

Geometric properties of two-dimensional coarsening with weak disorder

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Abstract – The domain morphology of weakly disordered ferromagnets, quenched from the high-temperature phase to the low-temperature phase, is studied using numerical simulations. We find that the geometrical properties of the coarsening domain structure, *e.g.* the distributions of hull-enclosed areas and domain perimeter lengths, are described by a scaling phenomenology in which the growing domain scale $R(t)$ is the only relevant parameter. Furthermore, the scaling functions have forms identical to those of the corresponding pure system, extending the “super-universality” property previously noted for the pair correlation function.

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Introduction. – Many aspects of the out-of-equilibrium relaxation of macroscopic systems still remain to be unveiled. The mechanism underlying coarsening phenomena, or the domain growth of two competing equilibrium phases after a quench from the disordered phase, is well understood [1]. However, an important part of the description of this process is phenomenological and, to a certain extent, qualitative.

A common feature of nearly all coarsening systems is that their dynamics are described by a dynamical scaling hypothesis with a single characteristic growing length scale $R(t)$ that depends on the problem at hand. This hypothesis states that the domain morphology is statistically the same at all times when lengths are measured in units of $R(t)$. As a result, all time dependences in correlation functions are encoded in R and lengths appear scaled by this typical length. The scaling hypothesis has been well verified experimentally, numerically as well as analytically in a small number of solvable simple cases. It is thus a well-established feature of phase ordering dynamics though no generic proof exists.

Most studies of coarsening dynamics focus on the growth law $R(t)$ and the scaling functions of various correlators and linear responses. An analytic determination of these functions remains open —apart from some toy

models such as the one-dimensional ferromagnetic chain or the $O(N)$ model in the large- N limit. A number of field-theoretic approaches to finding an approximate form of the scaling functions of two-point and two-time correlations have been proposed but none of them is fully successful [1].

In [2,3] we analysed the coarsening process in pure (not disordered) two-dimensional systems with non-conserved order parameter from a geometric point of view. We studied the probability distributions of hull-enclosed and domain areas and hull and domain-wall lengths. By deriving an analytic expression for the number density of hull-enclosed areas we proved that the scaling hypothesis holds for this quantity. With a few additional assumptions we extended these results to the statistical properties of domains and boundary lengths and we put them all to the numerical test. In this letter we extend these results to the coarsening dynamics of two-dimensional models with non-conserved order parameter under the effect of *weak quenched disorder*. By “weak disorder” we mean randomness that does not modify the character of the ordered phase (see [4] for a review). Recently there have been several studies [5–13] devoted to the coarsening dynamics of this kind of systems, although none of them addressed the geometric properties of the process.

An important property of domain growth in systems with weak disorder is the super-universality hypothesis that states that once the correct growing length scale is

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taken into account all scaling functions are independent of the disorder strength. This conjecture was first enunciated by Fisher and Huse [14] using a renormalization group approach. The idea is that the length scale at which the effects of quenched disorder are important is much smaller than the domain scale $R(t)$. The latter dominates the elastic energy. Thus, the dynamics of large structures is approximately curvature driven. Pinning at small scales modifies the scale factor, that is to say the growth law $R(t)$, but not the scaling functions. The validity of super-universality for the scaling function of the equal-time two-point correlation of several disordered systems including the random bond Ising model was checked numerically in [15–18]. The stringent test proposed in [19] that includes higher-order correlations was also passed numerically in [20].

In short, in this letter we continue our geometric study of coarsening in two-dimensional systems. We simulate the dynamics of the random bond Ising model and we pay special attention to the scaling and super-universality properties of the probability distribution of areas and borders.

Hull-enclosed-area distribution. – The ordering process of a magnetic system in its low-temperature phase is visualized in terms of domains or regions of connected aligned spins. Each domain has one external perimeter which is called the *hull*. The *hull-enclosed area* is the total area within this perimeter. In [2,3] we derived an exact expression for the hull-enclosed-area distribution of pure, curvature-driven, two-dimensional coarsening with non-conserved order parameter. For simplicity, in this paper we concentrate on the distribution of hull-enclosed areas, for which we have exact analytic results in the clean limit. A generalization to the domain area distribution is straightforward. Using a continuum description in which the non-conserved order parameter is a scalar field we found that the number of hull-enclosed areas per unit area, $n_h^p(A, t) dA$, with enclosed area in the interval $(A, A + dA)$, is

$$n_h^p(A, t) = \frac{2c_h}{(A + \lambda_h t)^2}. \quad (1)$$

$c_h = 1/8\pi\sqrt{3}$ is a universal constant that enters this expression through the influence of the initial condition and was computed by Cardy and Ziff in their study of the geometry of critical structures in equilibrium [21]. λ_h is a material-dependent constant relating the local velocity of an interface and its local curvature in the Allen-Cahn equation, $v = -(\lambda_h/2\pi)\kappa$ [22]. The expression (1) is valid for all quenches from equilibrium in the high-temperature phase, $T_0 > T_c$, to zero working temperature, $T = 0$. For a critical initial condition, $T_0 = T_c$, the same expression holds with $2c_h$ replaced by c_h . Equation (1) can be recast in the scaling form

$$n_h^p(A, t) = \frac{1}{(\lambda_h t)^2} f\left(\frac{A}{\lambda_h t}\right) \quad (2)$$

with $f(x) = \frac{2c_h}{(x+1)^2}$. In this way, scaling with the characteristic length scale, $R_p(t) = \sqrt{\lambda_h t}$, for coarsening dynamics with non-conserved order parameter in a pure system is demonstrated. The effects of a finite working temperature are fully encoded in the temperature dependence of the parameter λ_h , while the same scaling function $f(x)$ describes n_h^p [3] as suggested by the zero-temperature fixed-point scenario [1].

When quenched disorder is introduced the growing phenomenon is no longer fully curvature driven and domain-wall pinning by disorder becomes relevant. At early times, the system avoids pinning and evolves like in the pure case. Later, barriers pin the domain walls and the system gets trapped in metastable states from which it can escape only by thermal activation over the corresponding free-energy barriers. In spite of these differences, coarsening in ferromagnetic systems with quenched disorder also satisfies dynamic scaling [15–20] and a single characteristic length, $R(t, T, \varepsilon)$, can be identified (ε is a measure of the disorder amplitude). As a result of the competition between the curvature-driven mechanism and pinning by disorder, the coarsening process is slowed down and the characteristic radius of the domains depends on the disorder strength and it is smaller than the pure one, $R(t, T, \varepsilon) < R_p(t, T)$. Moreover, the super-universality hypothesis applied to the scaling function of the equal-time two-point correlation function in random ferromagnets was verified numerically in a number of works [15,16].

The scaling and super-universality hypotheses suggest that eq. (2) remains valid with the same scaling function $f(x) = 2c_h/(1+x)^2$ and $(\lambda_h t)^{1/2}$ replaced by $R(t, T, \varepsilon)$ for all 2d non-conserved order parameter coarsening processes in which the low-temperature ordered phase is not modified. More precisely, we expect that

$$n_h(A, t, T, \varepsilon) = R^{-4}(t, T, \varepsilon) f\left[\frac{A}{R^2(t, T, \varepsilon)}\right] \quad (3)$$

should be valid in all these cases. In this letter we test this hypothesis by following the dynamic evolution of the two-dimensional random bond Ising model (2d RBIM) defined by the Hamiltonian,

$$H = - \sum_{\langle i, j \rangle} J_{ij} \sigma_i \sigma_j, \quad (4)$$

where the J_{ij} are random variables uniformly distributed over the interval $[1 - \varepsilon/2, 1 + \varepsilon/2]$ with $0 < \varepsilon \leq 2$. This model has a second-order phase transition between a high-temperature paramagnetic phase and a low-temperature ferromagnetic phase. We simulate the dynamic evolution of a model defined on a square lattice with linear size $L = 10^3$, using a single-flip Monte Carlo technique and the heat-bath algorithm. Data are averaged over 10^3 samples. We show results for a random initial condition, $\sigma_i = \pm 1$ with equal probability to mimic an infinite-temperature equilibrium state, $T_0 \rightarrow \infty$. At the initial time $t = 0$ we set the working temperature to a low value and we follow

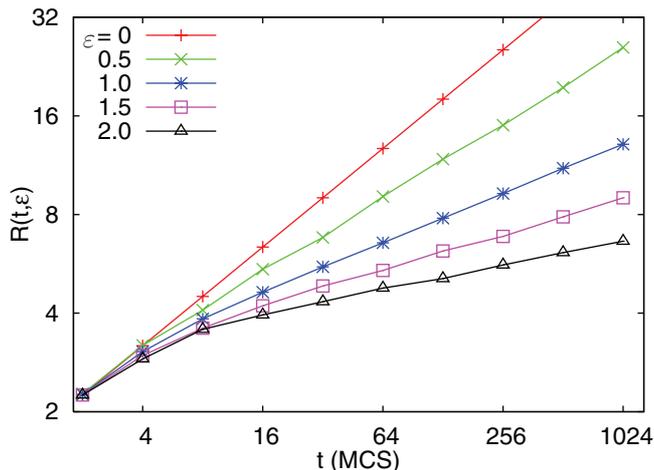


Fig. 1: (Colour on-line.) Log-log plot of the characteristic domain radius extracted from the collapse of the equal-time two-point correlation function against time after the quench. Different curves correspond to different values of the quenched disorder strengths. For comparison we include the non-disordered limit law $R(t, \varepsilon = 0) \propto \sqrt{t}$.

the evolution thereafter. We are interested in the effect of quenched randomness and we thus focus on a single working temperature, $T = 0.4$, at which the ordered equilibrium phase is ferromagnetic for all $0 \leq \varepsilon \leq 2$. In what follows we drop the T -dependence from R and n_h , so we simply denote them as $R(t, \varepsilon)$ and $n_h(A, t, \varepsilon)$.

We determine the growth law, $R(t, \varepsilon)$, from a direct measure of the spatial correlation function $C(r, t, \varepsilon)$:

$$C(r, t, \varepsilon) \equiv \frac{1}{N} \sum_{i=1}^N \langle s_i(t) s_j(t) \rangle_{|\vec{r}_i - \vec{r}_j| = r} \quad (5)$$

which also obeys dynamical scaling,

$$C(r, t, \varepsilon) \sim m^2(T, \varepsilon) f\left(\frac{r}{R(t, \varepsilon)}\right), \quad (6)$$

with $m(T, \varepsilon)$ the equilibrium magnetisation density. In fig. 1 we show $R(t, \varepsilon)$ as a function of t for several values of the quenched disorder strength along with the pure case. We extracted $R(t, \varepsilon)$ by collapsing the equal-time correlations data using eq. (6). After a few time steps all curves deviate from the pure $R(t, \varepsilon = 0) \propto \sqrt{t}$ law and the stronger the disorder strength ε , the slower the subsequent growth. For strong disorder the characteristic length reaches relatively modest values during the simulation interval, for instance, $R(t = 1024, \varepsilon = 2) \sim 7$ (measured in units of the lattice spacing). It is then quite hard to determine the actual functional law describing the late-time evolution of $R(t, \varepsilon)$ and it comes as no surprise that this issue has recently been a matter of debate [5–11, 23, 24]. Using arguments based on the energetics of domain-wall pinning, several authors [5, 23] have proposed a logarithmic growth law $R(t) \propto (\ln t)^{1/\psi}$ with an exponent ψ

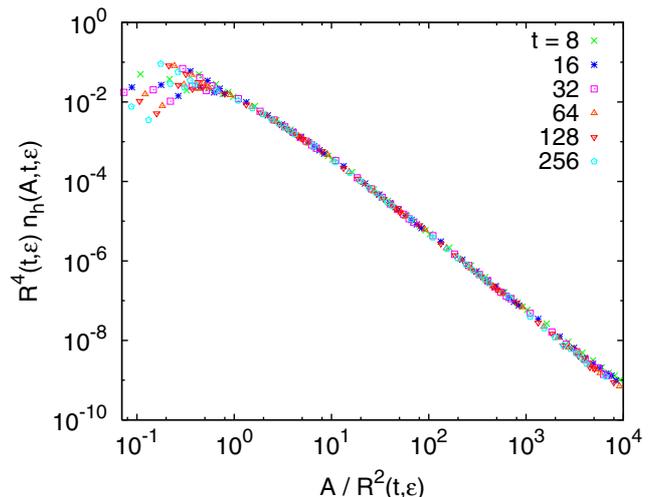


Fig. 2: (Colour on-line.) Number density of hull-enclosed areas for one value of the disorder strength, $\varepsilon = 2$, at several times. The data are shown in the scaling form and they collapse for $A/R^2 \gtrsim 1$.

characterizing the energy barriers. More recently, powerful Monte Carlo simulations [6–11, 24] suggest a power law $R(t) \propto t^\theta$ with an exponent θ that depends on T and ε . We are not concerned here with predicting the time dependence of R . Instead, we use the numerical values in fig. 1 to scale our data for the distributions of hull-enclosed areas and hull lengths as explained below.

In fig. 2 we display data for one disorder strength, $\varepsilon = 2$, taken at several times after the quench. Using the scaling form the data collapse for $A/R^2 \gtrsim 1$ over 8 decades in the vertical axis and 4 decades in the horizontal axis. Deviations are seen for small areas and we discuss their possible origin below. This analysis confirms the scaling hypothesis in the region of large areas, $A/R^2 \gtrsim 1$.

In fig. 3 we test the super-universality hypothesis as applied to the hull-enclosed-area distribution by presenting data for a single instant, $t = 256$ MCS, and several values of the disorder strength, $\varepsilon = 0, 0.5, 1, 1.5, 2$. Again, for areas such that $A/R^2 \gtrsim 1$ there is a very accurate data collapse, while for smaller areas deviations, that we discuss below, are visible. The solid black line represents the analytic prediction for the pure case that yields, under the super-universality hypothesis, the analytic prediction for all ε . The curve falls well on top of the data. In the tail of the distribution we see downward deviations that are due to finite-size effects already discussed in detail in [3].

Let us now examine several possible origins for the deviation of the numerical data from the analytic prediction at small areas, $A/R^2 \lesssim 1$, in figs. 2 and 3. The first source of problems could be the fact that we need to use a finite working temperature in the disordered case to depin the walls, while the analytic results are derived at $T = 0$. In the pure case the effect of temperature is to create thermal domains within the genuinely coarsening ones. Based

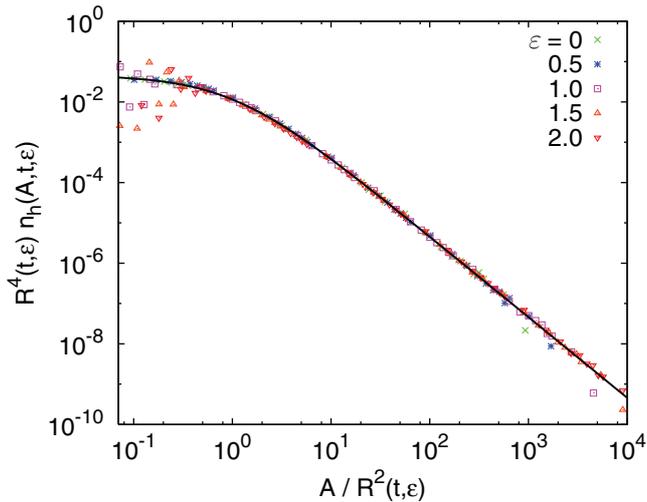


Fig. 3: (Colour on-line.) Number density of hull-enclosed areas at $t = 256$ MCs and several values of the disorder strength. The collapse of all data for $A/R^2 \gtrsim 1$ supports the validity of super-universality. The black solid line represents the analytic result for the pure case.

on this observation in [3] we explained that the small-area probability distribution may have an excess contribution coming from these thermal fluctuations. By extracting the contribution of the *equilibrium* distribution of thermal domains, that became important for temperatures $T \gtrsim 1$, we showed that the number density of hull-enclosed areas at $T > 0$ is given by the zero-temperature result once scaled by $R(T, t)$. In the present disordered case, the working temperature we use is too low to generate any thermal domains and thus temperature cannot be the source of deviations from the analytic form.

Another possible origin of the difference between numerical data and theoretical prediction is the fact that the analytic results are derived using a continuum field-theoretic description of coarsening, while numerical simulations are done on a lattice. In the presence of quenched randomness one can expect the effects of the lattice discretization to be more important than in the pure case, especially for relatively small structures. Moreover, disorder induces domain-wall roughening and the Allen-Cahn flat-interface assumption is not as well justified.

Finally, the super-universality hypothesis is argued for large structures only [14] and thus the small hull-enclosed areas are not really forced to follow it strictly when $A \lesssim R^2$.

Geometric structure of the domains. – In order to better characterise the geometric structure of the coarsening process we study the relation between hull-enclosed areas and perimeters. In fig. 4 we present the average of the scatter plot of the scaled hull-enclosed area, A/R^2 , against the corresponding scaled perimeter, p/R , in a log-log plot for several values of the disorder strength at $t = 256$ MCs. We observe a separation into two regimes at

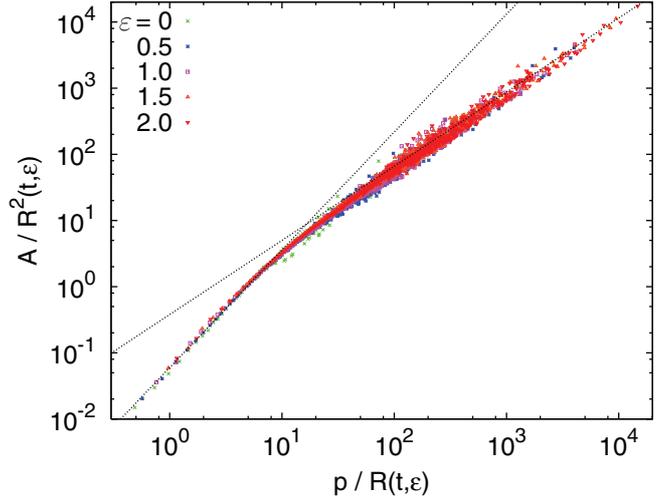


Fig. 4: (Colour on-line.) Averaged scatter plot of the scaled areas against perimeters for several values of the strength of the quenched randomness.

the breaking point $p^*/R \sim 10$ and $A^*/R^2 \sim 2$. In the upper and lower regime areas and perimeters are related by two power laws that *do not* depend on the disorder strength. More precisely [3],

$$\frac{A}{R^2(t, \varepsilon)} \sim \left[\frac{p}{R(t, \varepsilon)} \right]^\alpha \quad (7)$$

with

$$\begin{aligned} \alpha^> &= \alpha_0 = 1.12 \pm 0.10, & \text{for } \frac{p}{R(t, \varepsilon)} \gtrsim 10, \\ \alpha^< &= 1.83 \pm 0.10, & \text{for } \frac{p}{R(t, \varepsilon)} \lesssim 10. \end{aligned} \quad (8)$$

Note that the small value of $\alpha^>$ is not related to the existence of holes in these large structures, as suggested for domains in [25], since hull-enclosed areas do not have holes in them. The crossover between upper and lower regimes depends on time when observed in absolute value. In other words, during the coarsening process a characteristic scale $p^* \propto R(t, \varepsilon)$ develops such that hull-enclosed areas with perimeter $p > p^*$ have the same exponent $\alpha_0 \sim 1.12$ as in the initial condition before the quench [3] (because on such large length scales, the dynamics has not yet been effective), while hull-enclosed areas with smaller perimeter are more compact as indicated by the larger value of the exponent $\alpha^<$.

Hull length distribution. – We now examine the hull length distribution that is expected to follow the scaling and super-universality hypotheses. In fig. 5 we display the number density of hull lengths at $t = 256$ MCs and $\varepsilon = 0, 0.5, 1, 1.5, 2$. We show the data in the form suggested by the scaling hypothesis

$$n_h(p, t, \varepsilon) = R^{-3}(t, \varepsilon) g \left[\frac{p}{R(t, \varepsilon)} \right]. \quad (9)$$

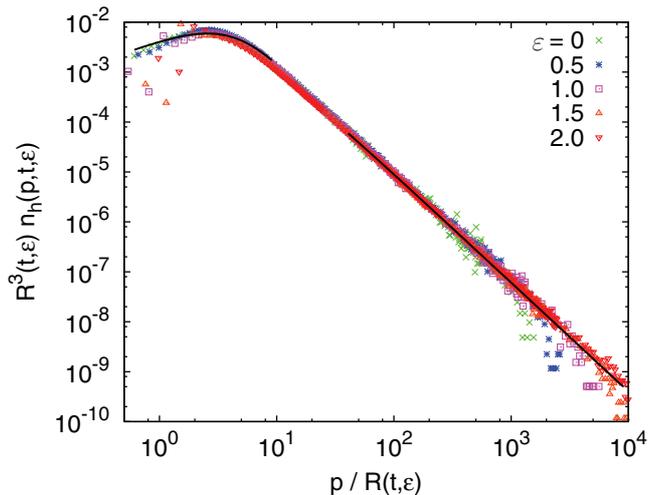


Fig. 5: (Colour on-line.) Number density of hull lengths at $t = 256$ MCs and several values of the disorder strength given in the key. The solid black line is the analytic prediction in eq. (10) and explained in the text. The collapse of all data for $p/R \gtrsim 2$ supports the validity of super-universality in this regime of lengths.

The data collapse on a master curve for all lengths satisfying $p/R \gtrsim 2$, a value that is roughly the location of the maximum in the distribution. The super-universality hypothesis holds in this regime of lengths.

Relating the distribution functions of areas and perimeters through a change of variables, as explained in [3], we find

$$R^3(t, \varepsilon) n_h(p, t, \varepsilon) \sim \frac{\alpha^{\gtrsim} c_h \left(\frac{p}{R(t, \varepsilon)} \right)^{\alpha^{\gtrsim} - 1}}{\left[1 + \left(\frac{p}{R(t, \varepsilon)} \right)^{\alpha^{\gtrsim}} \right]^2}. \quad (10)$$

The scaling function is thus the same as in the pure system [3] with two branches characterized by the exponents α^{\gtrsim} and α^{\lesssim} . As in the pure case, the part around the maximum of the distribution eq. (10) is characterized by the exponent α^{\lesssim} , while the tail of the distribution is well described by the exponent α^{\gtrsim} .

In fig. 6 we display the number density of hull lengths scaled with the typical radius R for one value of the disorder strength at several times after the quench. The upper and lower predictions are shown with solid black lines.

Conclusions. – In this letter we analysed the statistics of hull-enclosed areas and hull lengths during the coarsening dynamics of the 2d RBIM with a uniform distribution of coupling strengths. We found that the number densities of these observables satisfy scaling and super-universality for structures with $A/R^2 \gtrsim 1$ and $p/R \gtrsim 2$.

We showed that the analytic prediction for the number density of hull-enclosed areas derived for pure systems also describes the statistics of these quantities in the presence

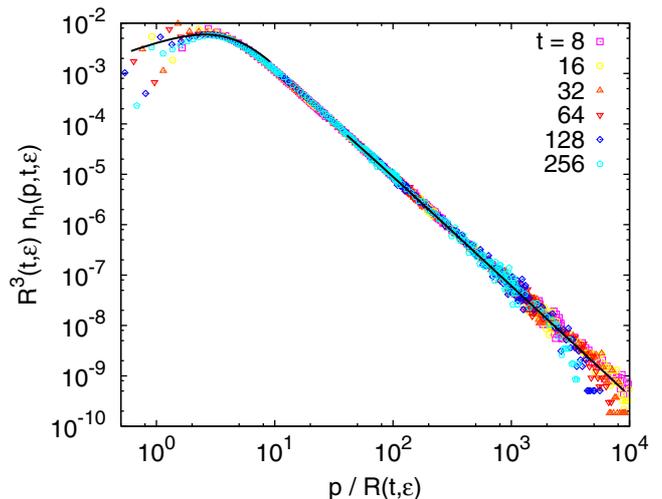


Fig. 6: (Colour on-line.) Scaling plot of the number density of hull lengths for a disorder with strength $\varepsilon = 1.0$ at different times. The solid black line is the analytic prediction in eq. (10).

of quenched ferromagnetic disorder. The geometric properties of the boundaries between phases are, in principle, more sensitive to quenched randomness than their interior. We showed, however, that the relation between areas and interfaces and, in consequence, the distribution of hull lengths are independent of the disorder strength also satisfying super-universality.

In previous work all numerical tests of the super-universality hypothesis in the RBIM have focused on the study the equal-time two-point correlation function [15–18,20]. The geometric approach that we have presented enables us to show that this property applies even to very small structures.

We also studied a number of related problems on which we report below:

- We verified that analogous results are obtained for different probability distributions of the coupling strengths as long as these remain ferromagnetic.
- We observed that the scaling plots do not depend on the working temperature while the latter is below the critical point.
- All our results can be easily extended to the study of domain areas and walls as done in [3] for the pure case.
- We checked that the super-universality hypothesis is also verified in $d = 1$ [3,26] for relatively long domains. Small deviations for very short ones are also observed in this case.

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