Balitsky–Kovchegov evolution equation

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Introduction

• When the center of mass energy in a collision is much bigger than the fixed hard scale (for example, in DIS, the photon virtuality is the hard scale and $s \gg Q^2$), parton densities increase with growing energy.

• This produces larger scattering amplitudes.

• In this regime of saturation, BFKL equation is not valid anymore.

• Using the $q\bar{q}$-dipole model, Kovchegov was able to derive a new equation.

• Kovchegov equation is not exact, but it is a mean field approximation of a hierarchy of equations derived previously by Balitsky.

• This equation is nonlinear and can be reduced to BFKL equation when the nonlinear term is disregarded.
For simplicity, we should consider deep inelastic scattering of a virtual photon on a hadron or nucleus.

In the dipole picture, the system is studied in the reference frame of the target (the hadron or the nucleus is at rest).

Therefore, all QCD evolution is included in the wave function of the incoming photon.

The incoming photon cannot directly strongly interact with the target.

The splitting of the photon into a quark-antiquark pair is then considered.

This heavy quark-antiquark pair is called *onium*.

Quarks have color, but the incident photon is colorless.

Then, the quark has color and the antiquark the correspondent anticolor.
In the figure, the photon splitting is shown.

- $z_1$ is the fraction of light cone momentum.
- $x_0$ and $x_1$ are quark and antiquark positions ($x_{01} = x_0 - x_1$).
• The quark or antiquark can emit soft gluons ($z_2 << z_1$).

• These soft gluons can also emit softer gluons.
Planar diagrams

- Each loop counts as $N_c$.
- Each vertex counts as $\alpha_s$.
- Therefore, planar diagrams are enhanced by $N_c^2$ compared to nonplanar diagrams of the same order $\alpha_s$. 
In the limit of large number of colors ($N_c \to \infty$), these soft gluons can be considered as quark-antiquark pairs.

Therefore, the original dipole ($x_{01}$) \textit{split}s into two new dipoles ($x_{02}$ and $x_{21}$).
The onium

- The original incoming photon can be represented now by an arbitrary number of dipoles.
- These dipoles do not interact with each other.
Onium-hadron scattering

- Now, the nucleus or hadron is not going to scatter off the photon, but off the complex structure of color dipoles:

- Two gluons must be exchanged to conserve color.
Single and multiple scattering

- Single dipole scattering leads to linear BFKL equation.

- Multiple dipole scatterings lead to nonlinear BK equation.
Fan diagrams

- Pomeron merging in the target can be understood as a multiple scattering between target and projectile.

- Therefore, all fan diagrams are included.
Diagrams not included

- Not all diagrams are included, however.
- Diagrams representing Pomeron merging (a) cannot be represented as a multiple scattering (b).
In the dipole picture, the $F_2$ structure function is given by:

$$F_2(x, Q^2) = \frac{Q^2}{4\pi^2\alpha_{em}} \int \frac{d^2x_{01}}{2\pi} \Phi(x_{01}, z)\sigma_{dip}(x_{01}, Y)$$  \hspace{1cm} (1)

where $\Phi(x_{01}, z) = \Phi_T(x_{01}, z) + \Phi_L(x_{01}, z)$ is the sum of the square of the transverse (T) and longitudinal (L) photon wavefunctions (more specifically, the wavefunction for a photon to go into a dipole)

$$\Phi_T(x_{01}, z) = \frac{2N_c\alpha_{EM}}{\pi} a^2 K_1^2(x_{01}a)[z^2 + (1 - z)^2]$$  \hspace{1cm} (2)

$$\Phi_L(x_{01}, z) = 4Q^2z^2(1 - z)^2 K_0^2(x_{01}a).$$  \hspace{1cm} (3)

with $a^2 = Q^2z(1 - z)$.

$\sigma_{dip}$ is the dipole-nucleus (or nucleon) cross section
The forward scattering amplitude

- The dipole cross section is given by

\[ \sigma_{dip}(x_0, Y) = 2 \int d^2b_{01}N(b_{01}, x_0, Y). \]  

(4)

- The quantity \( b_{01} \) is the impact parameter given by:

\[ b_{01} = \frac{x_1 + x_0}{2} \]  

(5)

- \( N(b_{01}, x_0, Y) \) is the propagator of the quark-antiquark pair through the nucleus related to the forward scattering amplitude of the quark-antiquark pair with the nucleus.
Balitsky–Kovchegov equation

- The Balitsky–Kovchegov equation is given by:

\[
\frac{d\mathcal{N}(b_{01}, x_{01}, Y)}{dY} = \frac{\bar{\alpha}_s}{2\pi} \int d^2x_2 \frac{x_{01}^2}{x_{20}x_{12}} \left[ \mathcal{N} \left( b_{01} + \frac{x_{21}}{2}, x_{20}, Y \right) + \mathcal{N} \left( b_{01} + \frac{x_{20}}{2}, x_{21}, Y \right) - \mathcal{N} \left( b_{01} + \frac{x_{12}}{2}, x_{20}, Y \right) \mathcal{N} \left( b_{01} + \frac{x_{20}}{2}, x_{21}, Y \right) \right]
\]  \hspace{1cm} (6)

- This equation evolves \( \mathcal{N}(b_{01}, x_{01}, Y) \).

- The evolution quantity is the rapidity \( Y \approx \ln 1/x \).

- Besides the evolution variable, one has 4 degrees of freedom (2 from \( b_{01} \) and 2 from \( x_{01} \)).

- It resumms all powers of \( \alpha_s \ln 1/x \).

- \( \bar{\alpha}_s = \alpha_s N_c / \pi \)
Balitsky approach

- A general feature of high-energy scattering is that a fast particle moves along its straight-line classical trajectory and the only quantum effect is the eikonal phase factor acquired along this propagation path.

- In QCD, for the fast quark or gluon scattering off some target, this eikonal phase factor is a Wilson line:

  - the infinite gauge link ordered along the straight line collinear to particle’s velocity $n^\mu$:

    $$ U_\eta(x_\perp) = \text{Pexp}\left\{ig \int_{-\infty}^{\infty} du \; n_\mu A^\mu(un + x_\perp) \right\}, \quad (7) $$

    - Here $A_\mu$ is the gluon field of the target, $x_\perp$ is the transverse position of the particle which remains unchanged throughout the collision, and the index $\eta$ labels the rapidity of the particle.

    - Repeating the argument for the target (moving fast in the spectator’s frame) we see that particles with very different rapidities perceive each other as Wilson lines.

    - Therefore, Wilson-line operators form the convenient effective degrees of freedom in high-energy QCD.
Balitsky approach

- At small $x_B = Q^2 / (2p \cdot q)$, the virtual photon decomposes into a pair of fast quarks moving along straight lines separated by some transverse distance.

- The propagation of this quark-antiquark pair reduces to the “propagator of the color dipole” $U(x_\perp)U^\dagger(y_\perp)$ - two Wilson lines ordered along the direction collinear to quarks’ velocity.

- The structure function of a hadron is proportional to a matrix element of this color dipole operator

$$\hat{U}^\eta(x_\perp, y_\perp) = 1 - \frac{1}{N_c} \text{Tr}\{\hat{U}^\eta(x_\perp)\hat{U}^{\dagger\eta}(y_\perp)\}$$

- sandwiched by target’s states.

- The gluon parton density is approximately

$$x_B G(x_B, \mu^2 = Q^2) \simeq \langle p|\hat{U}^\eta(x_\perp, 0)|p\rangle \bigg|_{x_\perp = Q^{-2}}$$

where $\eta = \ln \frac{1}{x_B}$.
Balitsky approach

- The energy dependence of the structure function is translated then into the dependence of the color dipole on the slope of the Wilson lines determined by the rapidity $\eta$.
- At relatively high energies and for sufficiently small dipoles we can use the leading logarithmic approximation (LLA) where $\alpha_s \ll 1$, $\alpha_s \ln x_B \sim 1$ and get the non-linear BK evolution equation for the color dipoles:

$$\frac{d}{d\eta} \hat{U}(x, y) = \frac{\alpha_s N_c}{2\pi^2} \int d^2 z \frac{(x - y)^2}{(x - z)^2(z - y)^2} [\hat{U}(x, z) + \hat{U}(y, z) - \hat{U}(x, y) - \hat{U}(x, z)\hat{U}(z, y)]$$

(10)
The operator equation creates an hierarchy of equations for the observables:

\[
\frac{d}{d\eta} \langle \hat{U}(x, y) \rangle = \frac{\alpha_s N_c}{2\pi^2} \int d^2 z \frac{(x - y)^2}{(x - z)^2(z - y)^2} \left[ \langle \hat{U}(x, z) \rangle + \langle \hat{U}(y, z) \rangle - \langle \hat{U}(x, y) \rangle \right. \\
\left. - \langle \hat{U}(x, z)\hat{U}(z, y) \rangle \right]
\]  

(11)

\[
\frac{d}{d\eta} \langle T_{xy} \rangle = \frac{\tilde{\alpha}_s}{2\pi} \int d^2 z \mathcal{M}(x, y, z) \left[ \langle T_{xz} \rangle + \langle T_{yz} \rangle - \langle T_{xy} \rangle - \langle T_{xz}T_{zy} \rangle \right]
\]  

(12)

In the mean field approximation, \( \langle T_{xz}T_{zy} \rangle = \langle T_{xz} \rangle \langle T_{zy} \rangle \), and BK equation is recovered.
The BFKL limit

- The last (and nonlinear) term is a direct consequence of taking into account multiple scatterings.

- BK equation is a closed equation, instead of Balitsky original hierarchy of equations.

- The mean field approximation of Balitsky first equation gives the BK equation.

- If we disregard the last term, we obtain the following linear equation that can be shown equivalent to BFKL equation:

\[
\frac{dN(b_{01}, x_{01}, Y)}{dY} = \frac{\bar{\alpha}_s}{2\pi} \int d^2x_2 \frac{x_{01}^2}{x_{20}^2 x_{12}^2} \left[ N\left( b_{01} + \frac{x_{21}}{2}, x_{20}, Y \right) + N\left( b_{01} + \frac{x_{21}}{2}, x_{20}, Y \right) - N\left( b_{01}, x_{01}, Y \right) \right]
\] (13)
BK versus AGL equation

- BK, AGL and GLR equations sum the same diagrams (for example):

- All three use the eikonal approximation, i.e., incoming photon momentum has large $q_+$. 

- However, BK equation does not consider large $Q^2$. It resums all terms $1/(2q_+q_-) \approx 1/Q^2$. 
BK in 0+1 dimensions

• If one neglects the spatial dependence \( (N(x_0, b_0, Y)) \), BK equation is given by the logistic equation:

\[
\frac{dN}{dY} = \omega (N - N^2) \quad \omega > 0. \tag{14}
\]

• This equation has two fixed points, one unstable \( (N = 0) \) and another stable \( (N = 1) \).

• For every initial condition in \( 0 < N \leq 1 \), for \( Y \rightarrow \infty \), \( N \rightarrow 1 \).

• In other words, there is saturation for high \( Y \) (small \( x \)).

• If the \( N^2 \) term is neglected, the fixed point at \( N = 1 \) disappears and \( N \rightarrow \infty \) for \( Y \rightarrow \infty \).
The logistic equation complete solution is \((C = N(Y = 0))\):

\[
N(Y) = \frac{Ce^{\omega Y}}{1 + C(e^{\omega Y} - 1)}.
\]

(15)

Solutions: linear (red) and nonlinear (blue) equations.
BK in 1+1 dimensions

- If impact parameter is disregarded and only sizes of dipoles are considered:

\[
\mathcal{N}(b, x, Y) \rightarrow \mathcal{N}(r, Y). \tag{16}
\]

- BK equation is then simplified:

\[
\frac{d\mathcal{N}(|x_{01}|, Y)}{dY} = \frac{\bar{\alpha}_s}{2\pi} \int d^2 x_2 \frac{x_{01}^2}{x_{20}^2 x_{12}^2} \left[ \mathcal{N}(|x_{20}|, Y) + \mathcal{N}(|x_{21}|, Y) 
- \mathcal{N}(|x_{01}|, Y) - \mathcal{N}(|x_{21}|, Y) \mathcal{N}(|x_{20}|, Y) \right] \tag{17}
\]

- Physically speaking, this represents the scattering on infinite and uniform nucleus.

- We note that the kernel already depends only on dipole sizes.
BK in 1+1 dimensions

• The same kind of solutions of BK equation in (0+1) dimensions is found for BK equation in (1+1) dimensions for fixed $r$.

• Solutions: linear (red) and nonlinear (blue) equations.
BK in 1+1 dimensions

- Solutions for different initial conditions in linear (above) and log (below) scales.
The figures shown previously can be divided into three regions in $r$:

- The small $r$ region, where nonlinear corrections are negligible;
- The large $r$ region, where nonlinear corrections dominate and $N \approx 1$;
- The region between the first two.

One can introduce the saturation scale:

$$ r < \frac{1}{Q_s(Y)} \rightarrow N \ll 1, $$

$$ r > \frac{1}{Q_s(Y)} \rightarrow N \approx 1. $$
Saturation scale

- One can get a saturation scale from BK solutions.

- Neglecting some corrections, it is given by:

\[ Q_s(Y) = Q_0 \exp(\bar{\alpha}_s \lambda Y) Y^{-\beta} \]

\[ \lambda \approx 2.4 \]

- The saturation scale is the dividing line between dense and dilute regions.

- The higher the rapidity the denser the system gets and partons start to re-interact.

- Also, saturation occurs earlier if the size of partons is bigger.
• Dilute and saturated regimes can be represented in 2-D graphs \((Y = \ln(1/x))\):
GBW model

- Linear growing and saturation properties of BK equation are roughly similarly found in the Golec-Biernat and Wusthoff (GBW) saturation model.
- One of the key ideas to GBW model is the assumption of a $x$-dependent radius:

$$R_0(x) = \frac{1}{Q_0} \left( \frac{x}{x_0} \right)^{\frac{1}{2}},$$

which scales the quark-antiquark separation $r$ in the dipole cross section:

$$\hat{r} = \frac{r}{2R_0(x)}.$$

- Then, the dipole cross section can be written as:

$$\sigma_{dip}(x, r^2) = \sigma_0 g(\hat{r}^2).$$

- $Q_0 = 1 \text{ GeV}$ sets the dimension.
GBW model: dipole cross section

- The function \( g(\hat{r}^2) \) chosen is:

\[
g(\hat{r}^2) = 1 - \exp[-\hat{r}^2]. \tag{21}
\]

- The dipole cross section is then:

\[
\sigma_{dip}(x, r) = \sigma_0 \left( 1 - \exp \left[ -\left( \frac{r}{2R_0} \right)^2 \right] \right)
\]

\[
= \sigma_0 \left( 1 - \exp \left[ -r^2 \frac{Q_0^2}{4} \left( \frac{x_0}{x} \right)^\lambda \right] \right) \tag{22}
\]

- The three parameters were fitted to \( \sigma_0 = 23mb, \lambda = 0.288 \) and \( x_0 = 3 \times 10^{-4} \).

- The cross section can be written as a function of \( Y \) also:

\[
\sigma_{dip}(Y, r) = \int d^2 b N(b, r, Y) = \sigma_0 \left[ 1 - \exp \left( -\frac{r^2 Q_s^2(Y)}{4} \right) \right]
\]

with \( Q_s^2(Y) = \exp(0.28(Y - Y_0)) \).
GBW model: $\sigma_{dip}$ behavior

The cross section behavior is shown in the figure:
Saturation and color transparency

- In GBW model, when \( r \ll \frac{1}{Q_s^2(Y)} \), the phenomenon of color transparency appears (small dipoles have small chance of interaction):

\[
\frac{\sigma(Y, r)}{\sigma_0} = \frac{r^2 Q_s^2(Y)}{4}.
\]

- When \( r \gg \frac{1}{Q_s^2(Y)} \), the cross section saturates:

\[
\frac{\sigma(Y, r)}{\sigma_0} \approx 1.
\]
GBW Results

The $\gamma^* p$-cross section for various energies (solid lines: quark mass of 140MeV; dotted lines: zero quark mass).
The results for the fit to the inclusive HERA data on $F_2$ for different values of the virtuality.
Saturation without unitarization

- Consider impact parameter dependence.
- Total inelastic cross section, in the saturation regime ($t = \ln(s/s0)$):

$$\sigma = \pi R^2(t). \quad (23)$$

- To satisfy the Froissart bound the radius $R(t)$ should grow at most linearly with $t$.
- However, at large rapidities the radius of the saturated region is exponentially large

$$R(t) = R(t_0) \exp\left[\frac{\alpha_s N_c}{2\pi} \epsilon(t - t_0)\right]. \quad (24)$$

Saturation without unitarization
**Geometric scaling** is a phenomenological feature of DIS which has been observed in the HERA data on inclusive $\gamma^* - p$ scattering, which is expressed as a scaling property of the virtual photon-proton cross section

$$\sigma_{\gamma^* p}(Y, Q) = \sigma_{\gamma^* p}(\tau), \quad \tau = \frac{Q^2}{Q_s(Y)}$$

where $Q$ is the virtuality of the photon, $Y = \log \frac{1}{x}$ is the total rapidity and $Q_s(Y)$ is the saturation scale

[Stasto, Golec Biernat and Kwiecinsky, 2001]
Geometric scaling

• Looking to the solutions of BK equation, one sees that they reach a universal shape independently of the initial condition.

• The solutions even look similar when calculated for different $Y$ (they appear to be shifted when $Y$ is changed).

• In terms of the scattering amplitude, the geometric scaling means that BK solution depends only on $rQ_s(Y)$ for asymptotic values of rapidity.

\[ \mathcal{N}(r, Y) = \mathcal{N}(rQ_s(Y)). \]

• Using $Q_s(Y) \approx Q_0\exp(\bar{\alpha}_s \lambda Y)$;

\[ \mathcal{N}(r, Y) = \mathcal{N}(Q_0\exp(\ln r + \bar{\alpha}_s \lambda Y)) \]

and the geometric scaling relation resembles as a traveling wave with velocity $\bar{\alpha}_s \lambda$, time $Y$ and spatial coordinate $\ln r$. 
$\sqrt{\tau} \sigma_{\gamma^* p}$ plotted versus the scaling variable $\tau$. 
Geometric scaling

Different values of scaling variable \textit{versus} experimental data.
Geometric scaling

Experimental data on $\sigma_{\gamma^*p}$ from the region $x > 0.01$ plotted versus the scaling variable $\tau = Q^2 R_0^2(x)$. 
BK equation in momentum space

- Performing the Fourier transform:

\[ \tilde{N}(k, Y) = \int \frac{d^2r}{2\pi} e^{-i\vec{k} \cdot \vec{r}} \frac{N(r, Y)}{r^2}, \]  

(25)

- BK equation is given by:

\[ \frac{d\tilde{N}(k, Y)}{dY} = \bar{\alpha}_s \int \frac{dk'}{k'} K(k, k') \tilde{N}(k', Y) - \bar{\alpha}_s \tilde{N}^2(k, Y). \]  

(26)

- The solution to the linear part (BFKL) in saddle point approximation is:

\[ k\tilde{N}'(k, Y) = \frac{1}{\sqrt{\pi \bar{\alpha}_s \chi''(0)Y}} e^{\bar{\alpha}_s \chi(0)Y} \exp \left( -\frac{\ln^2(k^2/k_0^2)}{2\bar{\alpha}_s \chi''(0)Y} \right) \]  

(27)

with \( \chi(0) = 4 \ln 2 \) and \( \chi''(0) = 28\zeta(3) \).

- The last term in the expression above presents the diffusion.
BK equation in momentum space

\[ \kappa \phi(k,y) \]

BFKL

Kovchegov

\[ k \geq 1 \text{ GeV} \]
BK equation in momentum space

- BFKL presents strong diffusion and can be interpreted as a random walk in $\ln k$ with rapidity as time.

- The Gaussian shapes of last figure are then expected, as well as the width increase of distributions.

- The initial condition used was

$$k\tilde{\mathcal{N}}(k, Y = 0) = \delta(k)$$  \hspace{1cm} (28)

- One sees that BK equation leads to a suppression of the diffusion into infrared.

- Therefore, one defines the saturation scale as the distribution peak position

$$Q_s(Y) = k_{\text{max}}(Y).$$

- Other way to see the same effects is using the distribution

$$\Psi(k, Y) = \frac{k\tilde{\mathcal{N}}(k, Y)}{k_{\text{max}}(Y)\mathcal{N}(k_{\text{max}}(Y), Y)}.$$
BK equation in momentum space

\[ \alpha_s = 0.2 \]
In the last figure, we see that BK equation shifts the contour plot towards higher values of transverse momenta.

Two regions are found: in the first, there is still diffusion, particularly for large $k$.

In the second, diffusion is suppressed and the contour lines are parallel.

This is an indication of geometric scaling, since straight lines can be parametrized by $\xi = \ln k/k_0 - \lambda Y + \xi_0$.

One sees also that for large $k$ BK and BFKL equations present similar solutions, but BFKL presents an unlimited increase with energy. On the other side, BK solutions are bounded.
Traveling waves

- The BK equation can be compactly written in momentum space also as

\[
\partial_Y \tilde{N}(\rho, Y) = \bar{\alpha} \chi(-\partial_\rho) \tilde{N} - \bar{\alpha} \tilde{N}^2. \tag{29}
\]

- The Mellin–transformed BFKL kernel \( \chi(\gamma) \) is given by:

\[
\chi(\gamma) = 2\psi(1) - \psi(\gamma) - \psi(1 - \gamma), \quad \text{with} \quad \psi(\gamma) = \frac{\Gamma'(\gamma)}{\Gamma(\gamma)}. \tag{30}
\]

- \( \chi(-\partial_\rho) \) is an integro-differential operator defined with the help of the formal series expansion:

\[
\chi(-\partial_\rho) = \chi(\gamma_0)1 + \chi'(\gamma_0)(-\partial_\rho - \gamma_0 1) + \frac{1}{2} \chi''(\gamma_0)(-\partial_\rho - \gamma_0 1)^2 \\
+ \frac{1}{6} \chi^{(3)}(\gamma_0)(-\partial_\rho - \gamma_0 1)^3 + \ldots \tag{31}
\]

for some \( \gamma_0 \) between 0 and 1, where we used the identity operator 1.
Munier and Peschanski showed that after the change of variables

\[ t \sim \bar{\alpha}Y, \quad x \sim \log(k^2/k_0^2), \quad u \sim \phi, \]  

\[ (32) \]

The approximation of BFKL kernel by the first three terms of the expansion:

\[ \chi(-\partial_\rho) = \chi(\gamma_0)1 + \chi'(\gamma_0)(-\partial_\rho - \gamma_0 1) + \frac{1}{2} \chi''(\gamma_0)(-\partial_\rho - \gamma_0 1)^2, \]

And a saddle point approximation:

\[ \chi(-\partial_\rho) = -\chi'(\gamma_c)\partial_\rho + \frac{1}{2} \chi''(\gamma_c)(-\partial_L - \gamma_c 1)^2, \]

BK equation is reduced to the Fisher and Kolmogorov-Petrovsky-Piscounov (FKPP) equation from non-equilibrium statistical physics:

\[ \partial_t u(x, t) = \partial_x^2 u(x, t) + u(x, t) - u^2(x, t), \]  

\[ (33) \]

where \( t \) is time and \( x \) is the coordinate. FKPP dynamics is called reaction-diffusion dynamics.
Traveling waves

- FKPP equation admits so-called traveling wave solutions.
- The position of a wave front \( x(t) = v(t)t \) for a traveling wave solution does not depend on details of nonlinear effects.
- At large times, the shape of a traveling wave is preserved during its propagation, and the solution becomes only a function of the scaling variable \( x - vt \).
In the language of saturation physics the position of the wave front is nothing but the saturation scale \( x(t) \sim \ln Q_s^2(Y) \) and the scaling corresponds to the geometric scaling \( x - x(t) \sim \ln k^2/Q_s^2(Y) \).

Summarizing: time \( t \rightarrow Y \); space \( x \rightarrow L \); wave front \( u(x - vt) \rightarrow \tilde{N}(L - vY) \) and traveling waves \( \rightarrow \) geometric scaling.
Traveling waves and saturation

- In the dilute regime, in which one has

\[
\tilde{T}(k, Y) \approx \frac{k^{2}}{Q_{s}^{2}(Y)} \exp \left[ -\log^{2} \left( \frac{k^{2}}{Q_{s}^{2}(Y)} \right) \frac{2\tilde{\alpha} \chi''(\gamma_{c}) Y}{\chi'(\gamma_{c})} \right], \tag{34}
\]

where

\[
\lambda = \min \frac{\chi(\gamma)}{\gamma} = \tilde{\alpha} \frac{\chi(\gamma_{c})}{\gamma_{c}} = \tilde{\alpha} \chi'(\gamma_{c}), \quad \tilde{\alpha} \equiv \frac{\alpha_{s} N_{c}}{\pi}. \tag{35}
\]

- Geometric scaling is obtained within the window

\[
\log \left( \frac{k^{2}}{Q_{s}^{2}(Y)} \right) \lesssim \sqrt{2\chi''(\gamma_{c})} \tilde{\alpha} Y. \tag{36}
\]
• As usual, to get the region of application of the leading-order evolution equation one needs to find the next-to-leading order (NLO) corrections.

• Unlike the DGLAP evolution, the argument of the coupling constant in LO BK equation is left undetermined in the LLA.

• Careful analysis of this argument is very important from both theoretical and experimental points of view.

• Balitsky calculated the quark contribution to NLO B equation *Phys.Rev.* D75, 014001 (2007).

• Balitsky and Chirilli calculated the gluon contribution to NLO B equation *Phys.Rev.* D77, 014019 (2008).
Next-to-leading order BK

- NLO result does not lead automatically to the argument of coupling constant in front of the leading term.

- In order to get this argument, it was used the renormalon-based approach.

- The running coupling was roughly found to be $\alpha_s(|x - y|)$, but:

$$\frac{\alpha_s((x - y)^2)}{2\pi^2} \frac{(x - y)^2}{X^2 Y^2} |x - y| \ll |x - z|, |y - z|$$

$$\frac{\alpha_s(X^2)}{2\pi^2 X^2} |x - z| \ll |x - y|, |y - z|$$

$$\frac{\alpha_s(Y^2)}{2\pi^2 Y^2} |y - z| \ll |x - y|, |x - z|$$

(37)
Next-to-leading order BK

Next-to-leading order BK:

\[
\frac{d}{d\eta} N(x, y) = \frac{\alpha_s N_c}{2 \pi^2} \int d^2 z \frac{(x - y)^2}{X^2 Y^2} \left\{ 1 + \frac{\alpha_s N_c}{4 \pi} \left[ \frac{11}{3} \ln(x - y)^2 \mu^2 - \frac{11}{3} \frac{X^2 - Y^2}{(x - y)^2} \ln \frac{X^2}{Y^2} + \frac{67}{9} - \frac{\pi^2}{3} - 2 \ln \frac{X^2}{(x - y)^2} \ln \frac{Y^2}{(x - y)^2} \right] \right\}
\times [N(x, z) + N(z, y) - N(x, y) - N(x, z)N(z, y)]
\]

\[
+ \frac{\alpha_s^2 N_c^2}{8 \pi^4} \int d^2 z d^2 z' \left\{ -\frac{2}{(z - z')^4} + \left[ \frac{X^2 Y'^2 + X'^2 Y^2 - 4(x - y)^2(z - z')^2}{(z - z')^4(X^2 Y'^2 - X'^2 Y^2)} \right] \right. \\
+ \frac{(x - y)^2}{X^2 Y'^2(z - z')^2} \ln \frac{X^2 Y'^2}{X'^2 Y^2} \left[ N(z, z') - N(x, z)N(z, z') - N(z, z')N(z', y) - N(x, z)N(z', y) + N(x, z)N(z, y) \right. \\
+ N(x, z)N(z, z')N(z', y)] \\
\]

\[
(136)
\]

- \( X = x - z, \ Y = y - z, \ X = x - z', \ Y = y - z' \).

- As we can infer, in the complete hierarchy, the single dipole evolution is related to the sextupole terms.
Beyond BK equation: Pomeron loops

(a)

(b)

(d)
Beyond BK equation: Pomeron loops

(a)

(c)

(e)
Beyond BK equation: Pomeron loops
Beyond BK equation: Fluctuations

- BK equation completely misses the effects of fluctuations in the gluon (dipole) number, related to discreteness of the evolution.
- Also, since BK equation uses a mean field approach, $\langle T^2 \rangle - \langle T \rangle^2 = 0$.
- After an approximation related to the impact parameter dependence, it can be shown that a Langevin equation for the event-by-event amplitude can rebuild the Pomeron loop hierarchy.
- This is formally the BK equation with a Gaussian white noise term.
- It lies in the same universality class as the stochastic FKPP equation (sFKPP):

$$\partial_t u(x, t) = \partial_x^2 u(x, t) + u(x, t) - u^2(x, t) + \sqrt{\frac{2}{N}} u(x, t)(u(x, t) - 1) \nu(x, t). \quad (38)$$

- White noise is defined as:

$$\langle \nu(x, t) \rangle = 0 \quad \langle \nu(x, t) \nu(x', t') \rangle = \delta(t - t') \delta(x - x'). \quad (39)$$

[E. Iancu, D.N. Triantafyllopoulos; Nucl.Phys.A 756, 419 (2005)]
Fluctuations

- Each realization of the noise means a single realization of the target in the evolution, which is stochastic, and this leads to an amplitude for a single event.
- Different realizations lead to a dispersion in the solutions and also in the saturation momentum $\rho_s \equiv \ln(Q_s^2(Y)/k_0^2)$.
- For each single event, the evolved amplitude shows a traveling-wave pattern, with speed of the wave smaller than the value predicted by BK equation:

$$\lambda^* \simeq \lambda - \frac{\pi^2 \gamma c \chi''(\gamma c)}{2 \ln(1/\alpha_s^2)}$$

(40)

- The saturation scale $Q_s(Y)$ is now a random variable whose average value is given by

$$\langle Q_s^2(Y) \rangle = \exp[\lambda^* Y].$$

(41)

- The dispersion in the position of the individual fronts is given by

$$\sigma^2 = \langle \rho_s^2 \rangle - \langle \rho_s \rangle^2 = D\alpha Y,$$

(42)
Diffusive scaling

- Geometric scaling:

\[
\langle T(\rho, \rho_s) \rangle = T (\rho - \langle \rho_s \rangle).
\]  

is replaced by diffusive scaling

\[
\langle T(\rho, \rho_s) \rangle = T \left( \frac{\rho - \langle \rho_s \rangle}{\sqrt{\alpha D Y}} \right).
\]
Fluctuations

Y = ln(1/x)

DY >> 1

ln <Q^2_s> = λY

diffusive scaling

ln <Q^2> = \gamma Y

gamma + DY

Low density

ln Λ^2_{QCD}

ln Q^2
Toy model

- Are fluctuations really important (in the context of a toy model)?
- At fixed coupling they are [E. Iancu, J.T. de Santana Amaral, G. Soyez, D.N. Triantafyllopoulos; Nucl.Phys. A786, 131 (2007)].
- However, at running coupling they are not [A. Dumitru, E. Iancu, L. Portugal, G. Soyez, D.N. Triantafyllopoulos; JHEP 0708:062 (2007)].
- This reflects the slowing down of the evolution by running coupling effects, in particular, the large rapidity evolution which is required for the formation of the saturation front via diffusion.
References

AGL equation can be derived from BK equation. Taking the equation derived by Kovchegov:

\[
\mathcal{N}(x_{01}, b_0, Y) = -\gamma(x_{01}, b_0) \exp \left[ -\frac{4\alpha_s C_F}{\pi} \ln \left( \frac{x_{01}}{\rho} \right) Y \right] \\
+ \frac{\alpha_s C_F}{\pi} \int_0^Y dy \exp \left[ -\frac{4\alpha_s C_F}{\pi} \ln \left( \frac{x_{01}}{\rho} \right) (y - Y) \right] \\
x \int \frac{d^2x_2}{\rho} \frac{x_{01}^2}{x_{02}^2 x_{12}^2} \left[ 2\mathcal{N}(x_{02}, b_0, Y) - \mathcal{N}(x_{02}, b_0, Y) \mathcal{N}(x_{12}, b_0, Y) \right].
\]

In the double logarithmic limit, in which the momentum scale of the photon $Q^2$ is larger than the momentum scale of the nucleus $\Lambda_{QCD}$, the large $Q^2$ limit of last equation reduces to:

\[
\frac{\partial \mathcal{N}(x_{01}, b_0, Y)}{\partial Y} = \frac{\alpha_s C_F}{\pi} x_{01}^2 \int_{x_{01}^2}^{1/\Lambda_{QCD}^2} \frac{d^2x_{02}}{x_{02}^2 x_{12}^2} \left[ 2\mathcal{N}(x_{02}, b_0, Y) - \mathcal{N}(x_{02}, b_0, Y) \mathcal{N}(x_{12}, b_0, Y) \right].
\]
The last equation can be derived again with respect to $\ln(1/x_{01}^2 \Lambda_{QCD}^2)$:

$$\frac{\partial N(x_{01}, b_0, Y)}{\partial Y \partial \ln(1/x_{01}^2 \Lambda_{QCD}^2)} = \frac{\alpha_s C_F}{\pi} [2 - N(x_{01}, b_0, Y)]N(x_{01}, b_0, Y). \quad (47)$$

In the physical picture for the dipole evolution in the DLA limit, the produced dipoles at each step of the evolution have much greater transverse dimensions than the parent dipoles.

The connection in this approximation between $N$ and the nuclear gluon function is:

$$N(x_{01}, b_0 = 0, Y) = 2 \left\{ 1 - \exp \left[ -\frac{\alpha_s C_F \pi^2}{N_c^2 S_{\perp}} x_{01}^2 AxG(x, 1/x_{01}^2) \right] \right\}. \quad (48)$$

From the above equations and using $x_{01} \approx 2/Q$, one obtains:

$$\frac{\partial N(x_{01}, b_0, Y)}{\partial Y} = \frac{\alpha_s C_F}{\pi} x_{01}^2 \int_{x_{01}^2}^{1/\Lambda_{QCD}^2} \frac{d^2 x_{02}}{x_{02}^2 x_{12}^2} [2N(x_{02}, b_0, Y)$$

$$- N(x_{02}, b_0, Y)N(x_{02}, b_0, Y)]. \quad (49)$$

This equation is precisely the AGL equation.
Light-cone kinematics

- Let $z$ be the longitudinal axis of the collision. For an arbitrary 4-vector $v^\mu = (v^0, v^1, v^2, v^3)$, the light-cone (LC) coordinates are defined as

$$
    v^+ \equiv \frac{1}{\sqrt{2}} (v^0 + v^3), \quad v^- \equiv \frac{1}{\sqrt{2}} (v^0 - v^3), \quad \mathbf{v} \equiv (v^1, v^2)
$$

(50)

- One usually writes $v_\perp \equiv |\mathbf{v}| = \sqrt{(v^1)^2 + (v^2)^2}$

- In these coordinates, $x^+ \equiv \frac{1}{\sqrt{2}} (t + z)$ is the LC time and $x^- \equiv \frac{1}{\sqrt{2}} (t - z)$ is the LC longitudinal coordinate.

- The invariant scalar product of two 4-vectors reads

$$
    p \cdot x = p^0 x^0 - p^1 x^1 - p^2 x^2 - p^3 x^3 \\
    = \frac{1}{2} (p^+ + p^-) (x^+ + x^-) - \frac{1}{2} (p^+ - p^-) (x^+ - x^-) - \mathbf{p} \cdot \mathbf{x} \\
    = p^- x^+ + p^+ x^- - \mathbf{p} \cdot \mathbf{x}
$$

(51)

- This form of the scalar product suggests that $p^-$ should be interpreted as the LC energy and $p^+$ as the LC longitudinal momentum.
**Light-cone kinematics**

- For particles on the mass-shell, $k^\pm = (E \pm k_z)/\sqrt{2}$, with $E^2 = (m^2 + k_z^2 + k^2)$

$$k^+ k^- = \frac{1}{2} (E^2 - k_z^2) = \frac{1}{2} (k^2 + m^2) \equiv m_\perp$$  \hspace{1cm} (52)

- One needs also the *rapidity*

$$y \equiv \frac{1}{2} \ln \frac{k^+}{k^-} = \frac{1}{2} \ln \frac{2k^+ \perp}{m^2_\perp}$$  \hspace{1cm} (53)

- Under a longitudinal Lorentz boost ($k^+ \rightarrow \beta k^+$, $k^- \rightarrow (1/\beta) k^-$, with constant $\beta$), the rapidity is shifted only by a constant, $y \rightarrow y + \beta$

- For a parton inside a right-moving (in the positive $z$ direction) hadron, we introduce the boost-invariant longitudinal momentum fraction $x$

$$x \equiv \frac{k^+}{P^+}$$  \hspace{1cm} (54)