UNIVERSIDADE FEDERAL DO RIO GRANDE DO SUL
INSTITUTO DE FÍSICA

Lista n.º 2 - FIS01051 – Tópicos Avançados em Física II: Introdução à Teoria de Campos
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> *What gets us into trouble is not what we don't know.*
> *It's what we know for sure that just ain't so.*
>
> Mark Twain

1. Considere a densidade lagrangiana do campo escalar

$$\mathcal{L} = \frac{1}{2}\left[\partial_\alpha \phi(\underset{\sim}{x})\,\partial^\alpha \phi(\underset{\sim}{x}) - \mu^2\phi^2(\underset{\sim}{x})\right]$$

onde $\phi(\underset{\sim}{x}) = \phi(\vec{x},t)$. Obtenha a equação de movimento, o conjugado do campo e a densidade hamiltoniana

$$\begin{aligned}(\Box + \mu^2)\,\phi(\underset{\sim}{x}) &= 0\\ \pi\,(\underset{\sim}{x}) &= \frac{1}{c^2}\,\dot{\phi}(\underset{\sim}{x})\\ \mathcal{H}(\underset{\sim}{x}) &= \frac{1}{2}\left[c^2\pi^2(\underset{\sim}{x}) + [\vec{\nabla}\phi(\underset{\sim}{x})]^2 + \mu^2\phi^2(\underset{\sim}{x})\right].\end{aligned}$$

2. Considere o Hamiltoniano $H = \int d\vec{x}\,\mathcal{H}$, obtido a partir da densidade hamiltoniana do exercício anterior e as relações de comutação abaixo

$$\begin{aligned}&[\phi(\vec{x},t),\pi(\vec{y},t)] = i\,\hbar\,\delta(\vec{x}-\vec{y})\\ &[\phi(\vec{x},t),\phi(\vec{y},t)] = [\pi(\vec{x},t),\pi(\vec{y},t)] = 0\,.\end{aligned}$$

Mostre que

$$[H,\phi(\underset{\sim}{x})] = -i\,\hbar\,c^2\,\pi(\underset{\sim}{x}) \quad , \quad [H,\pi(\underset{\sim}{x})] = i\,\hbar(\mu^2 - \vec{\nabla}^2\,)\phi(\underset{\sim}{x})$$

3. Considere o campo escalar real ser escrito da seguinte decomposição

$$\phi(\underset{\sim}{x}) = \phi^{(+)}(\underset{\sim}{x}) + \phi^{(-)}(\underset{\sim}{x})$$

onde

$$\phi^{(+)}(\underset{\sim}{x}) = \left[\frac{\hbar c^2}{(2\pi)^3}\right]^{\frac{1}{2}} \int \frac{d\vec{k}}{\sqrt{2\omega_{\vec{k}}}}\, a(\vec{k})\, e^{-i\,\underset{\sim}{k}\cdot\underset{\sim}{x}} \;\; ; \;\; \phi^{(-)}(\underset{\sim}{x}) = \left[\frac{\hbar c^2}{(2\pi)^3}\right]^{\frac{1}{2}} \int \frac{d\vec{k}}{\sqrt{2\omega_{\vec{k}}}}\, a^\dagger(\vec{k})\, e^{i\,\underset{\sim}{k}\cdot\underset{\sim}{x}}\,.$$

Lembrando que $[a(\vec{k}), a^\dagger(\vec{k'})] = \delta(\vec{k} - \vec{k'})$; $[a(\vec{k}), a(\vec{k'})] = 0$ e $a(\vec{k})|0\rangle = 0$. Mostre que

$$
\begin{aligned}
(a) \quad & [\phi^{(+)}(\underset{\sim}{x}), \phi^{(-)}(\underset{\sim}{y})] = i\hbar c \Delta^{(+)}(\underset{\sim}{x} - \underset{\sim}{y}) \\
(b) \quad & [\phi^{(-)}(\underset{\sim}{x}), \phi^{(+)}(\underset{\sim}{y})] = i\hbar c \Delta^{(-)}(\underset{\sim}{x} - \underset{\sim}{y}) \\
(c) \quad & [\phi(\underset{\sim}{x}), \phi(\underset{\sim}{y})] = i\hbar c \Delta(\underset{\sim}{x} - \underset{\sim}{y})
\end{aligned}
$$

onde

$$
\begin{aligned}
\Delta^{(+)}(\underset{\sim}{x} - \underset{\sim}{y}) &= -\Delta^{(-)}(\underset{\sim}{y} - \underset{\sim}{x}) = -\frac{ic}{2(2\pi)^3} \int \frac{d\vec{k}}{\omega_{\vec{k}}}\, e^{-i\underset{\sim}{k}\cdot(\underset{\sim}{x} - \underset{\sim}{y})} \\
\Delta(\underset{\sim}{x} - \underset{\sim}{y}) &= \Delta^{(+)}(\underset{\sim}{x} - \underset{\sim}{y}) + \Delta^{(-)}(\underset{\sim}{x} - \underset{\sim}{y})
\end{aligned}
$$

4. Sabendo as seguintes propriedades/definições das matrizes de Dirac:

$$
\begin{aligned}
\{\gamma^\alpha, \gamma^\beta\} &= 2\,\eta^{\alpha\beta} \\
\gamma^{0\,\dagger} &= \gamma^0 \quad ; \quad \gamma^{k\,\dagger} = -\gamma^k \quad ; \quad \left(\gamma^0\right)^2 = I \quad ; \quad \left(\gamma^k\right)^2 = -I \\
\gamma^5 &\equiv i\gamma^0\gamma^1\gamma^2\gamma^3 \quad ; \quad \gamma^{5\,\dagger} = \gamma^5 \quad ; \quad \left(\gamma^5\right)^2 = I \\
\eta^{\alpha\beta} &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}
\end{aligned}
$$

Mostre que:

(a) $\gamma_\mu \gamma^\mu = 4$
(b) $\gamma_\mu \gamma^\alpha \gamma^\mu = -2\gamma^\alpha$
(c) $\gamma_\mu \gamma^\alpha \gamma^\beta \gamma^\sigma \gamma^\mu = -2\gamma^\sigma \gamma^\beta \gamma^\alpha$
(d) $\gamma_\mu \gamma^\alpha \gamma^\beta \gamma^\mu = 4\,\eta^{\alpha\beta}$
(e) $\not{A}\,\not{A} = A^2$
(f) $\gamma_\mu \not{A}\,\gamma^\mu = -2\not{A}$
(g) $\mathrm{Tr}[\gamma^\alpha] = 0$
(h) $\mathrm{Tr}[\gamma^\alpha \gamma^\beta] = 4\eta^{\alpha\beta}$
(i) $\mathrm{Tr}[\gamma^5] = 0$
(j) $\mathrm{Tr}[\gamma^\alpha \gamma^5] = 0$
(k) $\mathrm{Tr}[\not{A}\,\not{B}] = 4(A_\mu B^\mu)$
(l) Se $(\gamma^\alpha \gamma^\beta \ldots \gamma^\mu \gamma^\nu)$ for o produto de um número ímpar de matrizes γ, então mostre que $\mathrm{Tr}[\gamma^\alpha \gamma^\beta \ldots \gamma^\mu \gamma^\nu] = 0$