QUANTUM MAGNETIC COLLAPSE AND MAGNETAR INSTABILITY

A. PÉREZ MARTíNEZ, H. PÉREZ ROJAS
Instituto de Cibernética, Matemática y Física
La Habana, Cuba

H. MOSQUERA CUESTA
Centro Brasileiro de Pesquisas Físicas, Laboratório de Cosmologia e Física Experimental de Altas Energias
22290-180 Rio de Janeiro, RJ, Brazil

A relativistic degenerate neutron gas in equilibrium with a background of electrons and protons in a magnetic field exerts its pressure anisotropically, having a smaller value perpendicular than along the magnetic field. For critical fields the magnetic pressure may produce the vanishing of the equatorial pressure of the neutron gas. Taking it as a model for neutron stars, the outcome could be a transverse collapse of the star. This fixes a limit to the fields to be observable in stable neutron star pulsars as a function of their density. The final structure left over after the implosion might be a mixed phase of nucleons and meson condensate, a strange star, or a "black cigar", but no magnetar at all.

1 Introduction. Anisotropic pressures

We argue that a gas of neutral particles having an anomalous magnetic moment (as a model for neutron stars (NSs)), when placed an in extremely strong magnetic fields has a non linear (ferromagnetic) response to the external field, and is also unstable due to the vanishing of the transverse pressure for fields strong enough. In this phenomenon quantum effects play an essential role due to the coupling of the particles' spins to the microscopic field B seen by the particles (spin-polarization). This problem was studied for the electron gas in[1]. Here we want to explain in more details the arising of anisotropic pressures, which is an essential point in understanding the problem. We shall give at first general arguments and concentrate later in the specific calculations in the one-loop approximation for the thermodynamic potential of the star configuration.

To fix ideas, there is some external magnetic field which we name as H. This field induces a magnetization $\mathcal{M}$ in the medium (the neutron gas making up the star) defined by $H = B - 4\pi\mathcal{M}(B)$. Obviously, B is the microscopic field seen by an electric charge or magnetic dipole, and in most cases in what follows we refer to B as the external magnetic field, in the sense of determining the dynamics of the particles.

The idea of local pressure anisotropy, that is, the occurrence of unequal principal stresses is a natural consequence of the spatial anisotropy introduced by external fields, as it is evident to everybody to occur in atmospheric or ocean pressures, where a preferred direction of increasing (radial) pressure is due to the (approximately) centrally-symmetric gravity force. This anisotropy is observed for instance when considering a macroscopic sphere in any of these fluids. In a small neighborhood of any point inside it, the pressure seems to be isotropic, but actually it is not so, and it only does not change as we move on isobar surfaces. But as we move across them, it changes slightly due to the momentum added by the external gravity force field.

In the case of a gas of electrically charged particles in an external constant magnetic field B, in classical electrodynamics, it is the Lorentz force $\mathbf{F} = e\mathbf{v} \times \mathbf{B}/c$ the source of an asymmetry in the pressures parallel and perpendicular to $\mathbf{B}$. By writing $e\mathbf{v} = \mathbf{j}\Delta V$, where $\Delta V = dx_1 dx_2 dx_3$, calling $f_i = F_i/\Delta V$ as the i-th component of the force density, and substituting $\mathbf{j} = c\nabla \times \mathcal{M}$, one has

$$f_i = -(\partial_i \mathcal{M}_s)B_s + (\partial_s \mathcal{M}_i)B_s \tag{1}$$

multiplying by $\Delta V = dx_1 dx_2 dx_3$ and assuming $B_s = B\delta_{s3}$ and $\partial\mathcal{M}_i/\partial x_3 = 0$ (actually it is also $\mathcal{M}_i = \mathcal{M}\delta_{i3}$), only the first term in (1) remains as nonzero and one gets back an expression for the force. For the pressure perpendicular to the field it results $p_\perp = -\mathcal{M} \cdot \mathbf{B}$. This is a classical effect and obviously $p_\perp$ must be added to the usual kinetic isotropic pressure. As in classical electrodynamics $\mathcal{M}$ is opposite to $\mathbf{B}$, then $\mathcal{M} \cdot \mathbf{B} < 0$, and $p_\perp > 0$. However, the opposite case occurs when $\mathcal{M}$ is parallel to $\mathbf{B}$, which occurs in the quantum case. Note also that in[2] the concept of anisotropic pressure in the case of an electron gas in a magnetic field is discussed. That paper refers to an earlier expression for $\mathcal{T}_{\mu\nu}$ obtained by Canuto and Chiu[4]. In[2] it is argued that the anisotropy in pressures is cancelled because the Lorentz force pressure counterbalances the transverse pressure obtained by the quantum relativistic contribution. We observe first that if care is taken with the sign in front of $\mathcal{M}$, the Lorentz force pressure actually expresses the same sort of pressure anisotropy which occurs in both classical and quantum cases, since it is due to the spatial anisotropy introduced by the external field. Second, to consider at the same time both the quantum and classical contributions to magnetization, and to sum up them is, definitely, nonsense.

Note also that in paper [3] based in the use of the classical stress tensor, claims are made against our results in [1], [19]. The classical stress tensor is usually derived starting from the Lorentz force [7]. However, when dealing with a quantum gas, the expression for the Lorentz force is no longer valid, since in it the effect of the magnetic field on the spin is not taken into account (the Lenz law is valid). In a degenerate quantum gas the spin coupling to the magnetic field (*which obviously does not obey the Lorentz force equation*) plays a dominant role and the use of the classical stress tensor may lead to contradictory statements. Obviously, if spin is ignored, one obtain classical results, as those of equation 2 of [3]. These are in correspondence with the classical collapse case considered below.

2 The energy-momentum tensor

Based on more fundamental grounds, one may write the general structure of the energy-momentum tensor of a system in an external field in the same way as one can guess the general structure of other tensors, as the polarization operator tensors[5]. In an external field $F_{\mu\nu}$, in addition to the basic vectors four-velocity of the medium, u_μ, and particle momentum k_μ, we have two extra vectors $F_{\mu\nu}k_\nu, F^2_{\mu\nu}k_\nu$, to form a basis of independent vectors. From them we may build a set of basic tensors, which together with the tensors $\delta_{\mu\nu}$, $F_{\mu\nu}$, $F^2_{\mu\nu}$ serves as a basis in terms of which we can expand any tensor structure related to the particle dynamics, in particular, the energy-momentum tensor. But to eliminate tensor structures containing off-diagonal terms which would correspond to unwanted shearing stresses in the rest frame $u = (0, 0, 0, u_4)$, we exclude some of them, i.e. $k_\mu k_\nu$, $k_\nu F_{\mu\lambda}k_\lambda$, $F_{\mu\nu}$, $u_\mu k_\nu$, or any of its combinations. By following the arguments used in[5], we conclude that we are left in the present case with three basic tensors:$\delta_{\mu\nu}, F^2_{\mu\nu}, u_\mu u_\nu$. Thus, the structure of the energy-momentum tensor is expected to be of form

$$\mathcal{T}_{\mu\nu} = a\delta_{\mu\nu} + bF^2_{\mu\nu} + cu_\mu u_\nu \qquad (2)$$

where $a = p$, is the isotropic pressure term, $b = \mathcal{M}/B$ and $c = U + p$. In the present case the second of these tensors can be written in a simpler form as $F^2_{\mu\nu} = -B^2\delta^{\perp}_{\mu\nu}$.

The coupling of the spin dipole moment of neutrons in an external magnetic field B produces a loss of rotational symmetry of the particle spectrum (in what follows we will consider B along the x_3 axis). From the spectrum, which is expressed in terms of B, by following the standard methods of temperature quantum field theory, we obtain the thermodynamical potential density

$\Omega = \Omega(B)$, and from it all the thermodynamic properties of the system, in particular its magnetization, $\mathcal{M}(B) = -\partial\Omega/\partial B$. We may write then

$$\Omega = -\int \mathcal{M} dB - p_0, \tag{3}$$

where $p_0 = \Omega(0)$ is the zero field pressure term. One must emphasize that in the quantum relativistic case Ω depends on B nonlinearly. One finds that the dependence of the energy spectrum upon momentum is not rotational invariant. This fact determines a reduction of the symmetry of the otherwise isotropic thermodynamic properties of the system, as pressure, which is expected to be axially-symmetric for the reasons pointed out above. By using Green functions method, it is found that the energy-momentum tensor of matter in an external constant magnetic field obeys the general structure (2), (i.e., since neutrons are composed from charged quarks), and thus we have[1],

$$\mathcal{T}_{\mu\nu} = (T\partial\Omega/\partial T + \sum \mu_i \partial\Omega/\partial\mu_i)\delta_{4\mu}\delta_{4\nu} + 4F_{\mu\lambda}F_{\nu\lambda}\partial\Omega/\partial F^2 - \delta_{\mu\nu}\Omega, \tag{4}$$

where i run over the species involved. Below we will take $i = n, p, e$. Expression (4) in the zero field limit reproduces the usual isotropic energy-momentum tensor $\mathcal{T}_{\mu\nu} = p\delta_{\mu\nu} - (p+U)\delta_{4\mu}\delta_{\nu 4}$. From Eq.(4) the spatial components are $\mathcal{T}_{33} = P_3 = -\Omega, \mathcal{T}_{11} = \mathcal{T}_{22} = P_\perp = -\Omega - B\mathcal{M}$.

By writing $\mathcal{M} = (H - B)/4\pi$, one may write formally $\Omega = -\frac{1}{8\pi}B^2 + \frac{1}{4\pi}\int HdB - p_0$. We remind that as $\Omega \equiv F - G$, the last expression is consistent with what would be obtained in the classical non-relativistic case[6] where $F = F_0 + \int HdB/4\pi$ is the Helmholtz free energy and $G = F + \int \mathcal{M}dB + p_0 = G_0 + B^2/8\pi$ as the Gibbs free energy. Due to our definition of $\mathcal{M}$, our last term is given in terms of B and not in terms of H. As we have also $-\Omega - B\mathcal{M} = -\frac{1}{8\pi}B^2 + \frac{1}{4\pi}\int BdH + p_0$, we observe then that one can write $\mathcal{T}_{ij}$ as

$$\mathcal{T}_{ij} = \mathcal{S}(B)_{ij} - \mathcal{T}^M_{ij}(B, H), \tag{5}$$

where $\mathcal{S}(B)_{ij} = \frac{1}{4\pi}[B_iB_j - \frac{1}{2}(B^2)\delta_{ij}]$ is the Maxwell stress tensor for the microscopic field B, and $\mathcal{T}^M_{ij}(B) = \frac{1}{4\pi}[H_iB_j - (\int BdH)\delta_{ij}]$ is the Minkowski tensor for nonlinear media. Note that it reduces to the usual expression for H depending linearly on B[7]. If $H = 0$, $B = 4\pi\mathcal{M}$ and $\mathcal{T}_{ij} = \mathcal{S}(B)_{ij}$.

Once we assume the existence of local anisotropy, two main questions arise: a) where does it come from? b) in what manner the overall properties of both local isotropy and anisotropy relate in between? These issues were extensively reviewed by Herrera and Santos[8] in a general relativistic approach, who present several physical mechanisms for its origin, in both extremely low and very high density systems, which may include astrophysical compact objects. In the case of highly dense systems, it was pointed out that "exotic" phase transitions could occur during gravitational collapse, the problem we are concern herewith in this *Talk*. More recently, Mak and Harko[9] present a class of exact solutions of Einstein's equations, corresponding to anisotropic stellar configurations which can describe realistic neutron stars.

Note, nonetheless, that Herrera and Santos[8,9] did not take into account the dynamical effects of the strong (and superstrong) magnetic fields supposed to exist in the core of canonical neutrons stars, see for instance[1,10,11,12]. Thence, the contention of this work is to address this open issue. The novel results obtained point out to the occurrence of new processes in the (relativistic) astrophysics of compact objects that were not manifest in previous papers.

3 Classical and Quantum Collapses

In Ref.[1] we found that a relativistic degenerate electron gas placed in a strong external magnetic field B is confined to a finite set of Landau quantum states. As the field is increased, the maximum Landau quantum number is decreased, favoring the arising of a paramagnetic or ferromagnetic response through a positive magnetization $\mathcal{M}$, up to the case in which only the ground state is occupied. The gas then becomes one dimensional, and in consequence the pressure transverse to the field vanishes for fields $B = -\Omega/\mathcal{M}$. Thus, the electron gas becomes unstable due to the decrease of the transverse pressure for fields strong enough, and the outcome is a collapse.

For neutrons, the magnetization is always positive, and (as in the electron gas case) is nonlinear, leading to a sort of ferromagnetic behavior. For fields strong enough the pressure transverse to the field, $P_{\perp} = -\Omega - B\mathcal{M}$, is considerably decreased and may vanish. If we assume that extremely magnetized NSs, or magnetars[13], have fields $H \sim 10^{15}$ G, and that inside the star B increases by following a dipole law $B(r') = B_{surf}/r^3$, we expect near its surface fields from $10^{16} - 10^{17}$G up to values of order 10^{20}G in its core, where it is maintained self-consistently, $(H = 0)$. For fields of this order of magnitude superdense matter composed of neutral particles having a magnetic moment may undergo a transverse collapse since $P_{\perp}$ vanishes. As discussed below, the resulting object (depending on the macroscopic properties of the imploding NS) may be a hybrid,[14] a strange star (through an induced phase transition[14,15]) but no any magnetar. We do not enter into the details of the mechanism for producing the self-consistent field, but a possible source is a condensate of the ρ vector meson or through neutron spin-spin ferromagnetic coupling (see below), or even by diquark condensation.

In the classical and diamagnetic cases $\mathcal{M} < 0$, then $H > B$ and $P_\perp > P_3$. This leads to the oblatening effect described above. But for the strong quantum case, the coupling of the spin magnetic dipole with B plays the main role, and $\mathcal{M} > 0$, leading to the para- or ferromagnetic effects. The situation then is reversed and $P_\perp$ is smaller than P_3 in the amount $B\mathcal{M}$ and it vanishes for $\frac{1}{8\pi}B^2 = \frac{1}{4\pi}\int BdH + p_0$, leading conversely, to a prolate configuration.

In classical electrodynamics[16] it is suggested that the pressure is given by the sum of the Maxwell stress tensor $\mathcal{S}_{\mu\nu}$ plus an isotropic pressure p_0 term. In the case of a constant magnetic field parallel to the x_3 axis, the total pressure tensor reads $\mathcal{T}_{ij} = p_0\delta_{ij} + \mathcal{S}_{ij}$ or $P_3 = p_0 - H^2/8\pi$ and $P_\perp = p_0 + H^2/8\pi$.

Obviously, this anisotropic pressure is to be compensated by the gravitational pressure by distributing also anisotropically matter in the body, leading to isobaric spheroidal or ellipsoidal surfaces, an effect which is similar to the oblateness of the Sun, Earth and planets due to the effective decrease of the transverse gravitational force-induced pressure by the centrifugal force[17]. Thus, in the classical case, for the extreme limit of flattening, one gets $P_3 = 0$ and $p_0 = B^2/8\pi$, and thence the body would collapse as a disk or a ring perpendicular to the field. Starting from general relativistic considerations it has been reported recently[10] the existence of a maximum field for having stationary configurations. This field induces a toroidal configuration, which is topologically equivalent to a ring. In the quantum case, for degenerate fermions, as $\mathcal{M} > 0$, it is $P_\perp = -\Omega - B\mathcal{M}$ which is decreased by increasing B. As the NS is in equilibrium under the balance of neutron and gravitationally-induced pressures, the body stretches along the direction of the magnetic field. Thus, for any density there are values of the field B strong enough, such that these pressures cannot compensate each other leading to a collapse perpendicular to the field for $P_\perp = 0$. This implosion is driven by the same mechanism described in[1] for electrons, where the transverse pressure vanishes when confined to the Landau ground state. We do not enter in the quantitative study of this collapse here, which would lead to a special sort of hybrid or strange stars,[14,15] or black cigars.

Our previous considerations are approximation-independent. For going to an specific model, we shall start in the next section from the free particle spectrum for neutral particles in a magnetic field.

4 Neutron Gas in a Magnetic Field

In dealing with a quantum gas in an external field we shall assume that the currents, which are classical sources of the field, are localized in a specific (relatively small) volume of space, out of the region occupied by the quantum system, or that the source of the external field H is not classical but quantum, as it may be produced from ferromagnetic coupling of spin of nucleons or Bose-Einstein condensation of charged particles having a magnetic moment, as some mesons and diquarks. Formally the problem is equivalent as to have the quantum gas placed among the poles of a big magnet which creates a magnetic field H.

As in the case of the electron-positron gas[1], the basic dynamics in our present case is described by the Dirac equation in the external field (in place of the Lorentz force), leading to the energy eigenvalue spectrum[18]. This energy depends on the microscopic magnetic field B through some interaction field in the initial Lagrangian. As in standardized models, the system is composed by fermions (in addition to neutrons, a certain amount of electrons and protons is demanded by Pauli's Principle) and bosons (photons, mesons) as quanta of the fields. The external field contributes with virtual particles, expressed through the Euler-Heisenberg vacuum terms arising in the regularization of the quantum vacuum terms appearing in the calculation of the basic statistical quantity, the thermodynamical potential Ω, which is built from the particle spectrum. The thermodynamical potential is the sum of two terms, $\Omega = \Omega_{st} + \Omega_0$, the finite statistical term Ω_{st} plus the vacuum field contribution Ω_0, which is divergent. In the process of regularization, it absorbs the classical field energy $B^2/8\pi$. The only way the field B can influence the particle dynamics, and in consequence the pressure, is through its coupling with them. This coupling is expressed by appropriate terms in the field Lagrangian which will appear later on in the particle spectra. Note in addition that if the coupling constant is turn to zero, the particles would not feel any pressure coming from the external field.

For free neutrons in a magnetic field we get the eigenvalues

$$E_n(p, B, \eta) = \sqrt{p_3^2 + (\sqrt{p_\perp^2 + m_n^2} + \eta q B)^2}, \tag{6}$$

where p_3, $p_\perp$ are respectively the momentum components along and perpendicular to the magnetic field B, $q = 1.91 M_n$, where M_n is the nuclear magneton, $\eta = 1, -1$ are the σ_3 eigenvalues corresponding to the two orientations of the magnetic moment (parallel and antiparallel) with respect to the field B. The expression (6) shows manifestly the change of spherical to axial symmetry with regard to momentum components.

The partition function $\mathcal{Z} = \mathrm{Tr}(\rho)$ is obtained from the density matrix describing the model $\rho = e^{-\beta \int d^3x(\mathcal{H}(\mathbf{x}) - \sum \mu_i N_i)}$. Here μ_i, $i = 1, 2, 3$ are the chemical potentials associated to lepton, baryon and electric charge conservation, (where $\mu_n = \mu_2$, $\mu_p = \mu_2 + \mu_3, \mu_e = \mu_1 + \mu_3$ and $\mu_v = \mu_1$). The thermodynamical potential can be written as $\Omega = -\beta^{-1} ln\mathcal{Z}$. If we assume the temperature $T = \beta^{-1} \sim 10^{-7}$ erg, degeneracy for fermions is guaranteed if $T \ll \mu_i - m_i$. Usually the eigenvalues of $\mathcal{H}$ contain the contribution from neutrons, protons, electrons and some meson species, and the densities are $N_i = -\partial\Omega/\partial\mu_i$, where $i = n, p, e...$ We name $\Omega = \sum_i \Omega_i$, $\mathcal{M} = \sum_i \mathcal{M}_i$ the total thermodynamical potential and magnetization, respectively.

A standard procedure is to work in the mean field approximation in which the meson fields σ, ρ, ω are taken as constant, as done in papers,[11,12] through which the mass spectrum of baryons is corrected and strong repulsive interactions between them is found. However, for simplicity we will keep the spectra in the tree approximation to obtain the one-loop approximation for Ω, and neglect the statistical contribution from meson terms in Ω as compared with those of fermions (since for them $m_i\beta \sim 10^3 - 10^4$) except for fields $B \leq B_{c\rho} = m_\rho^2/e \approx 10^{20}$G, since the contribution of ρ vector meson condensate to $\mathcal{M}$ becomes relevant and in analogy of $W^{\pm}$-s[1], leads to a self-consistent spontaneous magnetization $B = 4\pi\mathcal{M} = 2\pi e N_\rho \sqrt{m_\rho^2 - eB}$. This prevents vacuum instability for $B > B_{c\rho}$, but for such fields the magnetic pressure: $\int \mathcal{M}dB - \mathcal{M}B = -B^2/8\pi$ overwhelms the kinetic pressure term p_0 (of order 10^{36} dynes/cm^2) leading to $P_\perp < 0$, and the star is definitely unstable.

5 The thermodynamical potential

One can obtain an expression for the neutron thermodynamic potential in the one-loop approximation as $\Omega_n = \Omega_{sn} + \Omega_{0n}$, where

$$\Omega_{sn} = -\frac{1}{4\pi^2\beta} \int_0^\infty p_\perp dp_\perp dp_3 \ln\left[f^+(\mu_n, \beta) f^-(\mu_n, \beta)\right], \qquad (7)$$

where $f^{\pm}(\mu_n, \beta) = (1 + e^{-(E_n \mp \mu_n)\beta})$ accounts respectively for the contribution of particles and antiparticles (in the degenerate case, only particles contribute to Ω).

The neutron vacuum term (see Ref.[19]) has an Euler-Heisenberg-like form as

$$\Omega_{Vn} = \frac{1}{4\pi^2}\int_0^\infty dy y^{-3} e^{-(m_n^2+q^2B^2)y}[\cosh(qBmy) - 1 - (qBmy)^2/2!]$$

$$+\frac{qB}{2\pi^2}\int_0^\infty dy y^{-2}\int_0^\infty dw e^{-[(w+m_n)^2+q^2B^2]y} \quad (8)$$

$$[\sinh(2qB(w+m_n)y) - (2qB(w+m_n)y) + (2qB(w+m_n)y)^3/3!]$$

It can be shown[19] that the more significant term in (8) is the first one, which for fields of order 10^{17} G leads to $\Omega_{Vn} \sim 10^{30}$ erg·cm^{-3} and is negligible small as compared with Ω_{sn} up to $B \sim 10^{18}$ G. Thus, we neglect it in a first approximation in what follows.

Having an equation relating the chemical potentials, and demanding conservation of both baryonic number $N_n + N_p = N_B$ and electric charge $N_p + N_e = 0$, in principle one may solve exactly the problem in terms of the external field as a parameter. However, we shall focus our discussion on the properties of the equation of state. (Our expressions for the spectra and densities of neutrons and protons are similar to those of on neutron gas in a magnetic field[20], but we get different equations of state). In place of using (7), to calculate the thermodynamical quantities in the degenerate ideal gas it is simpler to calculate first the density $N_n = (2\pi)^{-2}\int dp_3 p_\perp dp_\perp \theta(\mu_n - E_n)$, where $\theta(z)$ is the step function Fermi distribution, and E_n is Eq.(6). From N_n we obtain $\Omega = -\int N_n d\mu_n$. We have thus

$$N_n = N_0 \sum_{\eta=1,-1}\left[\frac{f^3}{3} + \frac{\eta y(1+\eta y)f)}{2} - \frac{\eta y x^2}{2}s\right], \quad (9)$$

where $x = \mu_n/m_n$, ($m_n(x-1)$ is the usual Fermi energy), and $y = qB/m$. We define the functions $f \equiv f(x,\eta y) = \sqrt{x^2 - (1+\eta y)^2}$, $s \equiv s(x,\eta y) = (\pi/2 - \sin^{-1}(1+\eta y)/x)$, $L \equiv L(x,\eta y) = \ln(x + f(x,\eta y))/(1+\eta y)$. For Ω_n we get

$$\Omega_n = -\Omega_0 \sum_{\eta=\pm1}\left[\frac{xf^3}{12} + \frac{(1+\eta y)(5\eta y - 3)xf}{24} + \frac{(1+\eta y)^3(3-\eta y)}{24}L - \frac{\eta y x^3}{6}s\right] \quad (10)$$

while for the magnetization, given as $\mathcal{M}_n = -\partial\Omega_n/\partial B$, we have

$$\mathcal{M}_n = -\mathcal{M}_0 \sum_{\eta=\pm 1} \eta \left[\frac{(1-2\eta y)xf}{6} - \frac{(1+\eta y)^2(1-\eta y/2)}{3} L + \frac{x^3}{6} s \right] ,$$

where $N_0 = m_n^3/4\pi^2 \sim 2.04 \times 10^{39}$, $\Omega_0 = N_0 m_n \sim 3.0 \times 10^{36}$, and $\mathcal{M}_0 = N_0 q \sim 2.92 \times 10^{16}$, and one can write $\mathcal{M}_n = \mathcal{M}_n^+(\eta = -1) - \mathcal{M}_n^+(\eta = +1)$, and obviously, $\mathcal{M}_n \geq 0$. In the limit $B = 0$ Eq.(9) and Eq.(10) reproduce the usual density and thermodynamic potential of a relativistic Fermi gas at zero temperature. We see that $\mathcal{M}$ is a nonlinear function of B and we have the magnetic response as ferromagnetic (a fully ferromagnetic response would require to include also the spin-spin coupling contribution.

If we include both the normal and the anomalous magnetic moment for electrons, one can give a common formula for the spectrum of electrons and protons in the external field B as[18]:

$$E_{e,p} = \sqrt{p_3^2 + (\sqrt{2eBn + m_{e,p}^2} + \eta q_{e,p} B)^2}, \tag{11}$$

where $q_e = \alpha e/4\pi m_e$, $q_p = 2.79 M_n$. For neutrons, the critical field at which the coupling energy of its magnetic moment equals the rest energy is $B_{cn} = 1.57 \times 10^{20}$. For protons $B_{cp} = 2.29 \times 10^{20}$G, while for electrons, $B_{ce} = m_e^2/e \sim 10^{13}$G is the usual QED critical field. It was shown in [1] that for fields of that order and densities around 10^{30} cm^{-3} all electrons are in the Landau ground state, and the system show the instability which arises from the vanishing of the transverse pressure. For densities and magnetic fields above these critical values, the stability of the electron gas is doubtful (see Ref. [19] and references quoted therein).

For the proton gas, by defining $x_p = \mu_p/m_p$, $y_p = q_p/m_p$, $b = 2e/m_p^2$, then $y_p = 2.79e/2m_p^2$. We name also $g \equiv g(x_p, B, n) = \sqrt{x_p^2 - h(B,n)^2}$ and $h \equiv h(B,n) = (\sqrt{bBn+1} + \eta y_p B)$. Thus for the proton thermodynamical potential we get

$$\Omega_p = -\frac{eBm_p^2}{4\pi^2} \sum_n \sum_{\pm\eta} \left[x_p g - h^2 \ln(x_p + g)/h \right] , \tag{12}$$

and for its density: $N_p = \frac{eBm_p}{2\pi^2} \sum_n \sum_{\pm\eta} g(x_p, B, n)$. The magnetization is then given by

$$\mathcal{M}_p = \frac{em_p^2}{4\pi^2} \sum_n \sum_{\pm\eta} \left\{ x_p g - \left[h^2 + (\eta y_p + (bn/2\sqrt{bBn+1})) \right] \times \ln\left(x_p + g\right)/h \right\} ,$$

where the coefficients of these formulae are $N_0 = em_pB/2\pi^2 \sim 4.06 \times 10^{19}B$, $\Omega_0 = N_0 m_p B \sim 6.1 \times 10^{16}B$, and $\mathcal{M}_0 = N_0 m_p = \Omega_0/B$. The maximum occupied Landau quantum number n may be given as $n_{max} = (x_p - \eta y_p B)^2 - 1/bB$.

For $B \ll B_{cp}$, so that $y_pB \ll 1$, and $x_p \geq 1$, one can take approximately $n_{max} \sim (x_p^2 - 1)/bB$, and for fields large enough $n_{max} = 0$. As $x_p \sim x_n$, the proton density decreases with increasing B, favoring the inverse beta decay. For fields $B \sim m_p/q_p$ and $x_p \gg 1$, $n_{max} \geq 1$, and thus large Landau numbers are again occupied. However, for $x_n, x_p \geq 1$ the dominant longitudinal pressure, density and magnetization comes from the neutron gas.

6 Condition for zero transverse pressure and collapse

We observe at first that the limiting case: $P_\perp = 0$ can be figured out from the spectrum described by Eq.(6), since the contribution from $\eta = -1$ terms is dominant. Then, if one approximates the second term inside the square root as $[m_n + (p_\perp^2/2m_n - qB)]^2$, the term in parenthesis accounts for the transverse kinetic energy. This term decreases as B increases. By taking $p_\perp \sim p_F$, where p_F is the Fermi momentum, and equating the parenthesis to zero, we obtain a functional relation between μ_n and B leading to the vanishing of the transverse kinetic energy density, (the spectrum behaving in that region as that for unidimensional motion parallel to B, i.e, $E_n \simeq \sqrt{p_3^2 + m_n^2}$), and in consequence the transverse pressure vanishes. A more accurate result is obtained, however, from the equation: $\mathcal{T}_\perp = 0$.

In Figure 1 we have drawn the equation $P_{\perp n} = -\Omega - B\mathcal{M} = 0$ in terms of the variables N_n, B_{core}. We observe that there is a continuous range of values of N_n, and B_{core} for which the collapse takes place. The transverse compression of the whole mass of the star, due to flux conservation, lead to an increase of B and the mechanism of collapse is enhanced. In our previous calculations we have not considered spin-spin coupling, which would lead to Heisenberg ferromagnetism and would increase the magnetization to $\mathcal{M}' = \kappa\mathcal{M}$, where κ is the internal field parameter. If $\kappa >> 1$ our previous estimate is largely exceeded. As a rough approximation, we assume the exchange interaction J of order of the repulsive forces among neutrons [11,12] as few hundreds of MeV, and the number of nearest neighbors as $z \sim 10$. By dividing their product by the dipole interaction energy, one gets in the core $\kappa \simeq zJ/N_n q_n^2 \sim 10^4$, which means a spontaneous magnetization $B \sim \kappa\mathcal{M} = 10^{20}$G. This means our previous calculations are a lower bound of the more realistic case, and the vanishing of $P_\perp$ is expected to occur safely at values of B smaller than those

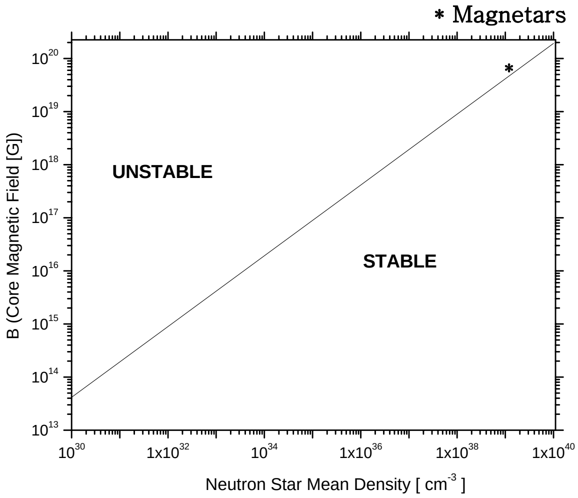

Figure 1. The instability condition: $P_\perp = 0$. A neutron star having a configuration such that its dynamical stage would be represented by a point above the central curve in this plot would be unstable to transversal collapse, since $P_\perp \leq 0$ there.

depicted in Fig.1. As above-mentioned, values of $B \sim 10^{20}$ G would be needed to keep self-consistently the field B throughout ρ condensation. For such fields the magnetic coupling of quarks with B becomes of the order of their binding energy through the color field producing a deconfinement phase transition leading to a quark (q)-star, a pressure-induced transition to uds-quark matter via ud-quark condensates, as discussed in[15,14].

7 Magnetar Formation and Stability

Next we briefly review the standard theory of magnetars and show why they cannot survive after reaching the claimed extremely strong magnetic fields (ESMF). We then present prospectives for a hybrid or strange star to appear as a remnant of the quantum magnetic collapse of a NS. According to Duncan and Thompson[13], NSs with very high dipole magnetic field strength, $B_D \sim [10^{14} - 10^{15}]$ G, may form when (classical) conditions for a helical dynamo action are efficiently met during the seconds following the core-collapse in a supernova (SN) explosion[13]. A newly-born NS may undergo vigorous convection during the first 30 s following its formation[21]. If the NS spins (differentially) sufficiently fast ($P \sim 1$ ms) the conditions are created for the $\alpha - \Omega$ dynamo action to be built (which may survive depletion due to turbulent diffusion). Collapse theory shows that some pre-supernova stellar cores could endow enough spin so as to rotate near their Keplerian equatorial velocity, the break-up spin: $\Omega_K \geq ([\frac{2}{3}]^3 G_N M/R^3)^{1/2}$, after core bounce. Under these conditions, fields as large as $B \sim 10^{17}(\frac{P}{1\text{ms}})$G may be generated as long as the differential rotation is dragged out by the growing magnetic stresses[23,13]. For this process to efficiently operate the ratio between the spin rate (P) and the convection overturn time scale (τ_{con}), the Rossby number R_0, should be ≤ 1 ($R_0 \gg 1$ should induce less effective mean-dynamos[13]). In this case an ordinary dipole $B_{sat} \sim [10^{12} - 10^{13}]$G may be built by incoherent superposition of small dipoles of characteristic size $\lambda \sim [\frac{1}{3} - 1]$km and $B_{sat} = (4\pi\rho)^{1/2}\lambda/\tau_{con} \simeq 10^{16}$G. At such fields, the huge rotational energy of a $f \geq 1$kHz NS is leaked out via *magnetic braking*, and an enormous energy is injected into the SN remnant which may explain the power of a *plerion.*

As shown above, at the end of the SN core collapse we are left with a NS with an ESMF strength and a huge matter density $\rho \sim [10^{14} - 10^{12}]$ gcm^{-3}. As illustrated in Figure 1, those are the conditions for the quantum instability to start to dominate the dynamics of the young neutron star pulsar. At this stage, the magnetic pressure inwards may overpass the star's energy density at its equator and the collapse becomes unavoidable. As the collapse proceeds, higher and higher densities are reached till the point the supranuclear density may reverse the direction of implosion. A hybrid or strange star may form. ¿From that moment, the sound wave generated at the core bounce builds itself into a shock wave travelling through the star at $V_{SW} \sim c/\sqrt{3}$ kms^{-1}. Although the ESMF strength could be quite large as the collapse advances, the huge kinetic energy ($E \sim 10^{51}$ erg, the mean energy obtained in simulations of SN driven by the *prompt shock*[21]) carried away by the shock wave may counterbalance it, and even surpass it, i.e., its *ram* pressure will equal the magnetic pressure:

$$\rho_{eject} V_{SW}^2 \geq \frac{B^2}{8\pi\mu_0}\left(\frac{R}{r_A}\right)^6 , \tag{13}$$

at the Alfvén radius:

$$r_A = \left(\frac{2\pi^2}{G\mu_0^2}\right)^{1/7} \left[\frac{B^4 R^{12}}{M\dot{m}^2}\right]^{1/7} , \qquad (14)$$

(see further details in[22]). Then the ESMF lines are pushed out, and finally broken, from $r_A \sim 80$ km onwards, into the SN remnant surroundings as a violent explosion that dissipates a large part of the magnetic flux ($\Phi \sim B^2 r_A^2$) trapped inside the magnetar magnetosphere. This is analogous to the mechanism operating during a solar flare or a coronal mass-ejection, where the very high B in the "Sun-Spot" is drastically diminished (see also Kluźniak and Ruderman 1998[23]). Although the process is quite fast, the large amount of matter ejected from the star at such large velocities drains out the dipole field of the remnant below the quantum electrodynamic limit of $B_{QED} \sim 10^{13}$G. Since all the differential rotation has been dragged up to build up the ESMF, then nothing else remains to make it to grow to its pre-collapse value. Thus no such ultra high B should reappear. We may be left with a sub-millisecond strange star[15] or a hybrid star[14] with "canonical" field strength.

8 Conclusions

We conclude by claiming that if a degenerate neutron gas is under the action of a superstrong magnetic field $B \leq B_c$, for values of the density typical of NS, its transverse pressure vanishes, the outcome being a transverse collapse. This phenomenon establishes a limit to the magnetic field expected to be observable in a neutron star pulsar, as a function of its density, and suggests a possible end in the evolution of highly magnetized neutron stars as a mixed phase of nucleons and ($\pi^{\pm}, \pi^0, K^{\pm}, K^0, \bar{K}^0, \sigma, \rho^{\pm}, \omega$) meson condensate, a strange star or a black cigar, but no any magnetar.

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