Lista II.4 - FIS01208 - Semestre 2010/1

1. Your friend from the Quantum Mechanics class argues the following: If a particle is in an eigenstate of a one-dimensional box of width L, then we know its energy exactly. We also know that the energy in the box is purely kinetic. Hence we know the particle's momentum exactly as well. This contradicts the Heisenberg uncertainty relation since the uncertainty in the particle position is finite ($\Delta x = L$). Punch a hole into your friend's argument.

2. A particle moving in one dimension has the wavefunction

$$\psi(x) = A \exp\left[i(ax - bt)\right]$$

where a and b are real constants.
a) What is the potential field V(x) in which the particle is moving?
b) If the momentum of the particle is measured, what value is found (in terms of a and b)?
c) In which momentum state is the particle most likely to be found?

3. We define the commutator $[A, B]$ between two operators A , B as the operator $[A, B] = AB - BA$.
a) Show that for any three operators A , B , C the relations $[A + B, C] = [A, C] + [B, C]$ and $[AB, C] = A[B, C] + [A, C]B$ hold.
b) Using either the position or the momentum representation, calculate the commutator $[x, p]$ between the position and momentum operators. (Remember that operators act on wavefunctions to the right.)
c) Show that $[x, p^2] = 2i\hbar p$.
d) Two operators are said to commute if $[A, B] = 0$. Does the position operator commute with the Hamiltonian for a particle in free space? Does the momentum operator commute with the Hamiltonian for a particle in free space?

4. A Hermitian operator is an operator that has only real eigenvalues.
a) Show that the Hamiltonian $H = \frac{p^2}{2m} + V(x)$ is a Hermitian operator.
b) Show that the expectation value $\langle H \rangle$ of the Hamiltonian operator in an energy eigenstate is equal to the eigenenergy of that state.
c) Show that the position operator in momentum space $x = i\hbar \frac{\partial}{\partial p}$ is a Hermitian operator.

5. A particle of mass m is prepared in the ground state of an infinite-potential box of size L extending from $x = 0$ to $x = L$. Suddenly, the wall at $x = L$ is moved to $x = 2L$ within a time Δt doubling the box size. You may assume that the wavefunction is the same immediately after the change, if the change happens fast enough.
a) How fast is fast enough? ($\Delta t = 0$ is a mathematician's answer and not good enough for a physicist.)
b) What is the probability that the particle is in the second $n = 2$ state of the new well, immediately after the change? (Note that the wavelength within the well, and hence the energy, for this state is the same as for the initial state in the old well.) Make sure that you use properly normalized wavefunctions for your calculations.
c) What is the probability that the particle would be found in the ground state of the new well?
d) What is the expectation value of the energy of the particle before and after the sudden expansion?

6. Consider a particle of mass m inside a box of size L with infinite walls,

$$V(x) = \begin{cases} 0 & \text{for } 0 \leq x \leq L \\ \infty & \text{elsewhere.} \end{cases}$$

We want to determine the time evolution of a wavefunction that is specified at $t = 0$. Assume that

$$\psi(x, t = 0) = C\left[2\sin(kx) + 3\sin(2kx) + \sin(3kx)\right]$$

where $k = \pi/L$.
a) Determine the normalization coefficient C so that $|\psi(x, t)|^2$, can be interpreted as a probability density.
b) Expand the wavefunction at the initial time $\psi(x, t = 0)$ in terms of the eigenfunctions $\phi_n(x)$ of the infinite box, i.e. determine the coefficients $c_n = \int_{-\infty}^{\infty} \phi_n^*(x)\psi(x, 0)\, dx$, so that you can write $\psi(x, 0)$ as a superposition of eigenstates of the infinite box.
c) Using the known time evolution of eigenstates (negative phase factor evolving at angular frequency find $\psi(x, t)$ an arbitrary later time t.

d) Is the motion periodic, is there a time T with $\psi(x,0) = \psi(x,T)$?
e) If a measurement of the particle's energy is performed, what will be the outcome (outcomes) and with what probability will those values be measured?
f) What is the average energy of the particle in the box? Does it change with time if no measurements are performed?
g) Does the particle's energy change if an energy measurement is performed? Comment on energy conservation.

7. Show that if $[A,H] = 0$ and $\partial A/\partial t = 0$, then $\langle \Delta A \rangle$ is constant in time.

8. Solve the time-independent Schrödinger equation with appropriate boundary conditions for an infinite square well of width L centered at $x = 0$, i.e.

$$V(x) = \begin{cases} 0 & \text{for } -L/2 \leq x \leq L/2 \\ \infty & \text{elsewhere.} \end{cases}$$

a) Check that the allowed energies are consistent with those derived in lecture for an infinite well of width a centered at $x = L/2$. Confirm that the wave functions $\phi(x)$ can be obtained from those found in lecture if one uses the substitution $x \to x - L/2$.
b) We define a parity operator P by $P\phi(x) = \phi(-x)$. Verify that the energy eigenstates for the above potential are also eigenstates of the parity operator. What are the corresponding eigenvalues? Is the same true for the eigenstates of the well centered at x=L/2?
c) Find a general condition either on the potential $V(x)$ or on the Hamiltonian that ensures that the energy eigenstates are simultaneously eigenstates of the parity operator.

9. To illustrate the idea that the zero point energy gets larger by going from macroscopic to microscopic systems, calculate the zero point energy for a particle in an infinite potential well for the following three cases:
a) a 100 g ball confined on a 5 m long line
b) an oxygen atom confined to a 2Å lattice
c) an electron confined to a 1Å atom.