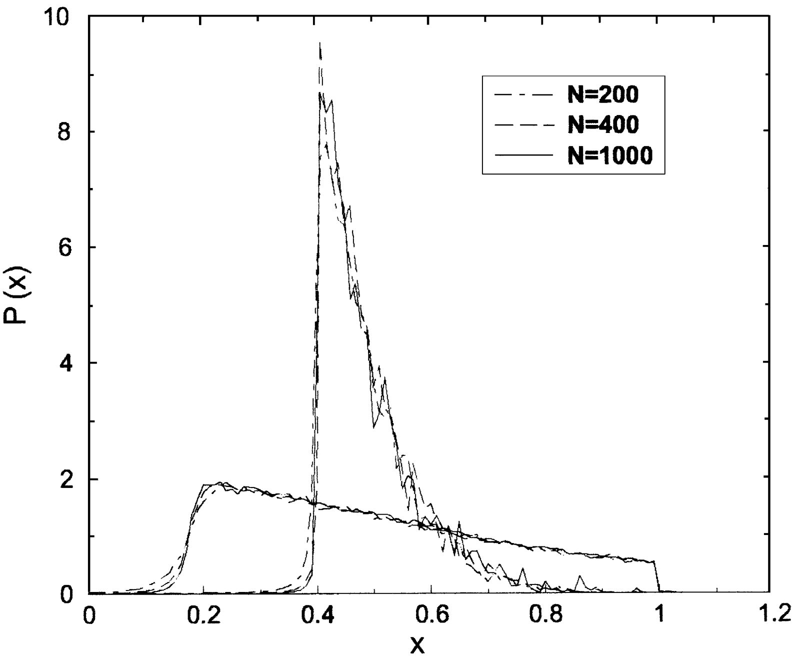

Fig. 1. Distribution of wealth. The plots show the wealth distribution $P(x)$ for the model with interactions to nearest neighbors (upper curves) and with global interactions (lower curves). In both cases we have superposed scaled data corresponding to $N = 200$, 400 and 1000. The histograms were built using 10^5 consecutive states, after a transient of 10^6 steps has elapsed.

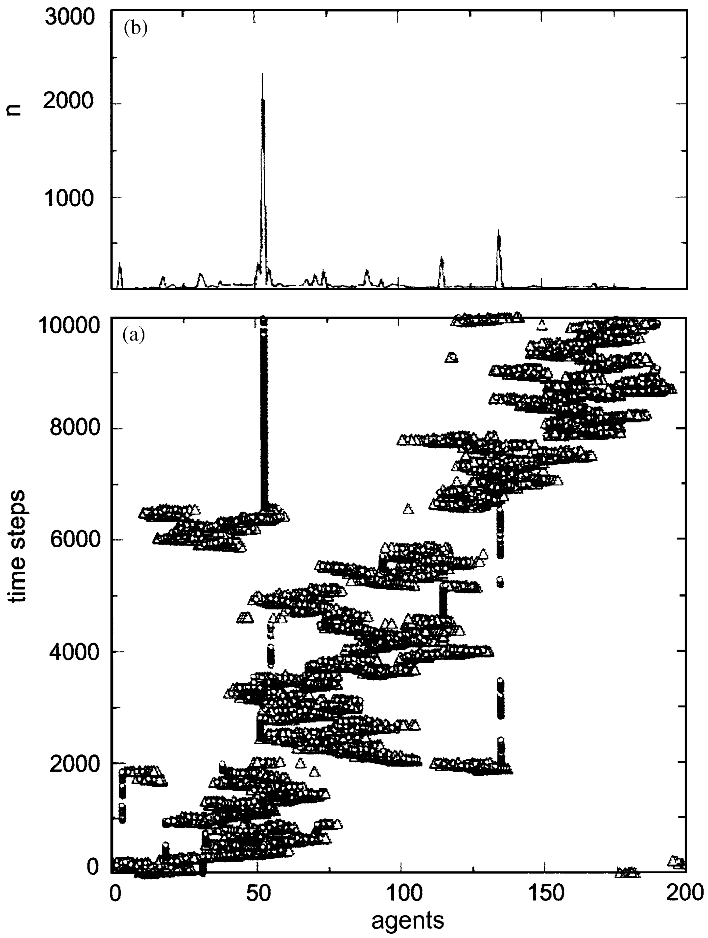

Fig. 2. (a) Time evolution of the positions of the maximum (circles) and minimum (triangles) wealth. (b) Number of time steps, n, that a given agent has had the maximum wealth. The results presented in this figure correspond to 10^4 time steps after a transient of 2×10^5 time steps.

In Fig. 2a we show the temporal evolution, in the SOC state, of the position of the system's minimum and maximum wealth. We can see that, while the site of minimum wealth is changing continuously, generating avalanches of wealth redistribution among neighbors, the richest site is stable over long periods of time. Indeed, when affected by an avalanche it can recover its status after a short time. These brief interruptions, usually produced by short-lived avalanches, are reflected as gaps in the maxima lines.

To quantify these observations, we present in Fig. 2b the statistics of the number of time-steps a site spends as absolute maximum. Clearly a few agents have spent most of the time as maxima, while the rest lurks somewhere in the middle class. We have also observed that not only the absolute maximum is stable, but also a privileged group, whose wealth is around the same value of the maximum, remains in its prosperous

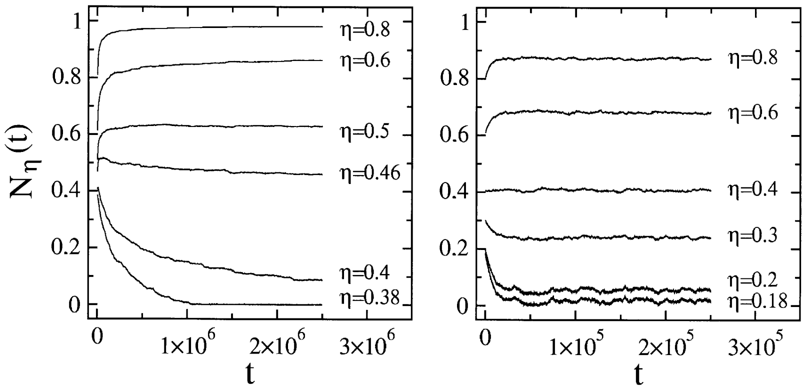

Fig. 3. The fraction of the system whose state is below $x=\eta$, $N_\eta(t)$, is displayed as a function of time, for selected values of η, as shown in the legend. In all cases $N=10000$. The left panel correspond to the local model and the right one to the global (mean-field) model.

position for quite a while. The composition and hierarchy of this privileged group is barely affected by the avalanches that produce the above mentioned gaps; a reordering takes place that is reverted after the avalanche ends. In the mean-field case, not represented in the figure, the rich sites are also stable, but over shorter periods of time.

So far we have focused our attention on the final state of the economy, that is, on the wealth distribution in the self-organized state. Let us now devote some time to discuss the process of wealth distribution/accumulation. To this end, in Fig. 3 we present the time evolution of N_η, the fraction of the agents whose wealth is $x(t) < \eta$, for different values of η. Since resources were distributed uniformly at the beginning $N_\eta(t=0) \approx \eta$. Slowly but steadily, for values of $\eta \leqslant \eta_T$ these fractions decrease, thus showing the speed of wealth redistribution in the system. As expected, the higher the value of η, the slower the progress. As can be clearly seen in the figure, all fractions with $\eta \leqslant \eta_T$ converge to zero, while for $\eta > \eta_T$ the fraction grows quickly to its asymptotic value. When the value of η is near 1, the fraction quickly converges to 1, reflecting the existence of small privileged groups.[2] We have also observed that the probability of

[2] This same argument can be carried out using the Gini Index and the Lorenz function. In brief, for any given wealth density function $f(x)$ with mean μ and distribution function $F(t)$, and a given number $\alpha \in [0,1]$, the Lorenz function $g(\alpha)$ is defined as the fraction of the total income which is attributable to the poorest 100α percentage of the society. It is easy to show that

$$g(\alpha) = \frac{1}{\mu} \int_0^{F^{-1}(\alpha)} x f(x)\,\mathrm{d}x \; . \tag{1}$$

The Gini index G can be then defined as

$$G = 2\int_0^1 (\alpha - g(\alpha))\,\mathrm{d}\alpha \; . \tag{2}$$

So normalized, $G=0$ corresponds to perfect equality and $G \to 1$ corresponds to maximum inequality. Since in our case the density function has a very simple form, this analysis is straightforward. We prefer the analysis in terms of N_η due to its intuitional appeal.

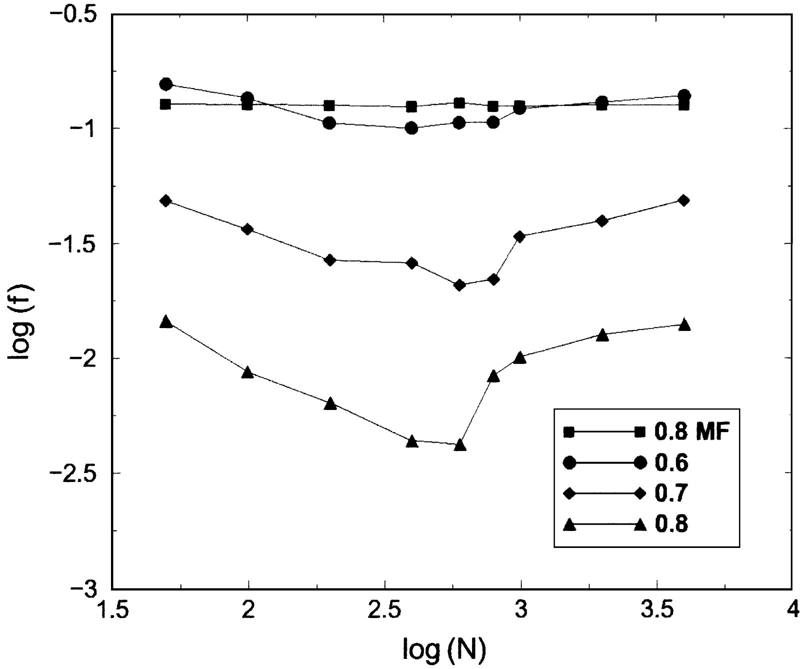

Fig. 4. The fraction of rich agents in the system as a function of N, for different values of the threshold. We have consider a transient of 10^6 time steps and performed an average over 10^5 time steps. Both nearest neighbors and mean field results are included.

one agent becoming wealthier in a time step decreases as time goes by, to finally converge to a finite value, $p \approx 0.76$. Both effects are a consequence of the fact that exchanges occur in a conservative manner.

Another interesting property of this model is that the fraction of rich agents does not depend on the size of the market. Indeed, turning our attention to Fig. 4 we can see that this fraction is essentially constant for all sizes of the economy. Moreover, the wealthy fraction in the global version in sensibly bigger than in the local case (for a given threshold), but, as it was also shown on Fig. 1 the poverty line is around 0.2, well below the "local" case.

4. Conclusions

From these results several conclusions may be drawn. First and foremost, conservative exchanges lead to an exponential wealth distribution, where the very few extremely rich agents are stable in time and the rest is just above threshold. This is tantamount to saying that the invisible hand [15] of redistribution works only among the middle class. Neither trade nor cost of debt, returns or tax on wealth are explicitly included in this model. This reinforces the role played by geography. As can be seen in Fig. 1, the a-geographic mean field solution generates a completely different wealth distribution. Indeed, the results for the globally coupled market are to be compared with those obtained in Refs. [12,13] for stochastic multiplicative market models. Moreover, they reinforce the conclusions presented in Ref. [14] regarding wealth condensation with a finite number of agents. Secondly, as Fig. 3 shows, economic progress in society is steady, even if slow.

At this point it is instructive to compare the results presented here with Pareto's law [3,16], which suggests that individual wealth follows a power law distribution for the high income part of the society. Our model exhibits exponential distribution in the local limit and an almost linear distribution in the mean field limit. As explained above, the later case corresponds to global interactions where there is a relatively great fraction of wealthy agents but, since the threshold is much lower than in the local interaction case, there is also a higher number of poorer agents. In brief, power laws seem to related to some degree of globalization in the market and favor a wide spectrum of wealth distribution, i.e., increase inequalities. In a sense, our local (not-globalized) model corresponds to a kind of feudal world, where local barons maintain their dominance for long periods of time, over a population of semi-poor agents.

All the calculations presented here have been performed with constant resources. This was done for simplicity and to study a parameter-free reference model. Indeed, had we taken a system in which total wealth increases linearly with time, the same results would apply provided one renormalizes the wealth of each agent at every time step, according to the new total wealth. In this way, the total wealth plays the role of a global scale parameter, but does no influence the dynamics of the system.

Summarizing, the model presented here provides a simple description of wealth redistribution in the early stages of human economic history, and indicates some of the possible driving forces beyond the market expansions that influenced this redistribution process. We believe these conclusions may be of interest in view of the present debate over the goods and evils of globalization.

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