The Color Glass Condensate and JIMWLK Equation

João Thiago de Santana Amaral
thiago.amaral@ufrgs.br

Instituto de Física, Universidade Federal do Rio Grande do Sul, Porto Alegre, Brazil

GFPAE - UFRGS
http://www.if.ufrgs.br/gfpae

Outline

- The Effective Theory for the Color Glass Condensate
- The Quantum evolution of the Color Glass Condensate: JIMWLK equation

Light-cone kinematics

- Let z be the longitudinal axis of the collision. For an arbitrary 4-vector $v^\mu = (v^0, v^1, v^2, v^3)$, the light-cone (LC) coordinates are defined as

$$v^+ \equiv \frac{1}{\sqrt{2}}(v^0 + v^3), \qquad v^- \equiv \frac{1}{\sqrt{2}}(v^0 - v^3), \qquad \boldsymbol{v} \equiv (v^1, v^2) \tag{1}$$

- One usually writes $v_\perp \equiv |\boldsymbol{v}| = \sqrt{(v^1)^2 + (v^2)^2}$
- In these coordinates, $x^+ \equiv \frac{1}{\sqrt{2}}(t+z)$ is the LC time and $x^- \equiv \frac{1}{\sqrt{2}}(t-z)$ is the LC longitudinal coordinate.
- The invariant scalar product of two 4-vectors reads

$$\begin{aligned} p \cdot x &= p^0x^0 - p^1x^1 - p^2x^2 - p^3x^3 \\ &= \frac{1}{2}(p^+ + p^-)(x^+ + x^-) - \frac{1}{2}(p^+ - p^-)(x^+ - x^-) - \boldsymbol{p}\cdot\boldsymbol{x} \\ &= p^-x^+ + p^+x^- - \boldsymbol{p}\cdot\boldsymbol{x} \end{aligned} \tag{2}$$

- This form of the scalar product suggests that p^- should be interpreted as the LC energy and p^+ as the LC longitudinal momentum.

Light-cone kinematics

- For particles on the mass-shell, $k^{\pm} = (E \pm k_z)/\sqrt{2}$, with $E = (m^2 + \boldsymbol{k}^2)$

$$k^+ k^- = \frac{1}{2}(E^2 - k_z^2) = \frac{1}{2}(\boldsymbol{k}^2 + m^2) \equiv m_\perp^2 \qquad (3)$$

- One needs also the *rapidity*

$$y \equiv \frac{1}{2} \ln \frac{k^+}{k^-} = \frac{1}{2} \ln \frac{2k^{+2}}{m_\perp^2} \qquad (4)$$

- Under a longitudinal Lorentz boost ($k^+ \to \beta k^+$, $k^- \to (1/\beta) k^-$, with constant β), the rapidity is shifted only by a constant, $y \to y + \beta$

- For a parton inside a right-moving (in the positive z direction) hadron, we introduce the boost-invariant longitudinal momentum fraction x

$$x \equiv \frac{k^+}{P^+} \qquad (5)$$

The McLerran-Venugopalan (MV) model

- Consider a nucleus in the infinite momentum frame (IMF) with momentum $P^+ \to \infty$ and with a nearly infinite transverse extent with a uniform matter distribution

- Partons which carry large fractions of momentum – *valence partons* – are Lorentz contracted to a distance $\sim 2R_A/\gamma$, with $\gamma = P^+/M = p^+/m_N$ ($p^+ = P^+/A$ is the longitudinal momentum of a single nucleon and $m_N = M/A$ is its mass)

- The *wee* partons with longitudinal momentum fractions $x \ll 1$ are delocalized in the x^- direction over relatively large distances $\sim 1/xp^+$

- For $x \ll A^{-1/3}$ these distances are much larger than the Lorentz contracted nuclear diameter; these partons 'see' the valence partons as an infinitely thin sheet of color charge

- The MV model assumes a simple kinematic distinction between wee and valence partons

The McLerran-Venugopalan (MV) model

- The wee partons also have very short lifetimes, given by the uncertainty and the LC dispersion relation

$$\Delta x^+ \sim \frac{1}{k^-} = \frac{2k^+}{m_\perp^2} \equiv \frac{2xP^+}{m_\perp^2} \tag{6}$$

which implies that the wee parton lifetime is much shorter than that of the valence partons: *the valence parton sources are static sources of color charge*

- Since the momenta of the valence partons are large, they are also recoilless sources of color charge; in this *eikonal approximation* the wee partons couple only to the plus component of the LC current

$$J^{\mu,a} = \delta^{\mu+}\delta(x^-)\rho^a(\boldsymbol{x}) \,, \tag{7}$$

where $\rho^a(\boldsymbol{x})$ is the valence quark color charge density in the transverse plane. . Actually, $\delta(x^-)\rho^a(\boldsymbol{x}) \to \rho^a(x^-, \boldsymbol{x})$, where $\rho^a(x^-, \boldsymbol{x})$ is localized near $x^- = 0$

How to generate $\rho^a(\boldsymbol{x})$?

- Nucleus interacting with an external probe which can resolve distances of size $\Delta\boldsymbol{x}$ in the transverse plane that are much smaller than the nucleon size $\sim \Lambda_{QCD}$.

- The small probe which has $x \ll A^{-1/3}$ simultaneously couples to partons from nucleons all along the nuclear diameter; since its transverse size is much smaller than the nucleon size, it sees them as sources of color charge

- The number ΔN of these sources can be estimated as the product $n\Delta S_\perp$ between the density $n \equiv N_c A/\pi R_A^2$ of valence quarks in the transverse plane and the area $\Delta S_\perp \sim (\Delta\boldsymbol{x})^2$ covered by the external probe

$$\Delta N \approx n\Delta S_\perp = \Delta S_\perp \frac{N_c A}{\pi R_A^2} \sim \frac{\Lambda^2_{\text{QCD}}}{Q^2} N_c A^{1/3} \qquad (8)$$

where Q^2 is the external resolution and $R_A = R_0 A^{1/3}$ with $R_0 \sim \Lambda_{\text{QCD}}$

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How to generate $\rho^a(\boldsymbol{x})$?

- If Q^2 is small enough to satisfy $\Lambda^2_{\text{QCD}} \ll Q^2 \ll \Lambda^2_{\text{QCD}} N_c A^{1/3}$, the area $\Delta S_\perp$ covers a large number $\Delta N \gg 1$ of valence quarks
- The valence quarks are then randomly distributed, in such a way that the total color charge seen by the probe

$$\langle \mathcal{Q}^a \rangle = 0, \qquad \langle \mathcal{Q}^a \mathcal{Q}^a \rangle = g^2 C_f \Delta N = \Delta S_\perp \frac{g^2 C_f N_c A}{\pi R_A^2}, \tag{9}$$

- One can treat this charge as classical since, when ΔN is large enough, we can ignore commutators of charges: $|\,[\mathcal{Q}^a, \mathcal{Q}^b]\,| = |\, i f^{abc} \mathcal{Q}^c \,| \ll \mathcal{Q}^2$
- In order to take the continuum limit, it is convenient to introduce the color charge densities $\rho^a(x^-, \boldsymbol{x})$ and

$$\rho^a(\boldsymbol{x}) \equiv \int dx^- \rho^a(x^-, \boldsymbol{x}) \tag{10}$$

- Then

$$\mathcal{Q}^a = \int_{\Delta S_\perp} d^2\boldsymbol{x}\, \rho^a(\boldsymbol{x}) = \int_{\Delta S_\perp} d^2\boldsymbol{x} \int dx^-\, \rho^a(x^-, \boldsymbol{x}), \tag{11}$$

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How to generate $\rho^a(\boldsymbol{x})$?

- One has ($C_f = (N_c^2 - 1)/2N_c$) :

$$\begin{aligned}
\langle \rho_a(\boldsymbol{x}) \rho_b(\boldsymbol{y}) \rangle_A &= \delta_{ab} \delta^{(2)}(\boldsymbol{x} - \boldsymbol{y})\, \mu_A^2, \qquad \mu_A^2 \equiv \frac{g^2 A}{2\pi R_A^2}, \\
\langle \rho_a(x^-, \boldsymbol{x}) \rho_b(y^-, \boldsymbol{y}) \rangle_A &= \delta_{ab} \delta^{(2)}(\boldsymbol{x} - \boldsymbol{y}) \delta(x^- - y^-)\, \lambda_A(x^-), \\
\int dx^- \, \lambda_A(x^-) &= \mu_A^2
\end{aligned} \tag{12}$$

where $\mu_A^2 \sim A^{1/3}$ is the average color charge squared of the valence quarks per unit transverse area and per color, and $\lambda_A(x^-)$ is the corresponding density per unit volume

- The nonzero correlators (12) are generated by the weight function ($\vec{x} = (x^-, \boldsymbol{x})$)

$$W_A[\rho] = \mathcal{N} \exp\left\{ -\frac{1}{2} \int d^3x \frac{\rho_a(\vec{x}) \rho_a(\vec{x})}{\lambda_A(x^-)} \right\} \tag{13}$$

which is a Gaussian in ρ_a, with a local kernel, which is gauge-invariant, so the variable ρ_a can be the color source in any gauge

Towards an effective theory

- The local Gaussian form of the weight function (13) is valid, by construction, for a large nucleus and with some restricted kinematical range, namely

$$\Lambda^2_{\text{QCD}} \ll Q^2 \ll \Lambda^2_{\text{QCD}} N_c A^{1/3}, \tag{14}$$

  for relatively small $x \ll A^{-1/3}$, but such that $\alpha_s \ln(1/x) \ll 1$

- For even smaller values of x or larger values of Q^2, the QCD quantum evolution cannot be neglected anymore, and the gluon distribution at the scale of interest is then dominated by the *quantum* color sources

- For sufficiently high Q^2 the system becomes very dilute, and the classical approximation breaks down

- Finally, the assumption that the valence quarks are uncorrelated must fail for transverse separations $\gtrsim \Lambda_{\text{QCD}}$

Towards an effective theory

- The color fields computed carry longitudinal momenta k^+ much lower than those of their sources, the valence quarks $\Rightarrow$ the weight function can be seen as part of an *effective theory* for gluon correlations at momenta k^+ smaller than some upper cutoff Λ^+
- However, for the classical approximations underlying the expression for the weight function to be valid, the value k^+ of interest should be not *much* smaller than Λ^+
  - Indeed, as we shall see, new color sources with momenta $p^+ < \Lambda^+$ are produced by radiation from the original sources at $p^+ \geq \Lambda^+$
- If the gap between k^+ and Λ^+ is relatively large, these new sources, which are mostly gluons, will completely dominate the physics at the scale k^+ of interest
- These new sources can be explicitly constructed by integrating out layers of quantum fluctuations in a renormalization group analysis, but the ensuing weight function is generally *not* a Gaussian
  - Still, the Gaussian $W_A[\rho]$ may be a good initial condition for this *quantum evolution*

The Color Glass Condensate (CGC)

- The effective theory depends upon the value of x at which we probe the hadron wavefunction
- Then, it is convenient to introduce the longitudinal momentum scale

$$\Lambda^+ = xP^+ \tag{15}$$

  and distinguish between 'fast' ($k^+ > \Lambda^+$) and 'soft' ($k^+ \leq \Lambda^+$) degrees of freedom

- The effective theory will be a theory for gluon correlations at the soft scale Λ^+ as obtained after having integrated out the fast modes
- This strong separation in longitudinal momenta implies that the small-x and large-x dynamics decouple from each other and can be treated separately
- Any observable pertinent to a small-x process can be first computed for a *fixed configuration* of the color sources, and the averaged over all the possible configurations with some *classical* probability distribution

The Classical Theory

- The structure of the classical theory is the same as in the MV model: the fast color sources are represented by a color current $J_a^\mu = \delta^{\mu +}\rho_a$ where ρ_a is static and random

- The small-x gluons are the color fields generated by this current according to the Yang-Mills equation

$$(D_\nu F^{\nu\mu})_a(x) = \delta^{\mu +}\rho_a(\vec{x}) \qquad (16)$$

where $D_\nu = \partial_\nu - igA^a_\nu T^a$ and $(T^a)_{bc} = -if_{abc}$

- All the interesting correlations are included in the functional weight function $W_{\Lambda^+}[\rho]$ (≥ 0),

$$\int \mathcal{D}[\rho]\, W_{\Lambda^+}[\rho] = 1, \quad \mathcal{D}[\rho] \equiv \prod_a \prod_{x^-} \prod_{\boldsymbol{x}} \mathrm{d}\rho_a(x^-, \boldsymbol{x}) \qquad (17)$$

- The observables are first evaluated on the solution $\mathcal{A}^\mu = \mathcal{A}^\mu[\rho]$ to the Yang-Mills equations, and then averaged over ρ

$$\langle \mathcal{O}[\mathcal{A}^\mu] \rangle_{\Lambda^+} = \int \mathcal{D}[\rho]\, W_{\Lambda^+}[\rho] \mathcal{O}[\mathcal{A}^\mu] \qquad (18)$$

The Quantum Calculations

- The quantum aspect refers to the evolution of the weight function with decreasing Λ^+
- The generating functional for gluon correlations at momenta $k^+ \leq \Lambda^+$

$$Z[j] = \int \mathcal{D}\rho \, W_{\Lambda^+}[\rho] \left\{ \frac{\int^{\Lambda^+} \mathcal{D}[\delta A] \, \delta(A^+) \, \mathrm{e}^{\, iS[A,\,\rho] - \int j \cdot A}}{\int^{\Lambda^+} \mathcal{D}[\delta A] \, \delta(A^+) \; \mathrm{e}^{\, iS[A,\,\rho]}} \right\} \tag{19}$$

- The 2-point function is given by (the symbol T denoting normal ordering in x^+)

$$\langle \mathrm{T} A^\mu(x) A^\nu(y) \rangle_{k^+} = \int \mathcal{D}\rho \, W_{\Lambda^+}[\rho] \left\{ \frac{\int^{\Lambda^+} \mathcal{D}[\delta A] \, \delta(A^+) \, A^\mu(x) A^\nu(y) \, \mathrm{e}^{\, iS[A,\,\rho]}}{\int^{\Lambda^+} \mathcal{D}[\delta A] \, \delta(A^+) \; \mathrm{e}^{\, iS[A,\,\rho]}} \right\} \tag{20}$$

- The averaging procedure involves two types of functional integrals

  `(i)` A *classical* integral over the color charge density ρ, which represents the fast partons with $k^+ \gg \Lambda^+$

  `(ii)` A *quantum* path integral over the gluon fluctuations δA^μ with momenta $k^+ \leq \Lambda^+$

The Quantum Calculations

- The total field A^μ at momenta $k^+ \leq \Lambda^+$ is the sum $A^\mu = \mathcal{A}^\mu[\rho] + \delta A^\mu$ between the classical field $\mathcal{A}^\mu$ and the *quantum fluctuations*

- The action $S[A, \rho]$ describes the dynamics of the soft gluons in the presence of the classical color charge ρ
  - In particular, when $k^+ \sim \Lambda^+$, the effects of the quantum fluctuations are negligible, and the path integral in the equation for the 2-point function can be evaluated in the saddle point approximation

$$\frac{\delta S}{\delta A^\mu} = 0 \Rightarrow A^\mu = \mathcal{A}^\mu[\rho] \tag{21}$$

and the 2-point function reduces to

$$\langle A_a^\mu(x^+, \vec{x}) A_b^\nu(x^+, \vec{y}) \rangle_{\Lambda^+} \;=\; \int \mathcal{D}\rho \; W_{\Lambda^+}[\rho] \; \mathcal{A}_a^\mu(\vec{x}) \mathcal{A}_b^\nu(\vec{y}) \tag{22}$$

The classical color field

- It is always possible find a solution to the classical equations of motion (EOM) $(D_\nu F^{\nu\mu})_a(x) = \delta^{\mu+}\rho_a(\vec{x})$ with the properties

$$F_a^{ij} = 0, \qquad A_a^- = 0, \qquad A_a^+ , \, A_a^i \; : \; \text{static} \, , \tag{23}$$

- The transverse fields A^i form a two-dimensional pure gauge; that is, there exists a gauge rotation $U(x^-, \boldsymbol{x}) \in \mathrm{SU}(N)$ such that

$$A^i(x^-, x_\perp) \; = \; \frac{i}{g} U(x^-, x_\perp) \, \partial^i U^\dagger(x^-, \boldsymbol{x}) \, . \tag{24}$$

(in matrix notations appropriate for the adjoint representation: $A^i = A_a^i T^a$, etc)

- We consider first the covariant (COV) gauge $\partial_\mu A^\mu = 0$. In this gauge, $\tilde{A}_a^\mu(x) \; = \; \delta^{\mu+}\alpha_a(x^-, \boldsymbol{x})$, with $\alpha_a(\vec{x})$ linearly related to the color source $\tilde{\rho}_a$ in the COV-gauge :

$$-\nabla_\perp^2 \alpha_a(\vec{x}) \; = \; \tilde{\rho}_a(\vec{x}) \, . \tag{25}$$

The classical color field

- The solution is:

$$\alpha_a(x^-, \boldsymbol{x}) = \int d^2\boldsymbol{y}\, \langle \boldsymbol{x} | \frac{1}{-\nabla_\perp^2} | \boldsymbol{y} \rangle\, \tilde{\rho}_a(x^-, \boldsymbol{y})$$
$$= \int \frac{d^2\boldsymbol{y}}{4\pi} \ln \frac{1}{(\boldsymbol{x}-\boldsymbol{y})^2\mu^2}\, \tilde{\rho}_a(x^-, \boldsymbol{y}), \quad (26)$$

- The classical solution in the LC-gauge $A^+ = 0$
  - This is of the form $\mathcal{A}_a^\mu = \delta^{\mu i}\mathcal{A}_a^i$ with $\mathcal{A}_a^i(x^-, \boldsymbol{x})$ a "pure gauge"
  - The gauge rotation $U(\vec{x})$ can be most simply obtained by a gauge rotation of the solution in the COV-gauge:

$$\mathcal{A}^\mu = U(\tilde{\mathcal{A}}^\mu + \frac{i}{g}\partial^\mu)U^\dagger, \quad (27)$$

where the gauge rotation $U(\vec{x})$ is chosen such that $\mathcal{A}^+ = 0$, i.e.,

$$U^\dagger(x^-, \boldsymbol{x}) = \mathrm{P}\exp\left\{ ig \int_{-\infty}^{x^-} dz^-\, \alpha_a(z^-, \boldsymbol{x}) T^a \right\} \quad (28)$$

The classical color field

- The explicit expression for the LC-gauge solution $\mathcal{A}^i$ in terms of the color source $\tilde{\rho}$ in the COV-gauge is

$$\langle A^i(x^+, \vec{x}) A^j(x^+, \vec{y}) \cdots \rangle_{\Lambda^+} = \int \mathcal{D}\tilde{\rho}\, W_{\Lambda^+}[\tilde{\rho}]\, \mathcal{A}^i_x[\tilde{\rho}]\, \mathcal{A}^j_y[\tilde{\rho}] \cdots . \tag{29}$$

- Recall that ρ has is localized near $x^- = 0$
  - In particular, it can be shown that

$$\mathcal{A}^i(x^-, \boldsymbol{x}) \approx \theta(x^-)\, \frac{i}{g}\, V_x(\partial^i V_x^\dagger) \tag{30}$$

where

$$V_x^\dagger = V^\dagger(\boldsymbol{x}) = U^\dagger(x^- \to \infty, \boldsymbol{x}) \equiv \mathrm{P} \exp\left\{ ig \int_{-\infty}^{\infty} dz^- \, \alpha(z^-, \boldsymbol{z}) \right\} \tag{31}$$

The gluon distribution

- $G(x, Q^2)dx$ the number of gluons in the hadron wavefunction having longitudinal momenta between xP^+ and $(x + dx)P^+$, and a transverse size $\Delta \boldsymbol{x} \sim 1/Q$
- In other terms, the *gluon distribution* $xG(x, Q^2)$ is the number of gluons with transverse momenta $k_\perp \lesssim Q$ per unit rapidity
- In the LC-gauge

$$xG(x, Q^2) = \frac{1}{\pi} \int \frac{d^2\boldsymbol{k}}{(2\pi)^2} \Theta(Q^2 - \boldsymbol{k}^2) \langle F_a^{i+}(\vec{k}) F_a^{i+}(-\vec{k}) \rangle, \tag{32}$$

- The “unintegrated gluon distribution”, or the “gluon occupation number” is defined as:

$$\varphi_\tau(\boldsymbol{k}) \equiv \frac{4\pi^3}{N_c^2 - 1} \frac{1}{\pi R^2} \frac{d^3N}{d\tau d^2\boldsymbol{k}} = \frac{1}{\pi R^2} \frac{\langle F_a^{i+}(\vec{k}) F_a^{i+}(-\vec{k}) \rangle}{N_c^2 - 1}, \tag{33}$$

where $\tau \equiv \ln\left(\frac{1}{x}\right) = \ln\left(\frac{P^+}{k^+}\right)$

The gluon distribution in the MV model

- Starting with the low density regime, when the atomic number A is not too high, so the corresponding classical field is weak and can be computed in the linear approximation

$$\mathcal{A}^i_a(k) \simeq -\frac{k^i}{k^+ + i\varepsilon}\frac{\rho_a(k^+, \boldsymbol{k})}{\boldsymbol{k}^2}, \qquad \mathcal{F}^{+i}_a(k) \simeq i\frac{k^i}{\boldsymbol{k}^2}\rho_a(\vec{k}), \tag{34}$$

and then

$$\langle \mathcal{F}^{i+}_a(\vec{k})\mathcal{F}^{i+}_a(-\vec{k})\rangle_A \simeq \frac{1}{\boldsymbol{k}^2}\langle \rho_a(\vec{k})\rho_a(-\vec{k})\rangle_A = \pi R_A^2 (N_c^2 - 1)\frac{\mu_A^2}{\boldsymbol{k}^2} \tag{35}$$

- The following estimates for the gluon density and distribution function are obtained

$$\varphi_A(\boldsymbol{k}) \simeq \frac{\mu_A^2}{\boldsymbol{k}^2}, \tag{36}$$

$$xG_A(x, Q^2) \simeq \frac{(N_c^2 - 1)R_A^2}{4\pi}\mu_A^2 \int_{\Lambda^2_{QCD}}^{Q^2} \frac{dk_\perp^2}{\boldsymbol{k}^2} = \frac{\alpha_s A N_c C_f}{\pi}\ln\frac{Q^2}{\Lambda^2_{QCD}},$$

with $\alpha_s = g^2/4\pi$; after taking into account quantum evolution, the actual scale for the screening of the infrared physics is not Λ_{QCD} but the *saturation scale* Q_s

Gluon saturation in a large nucleus

- One needs to recompute the gluon distribution by using the exact, non-linear solution for the classical field; By using $\mathcal{F}_a^{+i}(\vec{x}) = U^\dagger_{ab}(-\partial^i\alpha^b)$, one can express the relevant LC-gauge field-field correlator in terms of the color field in the COV-gauge:

$$\begin{aligned}\langle \mathcal{F}_a^{+i}(\vec{x})\mathcal{F}_a^{+i}(\vec{y})\rangle_A &= \Big\langle \partial^i\alpha^b(\vec{x})\partial^i\alpha^c(\vec{y})\Big\rangle \Big\langle U^\dagger_{ab}(\vec{x})U_{ca}(\vec{y})\Big\rangle \\ &= \delta(x^- - y^-)\langle \mathrm{Tr}\, U^\dagger(\vec{x})U(\vec{y})\rangle\big(-\nabla_\perp^2\gamma_A(x^-, \boldsymbol{x}-\boldsymbol{y})\big) \qquad (37)\end{aligned}$$

where

$$\gamma_A(x^-, k_\perp) \equiv \frac{1}{k_\perp^4}\,\lambda_A(x^-) \qquad (38)$$

- The trace

$$S_A(x^-, \boldsymbol{r} = \boldsymbol{x}-\boldsymbol{y}) \equiv \frac{1}{N_c^2-1}\langle \mathrm{Tr}\, U^\dagger(x^-, \boldsymbol{x})U(x^-, \boldsymbol{y})\rangle_A\ , \qquad (39)$$

can be explicitly computed as

$$S_A(x^-, \boldsymbol{r}) = \exp\Big\{ - g^2 N_c \int_{-\infty}^{x^-} dz^-\big[\gamma_A(z^-, \boldsymbol{0}) - \gamma_A(z^-, \boldsymbol{r})\big]\Big\}, \qquad (40)$$

Gluon saturation in a large nucleus

- The integrand is

$$\gamma_A(x^-,\mathbf{0}) - \gamma_A(x^-,\boldsymbol{r}) \,=\, \lambda_A(x^-) \int \frac{d^2\boldsymbol{k}}{(2\pi)^2}\,\frac{1}{\boldsymbol{k}^4}\,\left[1 - \mathrm{e}^{i\boldsymbol{k}\cdot\boldsymbol{r}}\right]. \tag{41}$$

- Ton leading-log accuracy, i.e., by keeping only terms enhanced by the large logarithm $\ln(1/\boldsymbol{r}^2\Lambda^2_{QCD})$, one obtains

$$S_A(x^-,\boldsymbol{r}) \,\simeq\, \exp\left\{-\,\frac{\alpha_s N_c}{4}\,\boldsymbol{r}^2\,\mu_A^2(x^-)\,\ln\frac{1}{\boldsymbol{r}^2\Lambda^2_{QCD}}\right\}, \tag{42}$$

and finally

$$\varphi_A(\boldsymbol{k}) = \int d^2\boldsymbol{r}\mathrm{e}^{-i\boldsymbol{k}\cdot\boldsymbol{r}}\,\frac{1-\exp\left\{-\frac{1}{4}\,\boldsymbol{r}^2 Q_A^2\,\ln\frac{1}{\boldsymbol{r}^2\Lambda^2_{QCD}}\right\}}{\pi\alpha_s N_c \boldsymbol{r}^2}, \tag{43}$$

where

$$Q_A^2 \,\equiv\, \alpha_s N_c \mu_A^2 \,=\, \alpha_s N_c \int dx^- \lambda_A(x^-) \,\sim\, A^{1/3}. \tag{44}$$

Gluon saturation in a large nucleus

- Let us first introduce the *saturation momentum* $Q_s(A)$ which is the scale separating between linear and non-linear behaviours and defined by the condition that, for $r_\perp = 2/Q_s(A)$, the exponent $\varphi_A(\boldsymbol{k})$ becomes of order one

$$Q_s^2(A) \simeq \alpha_s N_c \mu_A^2 \ln \frac{Q_s^2(A)}{\Lambda_{QCD}^2} \sim A^{1/3} \ln A \,. \tag{45}$$

- Two regimes can be distinguished:
  - At high momenta $k_\perp \gg Q_s(A)$, the integral is dominated by small $r_\perp \ll 1/Q_s(A)$, and can be evaluated by expanding out the exponential:

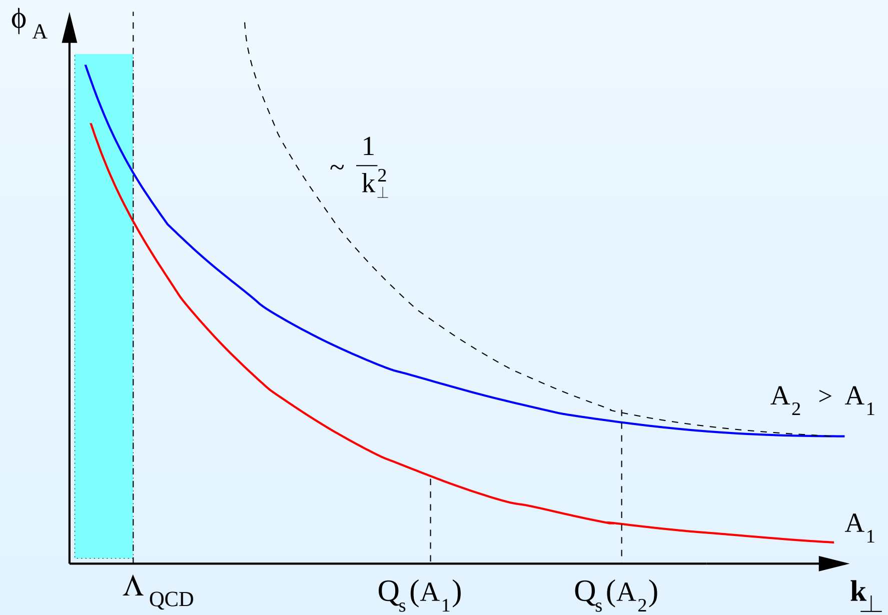
$$\varphi_A(k_\perp) \approx \frac{1}{\alpha_s N_c} \frac{Q_A^2}{k_\perp^2} = \frac{\mu_A^2}{k_\perp^2} \qquad \text{for} \quad k_\perp \gg Q_A . \tag{46}$$

  - At small momenta, $k_\perp \ll Q_s(A)$, the dominant contribution comes from large distances $r_\perp \gg 1/Q_s(A)$ and

$$\varphi_A(k_\perp) \approx \frac{1}{\alpha_s N_c} \ln \frac{Q_s^2(A)}{k_\perp^2} \qquad \text{for} \quad k_\perp \ll Q_A . \tag{47}$$

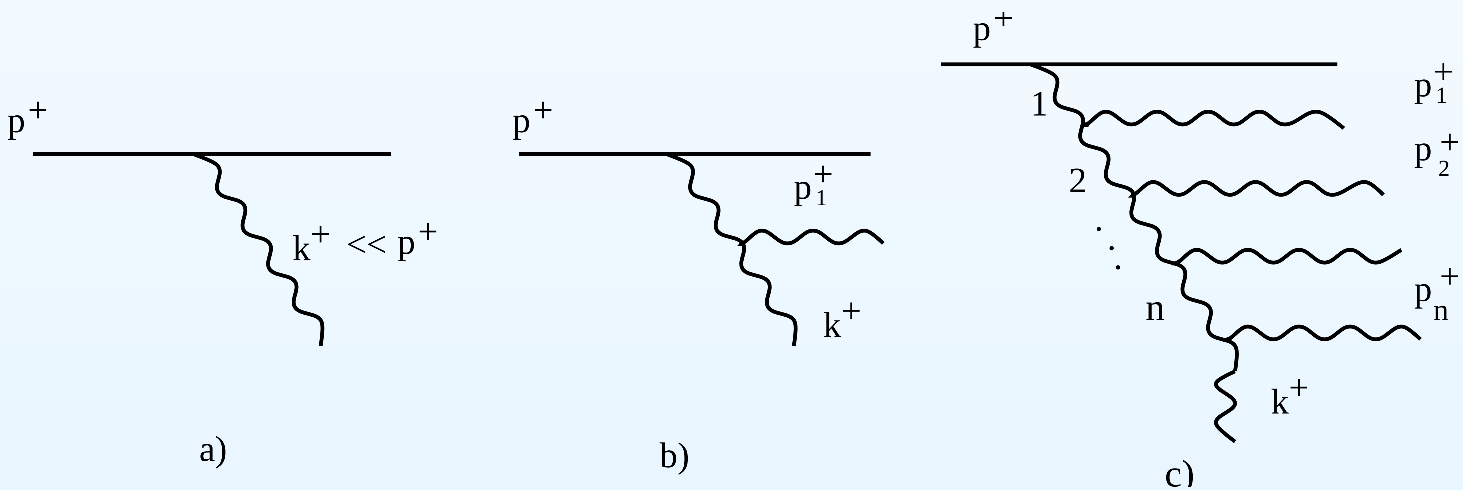
Gluon saturation in a large nucleus

- The distribution in $\varphi_A(k_\perp)$ given previously, which takes into account the non-linear effects in the classical Yang-Mills equations, rises only logarithmically as a function of both A and $1/k_\perp^2$: This is *saturation*
- At saturation, the gluon occupation factor is parametrically of order $1/\alpha_s$ and is the maximum density allowed by the repulsive interactions between the strong color fields $\bar{A}^i = \sqrt{\langle A^i A^i \rangle} \sim 1/g$
- When increasing the atomic number A, the new gluons are produced preponderantly at large transverse momenta $\gtrsim Q_s(A)$, where this repulsion is less important



BFKL evolution

- Consider the emission of a soft gluon with longitudinal momentum $k^+ = xP^+ \ll P^+$ by a fast moving parton (say, a valence quark) with momentum $p^+ = x_0 P^+$ and $1 > x_0 \gg x$
- As we have seen, the enhancement of the gluon distribution at small-x proceeds via the (BFKL) gluon cascades

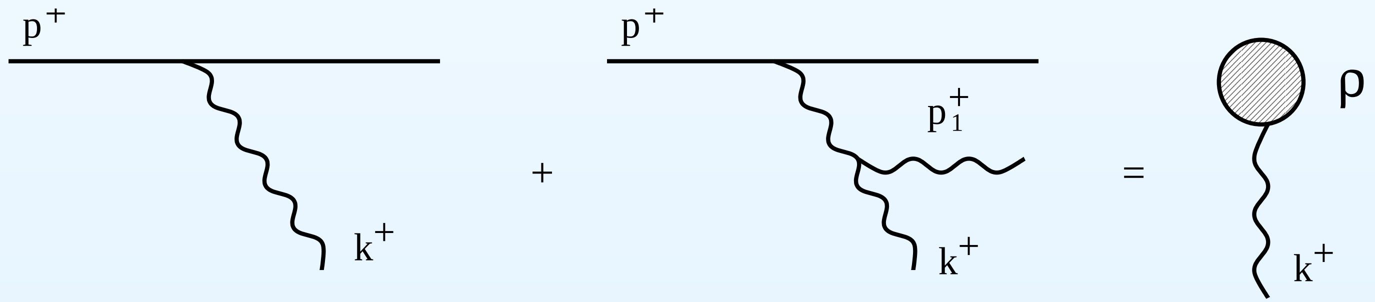


- In the cascade the successive gluons are strongly ordered in longitudinal momenta

$$p^+ \gg p_1^+ \gg p_2^+ \gg \cdots \gg p_n^+ \gg k^+ \tag{48}$$

Nonlinear evolution for the CGC

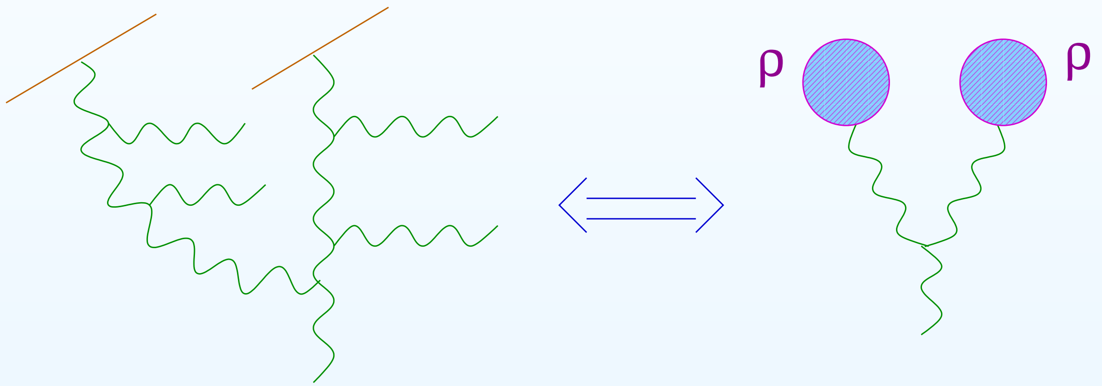
- Let us consider the first radiative correction, the one-gluon emission
- The typical contributions of this correction come from momenta p_1^+ such that $p^+ \gg p_1^+ \gg k^+$, that is, the condition of separation of scales is indeed satisfied for the intermediate gluon with momentum p_1^+ to be treated as a ‘frozen’ color source for the final gluon with momentum k^+
- The effect of this quantum correction is therefore simply to renormalize the *effective* color source at scale k^+



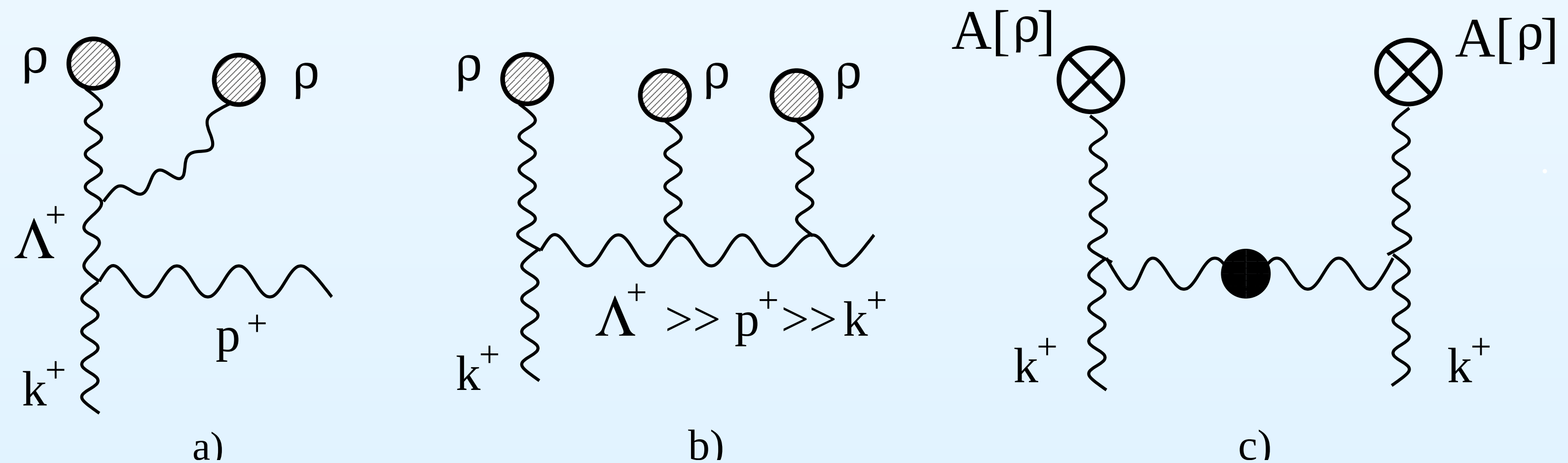
- By iterating this argument, a whole BFKL cascade can be included in the definition of the classical color source at the scale $\Lambda^+ = xP^+$ of interest

Nonlinear evolution for the CGC

- The fusion between two gluon cascades can be represented in the CGC theory as a nonlinear effect in the *classical* dynamics of the color fields generated by this effective source



- However, nonlinear effects are important also in the *quantum* evolution



Nonlinear evolution for the CGC

- The diagram (a) is an immediate generalization of the one-gluon emission; what is renormalized by the scattering off the "semi-fast" ($\Lambda^+ \gg p^+ \gg k^+$) quantum fluctuation is the classical field $\mathcal{A}^i[\rho]$ at scale Λ^+, which in turn is non-linear in ρ.

- The diagram (b) shows an additional source of non-linearity, arising from the propagation of the radiated gluon in the classical 'background' field $\mathcal{A}^i[\rho]$.

- If $\Lambda^+ = xP^+$ is small enough ($x \ll 1$), the classical field is very strong, $\mathcal{A}^i \sim 1/g$, and gluon rescatterings must be included to all orders in $\mathcal{A}^i$

- Both can be taken into account as the cut of the diagram (c). The classical field that enters the vertices is the fully non-linear solution $\mathcal{A}^i[\rho]$ and the propagator of the quantum gluon is computed to *all* orders in this background field

- The diagram (c) is manifestly a quantum correction to the 2-point function of the gauge fields at scale k^+, and is of order $\alpha_s \ln(\Lambda^+/k^+)$; for this to be computable in perturbation theory, the separation of scales between Λ^+ and k^+ must not be too large: $\Lambda^+ \gg k^+$, but $\alpha_s \ln(\Lambda^+/k^+) \ll 1$

Nonlinear evolution for the CGC

- The quantum modes must be integrated out in layers of p^+, within a renormalization group procedure. At each step in this procedure, one has to perform a one-loop quantum calculation, but with the exact background field propagator for the "semi-fast" gluons (the quantum gluons that are integrated out in that particular step).
- Such an all-order inclusion of the classical field effects permits one to resum not only the *large energy logarithms*, but also the dominant *high density effects*
- The condition that the new correlations induced by integrating out quantum fluctuations be reproduced by the CGC effective theory leads to a functional *renormalization group equation* (RGE) for the weight function $W_{\Lambda^+}[\rho] \equiv W_\tau[\rho]$

$$\frac{\partial W_\tau[\rho]}{\partial \tau} = \frac{1}{2} \int_{\boldsymbol{x}\boldsymbol{y}} \frac{\delta}{\delta \rho^a_\tau(\boldsymbol{x})} \chi_{ab}(\boldsymbol{x}, \boldsymbol{y})[\rho] \frac{\delta}{\delta \rho^b_\tau(\boldsymbol{y})} W_\tau[\rho] \,, \tag{49}$$

- This is the so-called Jalilian-Marian-Iancu-McLerran-Weigert-Leonidov-Kovner – *JIMWLK* – equation

The α-representation

- It is often preferable to use the 'α–representation', by performing the following change: $W_\tau[\alpha] \equiv W_\tau[\tilde{\rho} = -\nabla_\perp^2 \alpha]$; after this change of variables, the JIMWLK equation takes the following (Hamiltonian) form:

$$\frac{\partial W_\tau[\alpha]}{\partial \tau} = -HW_\tau[\alpha] \equiv \frac{1}{2}\int_{\boldsymbol{xy}} \frac{\delta}{\delta\alpha_\tau^a(\boldsymbol{x})}\,\eta^{ab}(\boldsymbol{x},\boldsymbol{y})[\alpha]\,\frac{\delta}{\delta\alpha_\tau^b(\boldsymbol{y})} W_\tau[\alpha] \tag{50}$$

where

$$\eta^{ab}(\boldsymbol{x},\boldsymbol{y}) = \frac{1}{\pi}\int \frac{d^2\boldsymbol{z}}{(2\pi)^2}\,\mathcal{K}_{\boldsymbol{xy},\boldsymbol{z}}\left\{1 + V_x^\dagger V_y - V_x^\dagger V_z - V_z^\dagger V_y\right\}^{ab}, \tag{51}$$

the kernel

$$\mathcal{K}_{\boldsymbol{xy},\boldsymbol{z}} \equiv \mathcal{K}(\boldsymbol{xy},\boldsymbol{z}) = \frac{(x^i - z^i)(y^i - z^i)}{(x_\perp - z_\perp)^2 (y_\perp - z_\perp)^2} \tag{52}$$

and

$$V_{\boldsymbol{x}}^\dagger \equiv \mathrm{P}\exp\left\{ig\int_{-\infty}^{\infty} dz^-\,\alpha(z^-,\boldsymbol{x})\right\} = \mathrm{P}\exp\left\{ig\int d\mathrm{y}\,\alpha_{\mathrm{y}}^a(\boldsymbol{x})t^a\right\} \tag{53}$$

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