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Wealth Condensation in a Multiplicative Random Asset Exchange Model

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Abstract

Random Asset Exchange (RAE) models, despite a number of simplifying assumptions, serve the purpose of establishing direct relationships between microscopic exchange mechanisms and observed economical data. In this work a conservative multiplicative RAE model is discussed in which, at each timestep, two agents "bet" for a fraction f of the poorest agent's wealth. When the poorest agent wins the bet with probability p, we show that, in a well defined region of the (p, f) phase space, there is *wealth condensation*. This means that all wealth ends up owned by only one agent, in the long run. We derive the condensation conditions analytically by two different procedures, and find results in accordance with previous numerical estimates. In the non-condensed phase, the equilibrium wealth distribution is a power law for small wealths. The associated exponent is derived analytically and it is found that it tends to -1 on the condensation interface. I turns out that wealth condensation happens also for values of p much larger than 0.5, that is under microscopic exchange rules that, apparently, favor the poor. We argue that the observed "rich get richer" effect is enhanced by the *multiplicative character* of the dynamics.

1 INTRODUCTION

In recent years, agent based systems have been put forward [1] in order to clarify the links between microscopic economic dynamics and observed wealth distributions. Empirical studies of the distribution of income for workers, companies and countries were first presented, a little more than a century ago, by Italian economist Vilfredo Pareto [2]. Pareto investigated personal income data for different European countries and found that the distributions of income and wealth follow a power law: the number $P(w)$ of agents with income larger than w is given $P(w) \propto w^{-\alpha}$, where the exponent α is called Pareto index. Later, more detailed, studies confirm and extend Pareto's early observations. For instance, data from Japan [3, 4], Italy [5], India [6], Brazil [7], the United States of America and the United Kingdom [8–10] are well fitted by log-normal or Gibbs forms for the low-to-middle income range, while a power law holds for the high-income end.

Random Asset Exchange (RAE) models consider an ensemble of economic agents exchanging a quantity called "wealth", according to pre-specified deterministic or stochastic rules. The precise meaning of wealth depends on the system that one has in mind. For instance, when thinking of countries in the world economy, the GNP (Gross National Product), or some function of macroeconomics indicators, could be a reasonable choice. In the case discussed by Dragulescu and Yakovenko [8] the agent's wealth may be interpreted as the amount of money a person has available for trading.

One of the original motivations for the introduction of agent-based systems was the need to explain the unevenness in wealth distribution observed in modern society [1]. In many theoretical models it was found that *condensation*, i. e. the concentration of all the available wealth in just one or a few agents, may happen. It is thus interesting to explore this phenomenon, and its links with the microscopic interaction rules. Rule modifications intended to produce more even distributions include, for example, increasing the probability of favoring the poorer agent in a transaction [11], introducing a cut-off that separates interactions between agents by their wealth [12, 13], or introducing "altruistic behavior" [14].

In the following we will concentrate on a Yard-Sale [15] model modified to include a poor-favoring factor [11]. Our main result is the analytical prediction of the conditions under which condensation will happen, in Section 3.3. We also calculate the small-wealth tail of the wealth distribution, in Section 4.

2 THE MODEL

The model we set to discuss is simple. A population of agents $i = 1, \ldots, N$, each possessing a wealth $w_i^t > 0$ at time t, interacts in discrete steps according to the following wealth-conserving rules: At each timestep, a pair of agents (i, j) is chosen at random, and an exchange interaction takes place between them. This interaction can be described as a *bet* for a fixed fraction f of the wealth of poorest among these two agents. Thus the amount of wealth at stake is $f \times \texttt{Min}(w_i, w_j)$. The poorest agent has a probability p to win the bet.

This dynamics can be argued to model wealth-conserving *trading* among agents. The parameter f determines the proportion of their wealth that the agents risk in each transaction, so that $\beta = (1 - f)$ is a measure of the "saving propensity" [16] defined by previous workers.

The interaction under consideration in this work, defined by the stochastic exchange of a finite fraction f of the *poorest* agent's wealth, has been termed *yard-sale* [1] exchange by Hayes [15], who discussed the case in which each agent wins with probability $1/2$. Our model is slightly more general since we assume the poorest agent to win with probability p.

Large values of p give less wealthy agents an increased chance to recover wealth through successful bets, and can represent the effect of a stronger bargaining will on their side, social conditions that favor the poor, or the fact that richer agents are more willing to pay high prices when purchasing goods. While large values of p will tend to distribute wealth homogeneously, small values of p are expected to produce concentration of wealth.

This model has been introduced recently by Iglesias et al. [11] (IGAV). In the IGAV study, $\beta = 1 - f$ is the "risk aversion" or "saving propensity" of the agent. A tentative phase-diagram was found numerically by IGAV in the $\beta - p$ space. IGAV found numerically that for any value of f there is a critical value $p^*(f)$ of p, below which all wealth concentrates in the hands of one agent in the long run. We will call this region the *condensed phase* [2]. If on the other hand, the probability p for the poor to win a bet is larger than $p^*(f)$, a stable non-trivial distribution of

[1] An alternative model assumes that the "bet" is done for a fixed fraction of the *loser's wealth*. This rule clearly disfavors agents with larger wealth, and thus has been termed *theft-and-fraud* exchange [15].

[2] Bouchaud and Mezard [17] discussed wealth condensation recently, but notice that in our case condensation means *one takes all*.

wealth exists in equilibrium. We call this region the *stable* or *non-condensed* phase. These two regimes are separated by a well defined interface $p^*(f)$ that we call the condensation interface. One surprising result of this simple trading model is the fact that there is always an f-dependent region $0.5 < p < p^*(f)$ where wealth condensation occurs even if $p > 0.5$, that is, under conditions that, a priori, seem to favor the poor. We will later see that the apparent advantage of poorer agents is in fact not such, and derive the conditions for a stable wealth distribution to exist.

3 POOREST AGENT ANALYSIS

Consider the case of an agent i who has became so poor in relative terms that, in almost every subsequent interaction with another member of the population, i will be the poorest. Thus it is always a fraction f of i's *own* wealth w_i, that is at stake in each "bet". This agent's wealth evolves according to the following multiplicative stochastic process [18]:

$$w_i^{t+1} = w_i^t \; \mu_t, \tag{3.1}$$

where μ_t is a random variable taking the values $(1+f)$ and $(1-f)$, respectively with probabilities p and $(1-p)$. Therefore after T interactions, k of which are favorable,

$$w_i^T = w_i^0 \; (1+f)^k (1-f)^{T-k}. \tag{3.2}$$

Obviously k, the number of bets won, is a random variable with binomial distribution

$$P(k,T) = \binom{T}{k} p^k (1-p)^{T-k}. \tag{3.3}$$

3.1 Expectation value of the poorest agent's wealth

The expectation value $< w_i^T >$ after T bets is easily calculated using (3.2) and (3.3), and turns out to be

$$< w_i^T >= w_i^0 \, (1 + (2p-1)f)^T \,. \tag{3.4}$$

Clearly $< w_i^T >$ *grows* in time if $p > 0.5$. This should not be taken to imply that $p > 1/2$ ensures that poor agents are able to restore their wealth. This would be in strong contradiction with the observed fact [11] that, for values of p not too large but still larger than $1/2$, all wealth ends up, with probability one, in the hands of one agent.
The solution of this puzzle comes with the realization that Eq. (3.4), although correct, does not describe the *typical* behavior of the poorest agent's wealth. The typical value of a random variable is given not by its *expectation value* but by the *most probable value* (MPV) of its distribution. Normally (i.e. for additive processes) these two estimators are similar. However for certain multiplicative processes [18] of which Eq. (3.2) is an example, it is well known that the expectation and most probable values differ greatly (i.e. exponentially in T). Thus one must resort to the MPV in order to get information about the typical outcome.

3.2 Most probable value of the poorest agent's wealth

The MPV of a random variable is the value which maximizes the probability. The binomial probability (3.3) is maximized for $k =< k >= pT$. Therefore

$$MPV(w_i) = w_i^0 \; (1+f)^{pT} (1-f)^{qT} = w_i^0 \; \theta^T, \tag{3.5}$$

where $\theta = (1+f)^p (1-f)^{1-p}$.

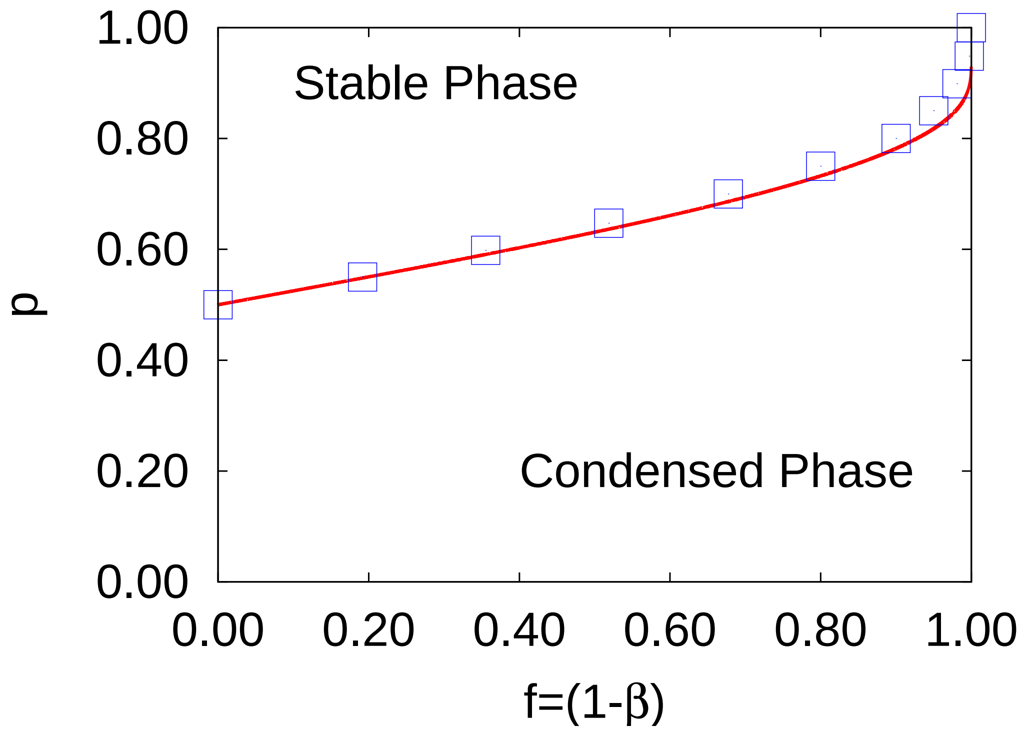

Fig. 1. If the probability p for the poor to win each bet is larger than $p^*(f)$ given by Eq. (3.6) (thick line), in equilibrium there is a non-trivial distribution of wealth $P(w)$ satisfying (4.1). For small wealths, $P(w)$ behaves as a power-law with an exponent given by (4.2). If $p < p^*$ in the long run all wealth ends up owned by just one agent. This we call *condensation* of wealth. The squares indicate our numerical estimate for the location of the condensation interface, obtained on systems of 10^5 agents after 10^8 trading events. The estimated errors are comparable to the symbol size.

3.3 Condensation interface

Condensation will occur if the poorest individual's wealth *decreases* with time, i.e. if $\theta < 1$. Equating θ to one, the critical probability

$$p^*(f) = \frac{\log(\frac{1}{1-f})}{\log(\frac{1+f}{1-f})}, \tag{3.6}$$

is obtained, that divides the condensed phase ($p < p^*(f)$) from the non-condensed phase ($p > p^*(f)$).

4 MASTER EQUATION

Following the approach of Ispolatov, Krapivsky and Redner [19] one can derive a master equation for the wealth-distribution $P(w)$. For reasons of space, we only quote here some results, leaving the details for a later publication.

In equilibrium, $P(w)$ is found to satisfy

$$\begin{aligned} P(x) &= \frac{p}{1+f}P(\frac{x}{1+f})\mathcal{Q}^{>}(\frac{x}{1+f}) + \frac{1-p}{1-f}P(\frac{x}{1-f})\mathcal{Q}^{>}(\frac{x}{1-f}) + \\ &+ (1-p)\int_0^{\frac{x}{1+f}} dyP(y)P(x-fy) + p\int_0^{\frac{x}{1-f}} dyP(y)P(x+fy). \end{aligned} \tag{4.1}$$

were $\mathcal{Q}^{>}(x) = \int_x^\infty P(y)dy$. We could not find a general solution for this integral equation. However in the special case $p = (1+f)/2$ one verifies easily that $P(x) = e^{-x}$ is an exact solution of (4.1).

On the other hand a small-x expansion of (4.1) is enough to derive the condensation interface, as follows. Assume that $P(x) \sim c\, x^{\rho-1}$ for $x \to 0$. Plugging this expression into (4.1), neglecting the last two integrals (as they contribute higher-order terms in x), and approximating $\mathcal{Q}^{>}(x) \approx 1$, one finds that ρ must satisfy

$$\frac{p}{(1+f)^\rho} + \frac{1-p}{(1-f)^\rho} = 1. \tag{4.2}$$

For small ϵ, the fraction $R^{<}(\epsilon)$ of agents whose wealth is smaller than ϵ is $R^{<}(\epsilon) = \int_0^\epsilon P(x)dx \approx \frac{c}{\rho}\,\epsilon^\rho$. Noticing that $R^{<}$ diverges when $\rho \to 0^+$, one concludes that the condition $\rho \to 0$ defines the location of the condensation interface. When $\rho \to 0$ Eq. (4.2) reads, to zeroth-order:

$$p^* \log(1+f) + (1-p^*)\log(1-f) = 0, \tag{4.3}$$

defining the location of the condensation interface $p^*(f)$, in accordance with Eq. (3.6).

5 DISCUSSION

This mechanism of wealth-condensation is interesting. One could be inclined to think that, if $p > 0.5$, then the poor are favored and no condensation should occur. This would certainly hold true if the "trading" mechanism were *additive*, e. g. if agents bet for a fixed amount of money. However, the fact that the poorest agents' dynamics is *multiplicative* changes the situation radically [15]. While for an additive random walk $p = 0.5$ is an equilibrium point that typically does not increase or decrease the player's wealth, for the multiplicative walk given by (3.1), $p = 0.5$ is a strongly unfavorable situation since, after T bets, his wealth will, typically, be multiplied by $(1-f^2)^{T/2} << 1$.

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