

ANALYTIC RESULTS FOR THE MAGNETIZED ELECTRON GAS

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We show that all the thermodynamic functions of a relativistic free electron gas in the background of a uniform magnetic field can be conveniently and simply expressed in terms of the Hurwitz zeta function. Known results on the behavior of the magnetized electron gas are rederived in the new formalism.

1 Introduction

The study of the thermodynamic properties of degenerate relativistic electron gases in strong magnetic fields is very relevant to understand the physics of various compact astrophysical objects, such as magnetic white dwarfs and neutron stars. Several aspects of this problem have been considered in the literature in the past 35 years^{1,2,3,4}.

Here we propose a novel analytical approach, which is completely relativistic, and makes no a priori assumptions about the magnitude of the magnetic field or the electron number density. We obtain simple closed-form expressions for the thermodynamic functions, which can be used as the starting point for expansions in different regimes or for numerical work. We reproduce, in a unified way, many of the results scattered in the literature.

Only a summary of the results is given here. More details can be found in reference⁵.

2 The System to be Studied

We shall consider a gas of electrons confined to a volume V in the background of a uniform magnetic field $\mathbf{B} = B\hat{\mathbf{z}}$. In order to maintain charge neutrality we assume that the whole system is immersed in a background of uniformly distributed positive charge. We make no assumptions on the size of the magnetic field, the electron number density n or the electron chemical potential μ , but we do assume that the temperature is small in comparison with

the chemical potential, so that the gas can be considered to be degenerate.

To first approximation we shall neglect the Coulombic interactions between the electrons. This is a good approximation in the ultra-relativistic limit, $\mu \gg m$ (m is the electron mass), the kinematical limit to be found in compact astrophysical objects such as neutron stars and, to a lesser extent, in white dwarfs. In this limit the ratio between the typical Coulombic repulsion energy between neighboring electrons and their typical kinetic energy is of the order of the electromagnetic coupling constant, α . In order then to work consistently to zeroth order in α , we shall also neglect the QED correction to the gyromagnetic ratio of the electron and simply work with $g = 2$.

Under the assumptions of the last paragraph the many-body problem simplifies enormously. Each electron moves independently of the rest, seeing only the background magnetic field. Its motion is described by Dirac's equation. For our thermodynamic purposes, only the energy spectrum of the one-electron system in the background of a uniform magnetic field is relevant. The solution to this problem is well known⁶. In Landau's gauge, in which the vector potential is everywhere perpendicular to some plane containing the magnetic field, the energy eigenstates are characterized by four quantum numbers: p_z (momentum component along \mathbf{B}), p_\perp (momentum component along \mathbf{A}), a non-negative integer $n = 0, 1, 2, \dots$ that labels the wavefunction in the direction perpendicular to the plane defined by \mathbf{B} and \mathbf{A} , and an integer $\lambda = \pm 1$ which denotes the spin parallel or antiparallel to \mathbf{B} . The energy levels can be described in terms of *Landau bands*,

$$E_j(p_z) = \sqrt{p_z^2 + m^2 + (2eB)j}, \quad (1)$$

where j is the non-negative integer defined by $j = n + (\lambda + 1)/2$. The energy levels (1) are highly degenerate, due to their independence on p_\perp and their dependence on n and λ only through the single combination j . The number of states $g_j(p_z)dp_z$ for given j and momentum along \mathbf{B} between p_z and $p_z + dp_z$, in a system of electrons confined to a finite cubic box of volume V , is given by

$$g_j(p_z)dp_z = g_j \frac{eB}{4\pi^2} V dp_z, \quad (2)$$

with $g_j = 1$ for $j = 0$ and $g_j = 2$ for $j \neq 0$. Natural units, for which the speed of light c , Planck's constant \hbar and Boltzmann's constant k are all equal to one, will be used throughout.

3 The Thermodynamic Potential

All the equilibrium thermodynamic properties of a system, such as the density, pressure or magnetization, can be derived from the grand potential function. For our ideal Fermi gas in a uniform background magnetic field this is given by

$$\Omega(T, B, V, \mu) = -T \sum_{\alpha} \ln \left(1 + e^{-(E_{\alpha} - \mu)/T} \right), \quad (3)$$

where μ is the chemical potential and the sum is over all 1-particle orbitals α . The B dependence is implicit in the orbital energies E_{α} and their associated level degeneracies. In terms of the grand potential density $\omega \equiv \Omega/V$ one has

$$p = -\omega, \quad n = -\partial\omega/\partial\mu, \quad \mathcal{M} = -\partial\omega/\partial B, \quad (4)$$

for the pressure, density and magnetization, respectively.

Written in full form, the object to be computed has the daunting form

$$\omega = -T \frac{eB}{4\pi^2} \sum_{j=0}^{\infty} g_j \int_{-\infty}^{\infty} dp_z \ln \left(1 + \exp \left[-\frac{\sqrt{p_z^2 + m^2 + (2eB)j} - \mu}{T} \right] \right). \quad (5)$$

4 The Density of States for the Relativistic Landau Problem

Instead of organizing the sum over energy eigenstates as in Eq.(5), in terms of Landau bands, labelled by the integer j , it is more convenient to recast the sum in terms of an integral over a single energy value, by finding the density of states in energy space, as is the standard procedure in statistical mechanics. The latter, $g(E)$, is defined so that $g(E)dE$ is the number of states with energies between E and $E + dE$.

In terms of the density of states, the expression for the grand potential density has the much simpler form

$$\omega = -T \int_m^{\infty} \frac{g(E)}{V} \ln \left(1 + e^{-(E - \mu)/T} \right) dE. \quad (6)$$

For the problem at hand, it is clear that for any given value of E only a finite number of bands will contribute states to $g(E)$, since they start at a ‘‘floor’’ equal to

$$E_j^{\min} = \sqrt{m^2 + (2eB)j}. \quad (7)$$

A straightforward calculation yields the following expression for $g(E)$:

$$g(E) = \frac{V}{4\pi^2} (2eB)^{1/2} E \sum_{j=0}^{\lfloor q_E \rfloor} g_j \frac{1}{\sqrt{q_E - j}}, \quad (8)$$

where we have defined the auxiliary dimensionless variable

$$q_E = \frac{E^2 - m^2}{2eB}, \quad (9)$$

whose integer part, $\lfloor q_E \rfloor$, labels the last Landau band whose floor is below E .

The finite sum in Eq. (8) can be expressed in closed form in terms of the Hurwitz zeta function⁷ $\zeta(\frac{1}{2}, q)$ as

$$\sum_{j=0}^{\lfloor q_E \rfloor} g_j \frac{1}{\sqrt{q_E - j}} = 2\mathcal{H}_{1/2}(q_E), \quad (10)$$

where

$$\mathcal{H}_z(q) \equiv \zeta(z, \{q\}) - \zeta(z, q+1) - \frac{1}{2}q^{-z}, \quad (11)$$

and where $\{q\} = q - \lfloor q \rfloor$ is the fractional part of q . Despite the minus signs in Eq. (11), the function $\mathcal{H}_{1/2}(q_E)$ must clearly be positive definite and monotonically increasing for $q_E > 0$. The oscillatory first term in Eq. (11) has, for $z = 1/2$, an integrable singularity at $\{q\} = 0$ which is related to the accumulation of momentum states at the floor of each Landau band.

Therefore, we have obtained the following closed form expression for the density of states for the relativistic Landau problem:

$$g(E) = \frac{V}{2\pi^2} (2eB)^{1/2} E \mathcal{H}_{1/2} \left(\frac{E^2 - m^2}{2eB} \right). \quad (12)$$

5 Finite Temperature From Zero Temperature

Although we could simply use our closed form expression (12) for the density of states in expression (6) for the grand potential density and obtain thereby a single variable definite integral expression for ω , which could be used, together with Eq.(4), to compute numerically the thermodynamic functions of the system, we show next that the analytic treatment of the problem can be extended much further.

In the first place, we remind the reader that any thermodynamic function of the form of Eq. (6) or

$$Q(T, \mu) = \int_m^\infty q(E)g(E) \frac{1}{e^{(E-\mu)/T} + 1} dE, \quad (13)$$

can be obtained in terms of its zero-temperature limit by a sort of convolution integral. For instance, if $\omega_0(E_F)$ denotes the zero-temperature limit of the grand potential density (6),

$$\omega_0(E_F) \equiv \lim_{T \rightarrow 0} \omega(T, \mu = E_F), \quad (14)$$

then one can show that

$$\omega(T, \mu) = \int_{-\frac{\mu-m}{T}}^\infty \omega_0(\mu + Tx) h(x) dx, \quad (15)$$

where $h(x)$ is the *Fermi-Dirac hump*,

$$h(x) = \frac{e^x}{(e^x + 1)^2} = \frac{1}{4 \cosh^2(x/2)}. \quad (16)$$

In the degenerate regime we are interested in $T \ll \mu - m$ and the lower limit of integration in Eq.(15) can be replaced by $-\infty$ with negligible error. Additionally, if the function $\omega_0(\mu + Tx)$ varies slowly under the hump (this may not be the case if there are oscillatory terms, as we will see below), then it can be expanded in a Taylor series around μ and the resulting terms integrated one by one to obtain the well-known small temperature expansion,

$$\omega(T, \mu) = \omega_0(\mu) + \frac{\pi^2}{6} T^2 \omega_0''(\mu) + \frac{7\pi^4}{360} T^4 \omega_0^{(4)}(\mu) \dots \quad (17)$$

6 Zero-temperature Results

Our next task is to compute the grand potential density in the limit of zero-temperature. At zero temperature, the chemical potential is simply called the Fermi energy, E_F . The logarithm in Eq. (6) simplifies to

$$\lim_{T \rightarrow 0} -T \ln \left(1 + e^{-(E-E_F)/T} \right) = \begin{cases} E - E_F & \text{for } E < E_F \\ 0 & \text{for } E > E_F \end{cases}, \quad (18)$$

so that

$$\omega_0(E_F, B) = \int_m^{E_F} \frac{g(E)}{V} E dE - E_F \frac{N}{V}, \quad (19)$$

where N is the total number of electrons contained in the volume V .

At this point it is convenient to introduce some dimensionless variables:

$$b \equiv \frac{B}{B_0} = \frac{2eB}{m^2}, \quad (20)$$

which measures the magnetic field strength in units of the natural strength $B_0 = m^2/2e \approx 2.2 \times 10^{13}$ Gauss (half of the Schwinger field); and

$$\varepsilon_F = E_F/m, \quad p_F = \sqrt{\varepsilon_F^2 - 1}, \quad (21)$$

which measure the Fermi energy and momentum in units of the electron mass, respectively. p_F corresponds to the maximum momentum in the direction of the magnetic field to be found in the gas at zero temperature (this state belongs to the band $j = 0$) and the integer $\lfloor p_F^2/b \rfloor$ labels the last occupied Landau band.

Substituting expression (12) into (19) we obtain the following closed expression for the zero-temperature grand potential density:

$$\omega_0(\varepsilon_F, b) = -\frac{m^4}{4\pi^2} b^{5/2} \int_0^{p_F^2/b} \frac{\mathcal{H}_{-1/2}(q)}{\sqrt{1+bq}} dq, \quad (22)$$

with

$$\mathcal{H}_{-1/2}(q) = \zeta(-\frac{1}{2}, \{q\}) - \zeta(-\frac{1}{2}, q+1) - \frac{1}{2}\sqrt{q}. \quad (23)$$

From this result and using (4) we obtain the number density as

$$n_0(\varepsilon_F, b) = \frac{m^3}{2\pi^2} b^{3/2} \mathcal{H}_{-1/2}\left(\frac{p_F^2}{b}\right), \quad (24)$$

and the magnetization as

$$\begin{aligned} \mathcal{M}_0(\varepsilon_F, b) = \frac{e m^2}{2\pi^2} & \left\{ b^{3/2} \int_0^{p_F^2/b} \mathcal{H}_{-1/2}(q) \left[\frac{2}{(1+bq)^{1/2}} + \frac{1}{2(1+bq)^{3/2}} \right] dq \right. \\ & \left. - b^{1/2} \frac{p_F^2}{\varepsilon_F} \mathcal{H}_{-1/2}\left(\frac{p_F^2}{b}\right) \right\}. \quad (25) \end{aligned}$$

The thermodynamic functions at zero-temperature display the well-known⁹ oscillatory behavior associated to the Landau band structure of the electron energy spectrum, as is shown in figure 1.

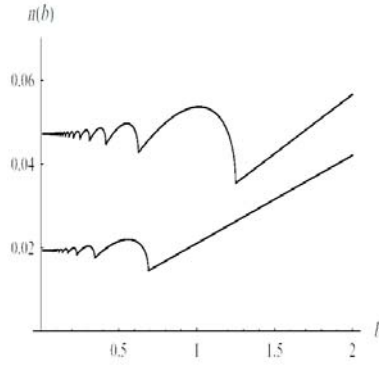


Figure 1. Electron number density at $T = 0$ as a function of b , for two fixed values of the Fermi energy, $\varepsilon_F = 1.3$ (lower curve) and $\varepsilon_F = 1.5$ (upper curve).

7 The Hurwitz Zeta Function

The Hurwitz zeta function $\zeta(z, q)$ is one of the least known of the special functions of mathematical physics. For $\text{Re } z > 1$ and $q \neq 0, -1, -2, \dots$ it is defined by

$$\zeta(z, q) = \sum_{n=0}^{\infty} \frac{1}{(n+q)^z}. \quad (26)$$

This defines an analytic function in the region $\text{Re } z > 1$, which can be analytically continued to the whole complex plane except at $z = 1$, where it has a simple pole with unit residue. Although the variable q can take any complex value except a non-positive integer, only real values of q are relevant for the work presented here, and we shall assume henceforth that q is real and nonnegative.

The resulting Hurwitz zeta function has several simple functional and analytic properties^{7,8}. Of prime importance for the results presented here is the functional relation,

$$\zeta(z, q) = \frac{1}{q^z} + \zeta(z, q+1), \quad (27)$$

whose iterated version,

$$\zeta(z, q) - \zeta(z, q + N + 1) = \sum_{n=0}^N \frac{1}{(n + q)^z}, \quad (28)$$

allowed us to express the density of states $g(E)$ in terms of the function $\mathcal{H}_{1/2}$ in Eq.(12).

Other properties of the Hurwitz zeta function that we found useful are the derivative formula

$$\frac{\partial}{\partial q} \zeta(z, q) = -z \zeta(z + 1, q), \quad (29)$$

the asymptotic expansion ($z \in \mathbf{C}, q \rightarrow \infty$)

$$\zeta(z, q) \sim \frac{1}{z-1} q^{1-z} + \frac{1}{2} q^{-z} + \sum_{k=0}^{\infty} \frac{B_{2k+2}}{(2k+2)!} \frac{(z)_{2k+1}}{q^{z+2k+1}}, \quad (30)$$

where B_k are the Bernoulli numbers and $(a)_k$ is the Pochhammer symbol, and the Fourier expansion ($\text{Re } z \leq 0$ y $0 \leq q \leq 1$)

$$\zeta(z, q) = \frac{2\Gamma(1-z)}{(2\pi)^{1-z}} \times \left\{ \sin\left(\frac{\pi z}{2}\right) \sum_{n=1}^{\infty} \frac{\cos(2\pi qn)}{n^{1-z}} + \cos\left(\frac{\pi z}{2}\right) \sum_{n=1}^{\infty} \frac{\sin(2\pi qn)}{n^{1-z}} \right\}. \quad (31)$$

Note that Riemann's zeta function $\zeta(z)$ is just a special case of the Hurwitz zeta function, $\zeta(z) = \zeta(z, 1)$.

8 Small Field Expansions

From the closed form expressions (22), (24) and (25), and with the aid of the various properties of the Hurwitz zeta function just shown, it is possible to obtain explicit expansions for the zero-temperature thermodynamic functions both in the regimes where the magnetic field is small, $b \ll 1$, or where there is a large number of Landau bands occupied, $p_F^2/b \gg 1$. These expansions are not completely trivial to obtain. Consider, for instance, the small magnetic field limit and focus on the grand potential density given by Eq.(22). First one must separate the oscillatory and the monotonic contributions, the first associated with the term $\zeta(-\frac{1}{2}, \{q\})$ in $\mathcal{H}_{-1/2}(q)$ and the latter with the other two terms in (11). For the monotonic contribution, a naïve binomial expansion of the

square root in the integrand will not work because of the diverging behavior of the resulting integrals as $p_F^2/b \rightarrow \infty$. A procedure to obtain the correct expansion for small b consists in first extracting out a sufficient number of leading terms in the asymptotic expansion of the zeta function for large q , such that the remainder is integrable in the limit $b \rightarrow 0$ [three terms will do for $\zeta(-1/2, q)$], and then integrating the subtracted terms explicitly. After all of this is done, the following expansions are found⁵:

$$n_0(\varepsilon_F, b) = \frac{m^3}{2\pi^2} \left\{ \frac{2}{3} p_F^3 + b^{3/2} \zeta\left(-\frac{1}{2}, \{p_F^2/b\}\right) + \frac{b^2}{24p_F} + O(b^4) \dots \right\}; \quad (32)$$

$$\omega_0(\varepsilon_F, b) = -\frac{m^4}{4\pi^2} \left\{ \frac{1}{2} \cosh^{-1}(\varepsilon_F) + \frac{1}{3} \varepsilon_F p_F^3 - \frac{1}{2} \varepsilon_F p_F + \frac{b^2}{12} \cosh^{-1}(\varepsilon_F) + \frac{2}{3\varepsilon_F} b^{5/2} \zeta\left(-\frac{3}{2}, \{p_F^2/b\}\right) + O(b^{7/2}) \dots \right\}; \quad (33)$$

$$\mathcal{M}_0(\varepsilon_F, b) = \frac{em^2}{2\pi^2} \left\{ -\frac{p_F^2}{\varepsilon_F} b^{1/2} \zeta\left(-\frac{1}{2}, \{p_F^2/b\}\right) + \frac{b}{6} \cosh^{-1}(\varepsilon_F) + \frac{4\varepsilon_F^2 + 1}{3\varepsilon_F^3} b^{3/2} \zeta\left(-\frac{3}{2}, \{p_F^2/b\}\right) + O(b^{5/2}) \dots \right\}. \quad (34)$$

We note that these expansions have a common generic structure. The oscillatory terms are non-analytic in the expansion variable b and they all have the common form

$$b^p \zeta(-s, \{p_F^2/b\}), \quad (35)$$

where $p = s$ or $p = s + 1$ and $s = \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \dots$. The first few relevant functions are plotted in figure 2.

We note also that the leading term in the magnetization at zero temperature is oscillatory, which indicates an oscillatory *de Haas - van Alphen* behavior of the magnetization for small fields⁹.

9 The Magnetization at Finite Temperature

In order to study the persistence of the *de Haas - van Alphen* effect as we increase the temperature, we must consider the magnetization at non-zero temperature. According to the discussion of section 5, the magnetization at

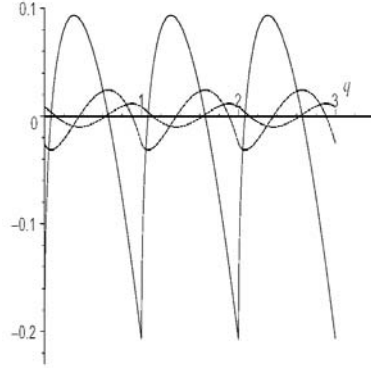


Figure 2. The functions $\zeta(z, \{q\})$, for $z = -1/2, -3/2, -5/2$ (large, medium and small amplitude, respectively).

finite temperature will be given by

$$\mathcal{M}(T, \mu, b) \simeq \int_{-\infty}^{\infty} \mathcal{M}_0(\mu + Tx, b) h(x) dx. \quad (36)$$

Approximating \mathcal{M}_0 by its leading term in Eq.(34) we find in the degenerate regime ($T \ll \mu$),

$$\mathcal{M}^{(\text{osc})}(T, \mu, b) \simeq -\frac{em^2 p_F^2}{2\pi^2 \mu} b^{1/2} \int_{-\infty}^{\infty} \zeta(-\frac{1}{2}, \{p_F^2/b + 2x \mu T/b\}) h(x) dx, \quad (37)$$

which shows that the parameter that controls the size of the oscillations is $\mu T/b$. In fact, one can use the Fourier expansion of the Hurwitz zeta function given in Eq.(31) to obtain the explicit series expansion

$$\mathcal{M}^{(\text{osc})}(T, \mu, b) \simeq -\frac{em^2 p_F^2 T}{\pi \sqrt{2b}} \sum_{r=1}^{\infty} \frac{\sin(2\pi r p_F^2/b - \pi/4)}{\sqrt{r} \sinh(4\pi^2 r \mu T/b)} + \dots \quad (38)$$

This result is the relativistic generalization of Landau's result^{10,11} for the non-relativistic electron gas. This generalization has been obtained before in Refs.^{3,4}. We see that the oscillations are damped out as soon as the parameter $\mu T/b$ becomes of order 10^{-1} , that is, when the temperature starts becoming of the order of the spacing between the Landau bands that start near to the Fermi surface, b/μ . For higher values of $\mu T/b$ it is the monotonic piece of the magnetization which dominates. This behavior is shown in figure 3.

For typical neutron stars, reasonable values of the physical parameters are $b \lesssim O(1)$, $\mu \simeq p_F \sim O(10^2)$, $T \gtrsim O(10^{-2})$. Hence $\mu T/b \gtrsim O(1)$ and the amplitude of the oscillatory magnetization is completely negligible.

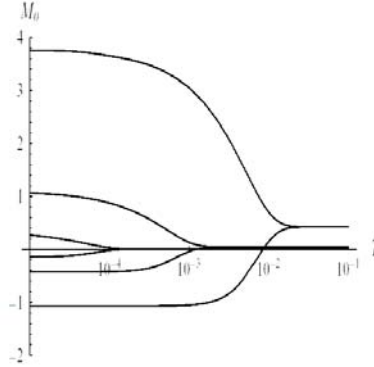


Figure 3. Upper and lower envelopes of the magnetization as a function of the temperature, for fixed chemical potential $\mu = 100$, and magnetic fields $b = 0.1, 1$ and 10 (lower, medium and upper curve, respectively).

10 Conclusions

We have developed a closed analytical approach to solve the thermodynamics of a free gas of electrons immersed in an uniform magnetic field of arbitrary magnitude. The method is completely relativistic and particularly useful in the case of a degenerate gas, a likely situation to be met in the interior of white dwarfs, neutron stars, and magnetars. A central role is played by one of the least known of the special functions of mathematical physics, the Hurwitz zeta function. Our work provides a unified derivation of several of the results found scattered in the literature. We reproduce the *de Haas-van Alphen* behavior of the magnetization in the relativistic gas and the dilution of it at high temperatures.

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