

VACUUM PRESSURES AND ENERGY IN A STRONG MAGNETIC FIELD

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We study vacuum in a strong magnetic field. It shows a nonlinear response, as a ferromagnetic medium. Anisotropic pressure arises and a negative pressure is exerted in the direction perpendicular to the field. The analogy of this effect with the Casimir effect is analyzed. Vacuum transverse pressure is found to be of the same order of the statistical pressure for $B \sim 10^{15}G$ and $N \sim 10^{33} \text{electrons/cm}^3$. Vacuum interaction with the field is studied also for $B \sim 10^{16}G$ and larger, including the electron anomalous magnetic moment. We estimate quark contributions to the vacuum behavior.

1 Introduction

There is an analogy among certain boundary conditions and the effect of external fields. Some boundary conditions lead to new physical effects as it is for instance the anisotropic box created by two parallel metallic plates of length a separated by a distance d , placed in vacuum. For $d \ll a$ an attractive force appears in between the plates, as produced by the zero point vacuum energy of electromagnetic modes, leading to the well-known Casimir effect^{1,2}.

The Casimir effect shows that if one breaks the symmetry in a region of space, the energy E_V of the vacuum modes results distributed anisotropically. We may consider the vector \mathbf{d} perpendicular to the plates as characterizing the symmetry breakdown. Only vacuum modes of momentum

$$p_{nd} = \frac{2\pi n\hbar}{d}, \quad n = 1, 2, \dots$$

in the direction of symmetry breakdown are allowed inside the cavity; then

$$E_V = \sqrt{p_1^2 + p_2^2 + p_{nd}^2}.$$

A negative pressure dependent on d arises inside the plates, and perpendicular to them.

On the other hand, it is known that in an external magnetic field B , the momentum of an electron (positron) in the direction perpendicular to the field is quantized $p_{\perp} = \sqrt{2eBn}$. It is interesting to inquire then if an effect similar

to the Casimir one is produced by the zero point electron-positron energy of vacuum in an external constant magnetic field.

2 Vacuum Properties in a Constant Magnetic Field

The electron-positron zero point vacuum energy in an external electromagnetic field was obtained by Heisenberg and Euler³. In the specific case of an external constant magnetic field, this expression results as the pure vacuum term contribution when calculating the tadpole term of the thermodynamic potential Ω in a medium⁵. In this calculation one starts from the energy eigenvalues of the Dirac equation for an electron (positron) in a constant magnetic field B ,

$$\varepsilon_n = \sqrt{p_3^2 + m^2 + 2eBn}, \quad (1)$$

where $n = 0, 1, 2, \dots$, are the Landau quantum numbers, p_3 is the momentum component along the magnetic field (we consider B parallel to the third axis) and m is the electron mass. The general expression for Ω contains two terms

$$\Omega = \Omega_{ST} + \Omega_V. \quad (2)$$

The first one is the quantum statistical contribution which vanishes in the limit of zero temperature and zero density. The second term accounts for the zero-point energy: it contains the contribution coming from the virtual electron-positron pairs created and annihilated spontaneously in vacuum and interacting with the field B . In the one loop approximation, where no radiative corrections are considered, it has the expression

$$\Omega_V = -\frac{eB}{4\pi^2} \sum_{n=0}^{\infty} \alpha_n \int_{-\infty}^{\infty} dp_3 \varepsilon_n, \quad (3)$$

with $\alpha_n = 2 - \delta_{n0}$. As can be observed, the quantity in Eq. (3) is divergent. After regularization it leads to the Euler-Heisenberg expression

$$\Omega_V = \frac{e^2 B^2}{8\pi^2} \int_0^{\infty} e^{-m^2 x/eB} \left[\frac{\coth x}{x} - \frac{1}{x^2} - \frac{1}{3} \right] \frac{dx}{x}. \quad (4)$$

We observe that the vacuum term regularization demands the addition of a negative infinite term proportional to B^2 which absorbs the classical energy term $B^2/8\pi$. The vacuum thermodynamic potential is actually negative, however, this is not any problem if we interpret it according to the general

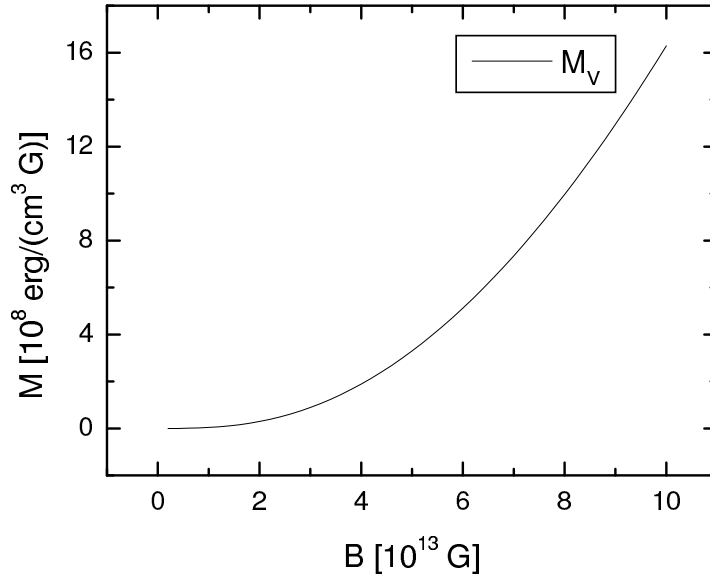


Figure 1. Vacuum magnetization \mathcal{M}_V for magnetic fields $B \sim 10^{13}G$.

energy-momentum tensor expression in a constant magnetic field (see Ref.⁴) for the case of zero temperature and zero chemical potential,

$$\mathcal{T}_{0\mu,\nu} = 4F_{\mu\rho}F_{\nu\rho}\partial\Omega_V/\partial F^2 - \delta_{\mu\nu}\Omega_V, \quad (5)$$

leading to a positive pressure $P_{V3} = -\Omega$ along the magnetic field B , and to $P_{V\perp} = -\Omega - B\mathcal{M}$ in the direction perpendicular to the field, where $\mathcal{M}_V = -\frac{\partial\Omega_V}{\partial B}$ is the magnetization, which is obtained from (4) as

$$\mathcal{M}_V = -\frac{2\Omega_V}{B} - \frac{em^2}{8\pi^2} \int_0^\infty e^{-m^2 x/eB} \left[\frac{\coth x}{x} - \frac{1}{x^2} - \frac{1}{3} \right] dx. \quad (6)$$

It is easy to see that the magnetization (6) is a positive quantity. Moreover, it has a non-linear dependence on the field B , as is shown in Fig. 1. That is why we state that the vacuum has ferromagnetic properties, although in our present one-loop approximation we do not consider the spin-spin interaction between virtual particles.

Concerning the transverse pressure, $P_{V\perp} = -\Omega - \mathcal{M}B$, we get

$$P_{V\perp} = \Omega_V + \frac{m^2 e B}{8\pi^2} \int_0^\infty e^{-m^2 x/eB} \left[\frac{\coth x}{x} - \frac{1}{x^2} - \frac{1}{3} \right] dx. \quad (7)$$

$P_{V\perp}$ is negative and it may lead to some effects for small as well for high fields.

3 The Low Energy Limit $eB \ll m^2$

Fields currently achieved in laboratories are very small if compared with the critical field $B_c = m^2/e \sim 10^{13}$. When this limit condition holds $B \ll m^2/e$, one can write,

$$\Omega_V \approx -\frac{(eB)^4}{360\pi^2 m^4}, \quad (8)$$

and in consequence

$$P_{V\perp} \approx -\frac{(eB)^4}{120\pi^2 m^4}. \quad (9)$$

In usual units it reads

$$P_{V\perp} \approx -\frac{\pi^2 \hbar c}{120 b^4}, \quad (10)$$

where the characteristic parameter $b(B)$ is

$$b(B) = \frac{2\pi\lambda_L^2}{\lambda_C}. \quad (11)$$

Here, λ_L is the magnetic wavelength

$$\lambda_L = \sqrt{\frac{\hbar c}{2eB}} \quad (12)$$

and λ_C is the Compton wavelength $\lambda_C = \frac{\hbar}{mc}$. It is easy to see that λ_L^2 coincides with the area corresponding to one magnetic quantum flux $\Phi_o = \frac{\hbar c}{2e}$. Pressure then is a function of the field dependent parameter $b(B)$, which is determined by the ratio between the Compton wavelength and the ‘‘one fluxon area’’.

The expression for the transverse pressure (10) looks similar to the expression for the negative pressure due to the Casimir effect between parallel metallic plates ¹

$$P_C = -\frac{\pi^2 \hbar c}{240 d^4}. \quad (13)$$

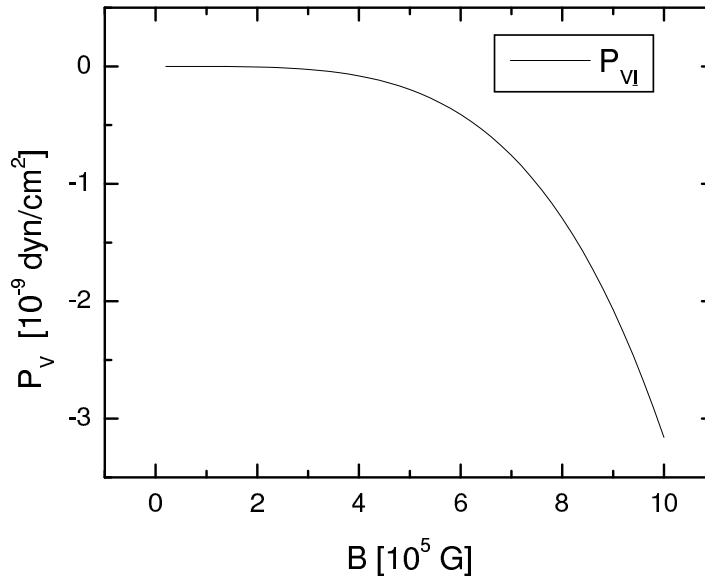


Figure 2. Vacuum transverse pressure for small fields $B \sim 10^5 G$.

$P_{V\perp}$ is then a Casimir like pressure. For small fields, of order $10 - 10^3 G$, it is negligible as compared with the usual Casimir pressure. But for larger fields, e.g. for $B \sim 10^5$ G it becomes larger; one may obtain then pressures up to $P_{V\perp} \sim 10^{-9} \text{ dyn/cm}^2$ (Fig. 2). For a distance between plates $d = 0.1 \text{ cm}$, gives $P_C \sim 10^{-14} \text{ dyn/cm}^2$, i.e., five orders of magnitude smaller than $P_{V\perp}$. This suggests that vacuum interaction with the magnetic field may produce observable effects for fields, which can be realized in nature or in terrestrial laboratories.

4 Vacuum Pressure for Strong Fields

At this point, thinking in possible astrophysical consequences, it is useful to analyze the behavior of transverse vacuum pressure for large magnetic fields, i.e., for fields of the order of the critical one $B_c = m_e^2/e \sim 10^{13} G$ and larger. Fields of these orders can be generated due to gravitational and rotational effects in stellar objects⁸, where the electron-positron gas plays an

important role. We use a model of white dwarf, in which the main contribution to thermodynamic magnitudes comes from the electron sector, described as a degenerate quantum gas. There is also a nuclei background, which compensates the electrical charge, but it behaves like a classical gas, and leads to quantities negligible small as compared with those of the electron gas.

The statistical contribution of electrons to the transverse pressure, in the degenerate limit, is

$$P_{ST\perp} = \frac{2(eB)^2}{\pi^2} \sum_{n=1}^{n_\mu} n \ln \frac{\mu + \sqrt{\mu^2 - m^2 - 2eBn}}{\sqrt{m^2 + 2eBn}}, \quad (14)$$

where n is the Landau state number, and μ is the chemical potential, related to the electron density through the expression

$$N = \frac{eB}{2\pi^2} \sum_{n=0}^{n_\mu} \alpha_n \sqrt{\mu^2 - m^2 - 2eBn}, \quad (15)$$

and n_μ is an integer

$$n_\mu = I\left(\frac{\mu^2 - m^2}{2eB}\right). \quad (16)$$

In the extreme case of sufficiently strong magnetic field $2eB > \mu^2 - m^2$, all the electron system is in the Landau ground state $n = 0$. But it is easy to note that this state does not contribute to the statistical pressure. In other words, when the electrons are confined to the Landau ground state, the total transverse pressure is equal to the vacuum one and is negative. It causes system instability. It means that there is a limit value of magnetic field for which the system stability may be preserved, and it depends on the electron density

$$B_{lim} = AN^{2/3}, A = \frac{(2\pi^4)^{1/3}}{e}. \quad (17)$$

For fields $B = B_{lim}$ the instability is produced, and the transverse pressure becomes negative due to the vacuum term. For less fields, that is for $B < B_{lim}$, we must consider both contributions: statistical and vacuum one. For typical densities of a white dwarf $N \sim 10^{30} \text{electrons/cm}^3$ we find $B_{lim} \sim 10^{13} G$. In this particular case, if $B < B_{lim}$ we get $P_\perp \approx P_{ST\perp} \sim 10^{23} - 10^{24} \text{dyn/cm}^2$, and $P_\perp = P_{V\perp} \sim 10^{19} - 10^{22} \text{dyn/cm}^2$ for $B = B_{lim}$. We conclude that when $B = B_{lim}$ the star collapses. It agree with the results obtained in ⁴.

A different situation is produced for larger fields. For $B \sim 10^{15} G$, pressures of order $P_{V\perp} \sim 10^{25} - 10^{28} \text{dyn/cm}^2$ are found, and thus, *it may be of the same order than the statistical term*, although densities

$N \sim 10^{33} \text{ electrons/cm}^3$ are required to keep $P_{ST\perp} \neq 0$. The vacuum contribution becomes of the same importance of the statistical one and must be taken into account in any analysis of the system stability. In fact for these field intensities, it may happen that the total transverse pressure may not vanish, and even become negative, although the electron system were not confined to the Landau ground state $n = 0$.

5 Ultra Strong Fields $eB \gg m^2$

In the previous section we see that the effects of vacuum interaction with the magnetic field can not be neglected if we compare them with the effects produced due to the same interaction of the real particle system, for $B \sim 10^{15}G$. We can expect then, that the vacuum role becomes more and more relevant for larger fields. One way of handling this problem is by including the effect of radiative corrections through an anomalous magnetic moment, and modifying consequently the energy spectrum, which would appear as a solution of the Dirac equation for a charged particle with anomalous magnetic moment ⁹

$$\varepsilon_{n,\eta}^a = \sqrt{p_3^2 + (\sqrt{m^2 + (2n + \eta + 1)eB} + \eta \frac{\alpha}{2\pi} \mu_B B)^2}, \quad (18)$$

for $n = 0, 1, 2, \dots$, where μ_B is the Bohr magneton, and $\eta = 1, -1$ are the σ_3 eigenvalues corresponding to the two orientations of the magnetic moment with respect to the field B . The vacuum thermodynamic potential has now the form

$$\Omega_V = -\frac{eB}{4\pi^2} \sum_{\eta=1,-1} \sum_{n=0}^{\infty} \int_{-\infty}^{\infty} dp_3 \varepsilon_{n,\eta}^a. \quad (19)$$

After regularization, it transforms in to

$$\Omega_V = \frac{(eB)^2}{8\pi^2} \sum_{k=0}^{\infty} \frac{4^k}{(2k)!} \left(\left(\frac{\alpha}{4\pi} \right)^2 \frac{eB}{m^2} \right)^{2k} \Lambda_k, \quad (20)$$

where

$$\Lambda_0 = \int_0^{\infty} \frac{dx}{x^2} e^{-\left(\frac{m^2}{eB} + \left(\frac{\alpha}{4\pi}\right)^2 \frac{eB}{m^2}\right)x} \left(\coth x - \frac{1}{x} - \frac{1}{3}x \right), \quad (21)$$

and

$$\Lambda_k = \int_0^{\infty} dx e^{-\left(\frac{m^2}{eB} + \left(\frac{\alpha}{4\pi}\right)^2 \frac{eB}{m^2} + 1\right)x} \left(\frac{1}{\sinh x} - \frac{1}{x} \right) \left(\left(\frac{4\pi}{\alpha} \right)^2 \frac{m^2}{eB} \frac{d}{dx} - 1 \right)^k x^{2(k-1)}, \quad (22)$$

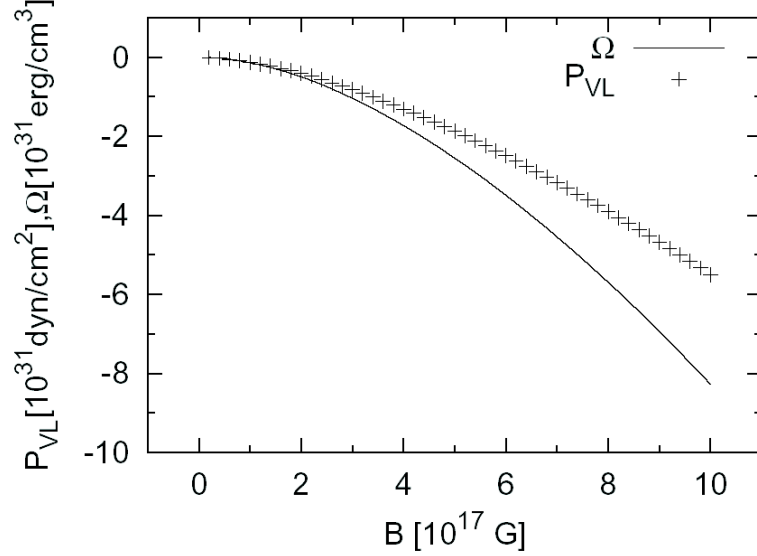


Figure 3. Vacuum transverse pressure and energy dependence on the field, taking into account the electron anomalous magnetic moment.

for $k \neq 0$. Due to the smallness of the factor $\frac{\alpha}{4\pi} \sim 10^{-4}$, in the series expansion (20) we can neglect all terms Λ_k with $k \neq 0$, except for fields $B \sim 10^{-1}(\frac{4\pi}{\alpha})^2 B_c$, being $(\frac{4\pi}{\alpha})^2 B_c \sim 10^{20} G$, when more terms should be include.

Magnetization and transverse pressure become

$$\mathcal{M}_V = -\frac{2\Omega_V}{B} - \frac{e^2 B}{8\pi^2} \left(\frac{m^2}{eB} - \left(\frac{\alpha}{4\pi} \right)^2 \frac{eB}{m^2} \right) \int_0^\infty \frac{dx}{x} e^{-\left(\frac{m^2}{eB} + \left(\frac{\alpha}{4\pi} \right)^2 \frac{eB}{m^2} \right) x} \left(\coth x - \frac{1}{x} - \frac{x}{3} \right), \quad (23)$$

$$P_{V\perp} = \Omega_V + \frac{(eB)^2}{8\pi^2} \left(\frac{m^2}{eB} - \left(\frac{\alpha}{4\pi} \right)^2 \frac{eB}{m^2} \right) \int_0^\infty \frac{dx}{x} e^{-\left(\frac{m^2}{eB} + \left(\frac{\alpha}{4\pi} \right)^2 \frac{eB}{m^2} \right) x} \left(\coth x - \frac{1}{x} - \frac{x}{3} \right). \quad (24)$$

It must be noted that for $B > \frac{4\pi}{\alpha} \frac{m^2}{e} \sim 10^{16} G$ although p_\perp remains negative, it grows more slowly than for fields $B < \frac{4\pi}{\alpha} \frac{m^2}{e}$, and its absolute value is smaller than p_3 (Fig. 3).

6 A Quark Gas

Fields of very high orders $B \sim 10^{17} - 10^{20}G$ are expected to be found in the cores of extremely magnetized neutron stars. But electron system is hardly in equilibrium for this range of fields, and for these stellar objects the neutron and proton gases are the most important. The last suggests that the interaction of quark-antiquark virtual pairs with the magnetic field should be studied. As the interaction energy with the magnetic field may become comparable with the color field interaction among quarks, we may consider the quark interaction with the magnetic field in the similar way as the electron gas. We may write the electromagnetic vacuum energy of the quark-antiquark field in presence of the magnetic field. The main contribution comes from the u-type quark, which has a smaller mass than other quarks. For u-type quark we can write

$$\Omega_V^u = -\frac{q_u B}{8\pi^2} \int_0^\infty e^{-\left(\frac{m_u^2}{q_u B} + \left(\frac{\alpha_u}{4\pi}\right)^2 \frac{q_u B}{m_u^2}\right)x} \left(\coth x - \frac{1}{x} - \frac{1}{3}x\right) \frac{dx}{x^2}, \quad (25)$$

where $q_u = \frac{2}{3}e$ and m_u are the u-quark charge and mass, respectively, and $\alpha_u = q_u^2/\hbar c$. Expression (25) for the vacuum quark-antiquark electromagnetic energy is valid for fields up to $B \sim 10^{19}G$. Magnetization has the form

$$\mathcal{M}_V^u = -\frac{2\Omega_V^u}{B} - \frac{q_u^2 B}{8\pi^2} \left(\frac{m_u^2}{q_u B} - \left(\frac{\alpha_u}{4\pi} \right)^2 \frac{q_u B}{m_u^2} \right) \mathcal{I} \quad (26)$$

and transverse pressure is

$$p_{V\perp}^u = \Omega_V^u - \frac{(q_u B)^2}{8\pi^2} \left(\frac{m_u^2}{q_u B} - \left(\frac{\alpha_u}{4\pi} \right)^2 \frac{q_u B}{m_u^2} \right) \mathcal{I}$$

with

$$\mathcal{I} \equiv \int_0^\infty e^{-\left(\frac{m_u^2}{q_u B} + \left(\frac{\alpha_u}{4\pi}\right)^2 \frac{q_u B}{m_u^2}\right)x} \left(\coth x - \frac{1}{x} - \frac{x}{3}\right) \frac{dx}{x}. \quad (27)$$

The transverse vacuum pressure gives values of the order of $p_{V\perp}^u \sim 10^{29} - 10^{31} \text{ dyn/cm}^2$ for fields $B \sim 10^{17}G$, and $p_{V\perp}^u \sim 10^{31} - 10^{33} \text{ dyn/cm}^2$ for fields $B \sim 10^{19}G$, as is shown in Fig. 4.

7 Conclusions

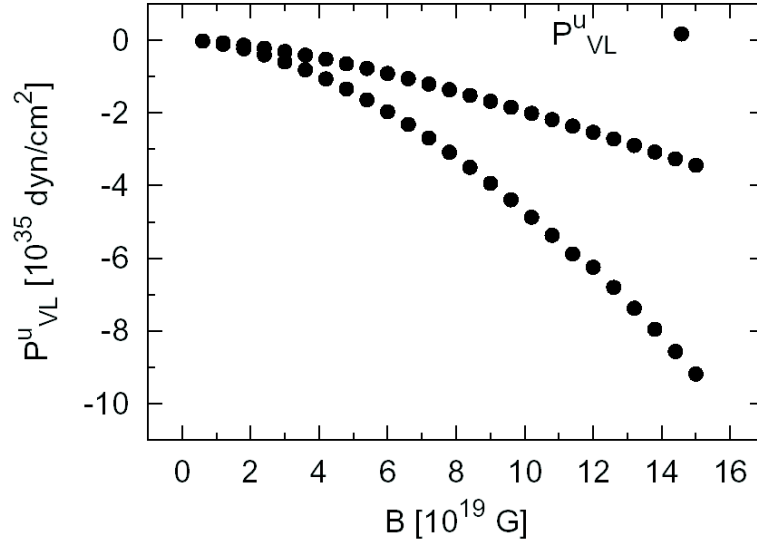


Figure 4. Transverse pressure exerted by the u-type quark-antiquark virtual pairs.

A magnetic field modify the electron-positron zero-point energy of vacuum, leading to new physical effects. The vacuum shows a nonlinear response, as a ferromagnetic medium. A negative pressure, exerted in the direction perpendicular to the field, appears and it is a Casimir like force, which must produce observable effects for fields $B \sim 10^5 G$. The electron-positron zero-point energy leads to a transverse pressure of similar order of the statistical one for fields $B \sim 10^{15} G$, and densities $N \sim 10^{33} \text{electrons/cm}^3$ or larger would be required in order to avoid the collapse.

Vacuum shrinks perpendicular to the magnetic field. This leads us to conclude that nuclear and/or quark matter are unstable for fields $B \sim m_n^2/e$, m_n being the nucleon mass. As the contribution of the transverse quark pressure is smaller than its longitudinal term, the vacuum negative transverse pressure makes the system to implode.

Appendix

We start from the expression (19) for Ω_V . Using the integral representation

$$a^{1/2} = \frac{1}{2}\pi^{-1/2} \int_0^\infty dt t^{-3/2} (1 - e^{-at}), \quad (28)$$

and taking $a^{1/2} = \varepsilon_{n,\eta}^a$, we can perform the Gaussian integral on p_3 . We obtain

$$\Omega_V(\varepsilon) = \frac{eB}{8\pi^2} \sum_{\eta=1,-1} \sum_{n=0}^\infty \int_\varepsilon^\infty \frac{dt}{t^2} e^{-(\sqrt{m^2+(2n+\eta+1)eB+\eta\frac{\alpha}{4\pi}\frac{eB}{m}})^2 t}, \quad (29)$$

where we have introduced a quantity ε in order to regularize the divergent term dependent on a in Eq. (28). Performing the sum over η , we can write

$$\begin{aligned} \Omega_V(\varepsilon) = & \frac{eB}{8\pi^2} \int_\varepsilon^\infty \left[e^{-(m-\frac{\alpha}{4\pi}\frac{eB}{m})^2 t} \right. \\ & \left. + 2 \sum_{n=1}^\infty e^{-(m^2+2neB+(\frac{\alpha}{4\pi}\frac{eB}{m})^2)t} \cosh \frac{\alpha}{2\pi} \frac{eB}{m} t \sqrt{m^2+2neB} \right] \frac{dt}{t^2}. \end{aligned} \quad (30)$$

Substituting the series expansion of $\cosh \frac{\alpha}{2\pi} \frac{eB}{m} t \sqrt{m^2+2neB}$, we get

$$\Omega_V(\varepsilon) = \frac{eB}{8\pi^2} \int_\varepsilon^\infty \frac{dt}{t^2} (e^{-(b_0-g)^2 t} + 2 \sum_{n=1}^\infty \sum_{k=0}^\infty \frac{(2gb_n t)^{2k}}{2k!} e^{-(b_n^2+g^2)t}), \quad (31)$$

where $b_n = \sqrt{m^2+2neB}$ and $g = \frac{\alpha}{4\pi} \frac{eB}{m}$. Using the fact that

$$b_n^2 \int_\varepsilon^\infty dt e^{-(b_n^2+g^2)t} t^{2(k-1)} = \int_\varepsilon^\infty dt e^{-(b_n^2+g^2)t} \left[\frac{d}{dt} - g^2 \right] t^{2(k-1)}, \quad (32)$$

and performing the sum over n , we find that

$$\begin{aligned} \Omega_V(\varepsilon) = & \frac{eB}{8\pi^2} \int_\varepsilon^\infty \frac{dt}{t^2} (e^{-(b_0-g)^2 t} \\ & + 2 \frac{eB}{8\pi^2} \sum_{k=0}^\infty \frac{(2g)^{2k}}{2k!} \int_\varepsilon^\infty dt e^{-(b_0^2+g^2)t} e^{-eBt} \frac{1}{\sinh eBt} \left[\frac{d}{dt} - g^2 \right] t^{2(k-1)}). \end{aligned} \quad (33)$$

Due to the smallness of $\frac{\alpha}{2\pi} \sim 10^{-4}$, we can approximate in Eq. (33) $1 \pm \frac{\alpha}{2\pi} \approx 1$ and write

$$e^{-2b_0 g t} + e^{-eBt} \frac{1}{\sinh eBt} \approx \coth eBt. \quad (34)$$

Finally, subtracting to

$$\coth eBt$$

and

$$\frac{1}{\sinh eBt}$$

the first terms in their series expansion, we can take $\epsilon \rightarrow 0$, and obtain the expression (20), where we put $eBt = x$.

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