Balitsky–Kovchegov evolution equation

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Introduction

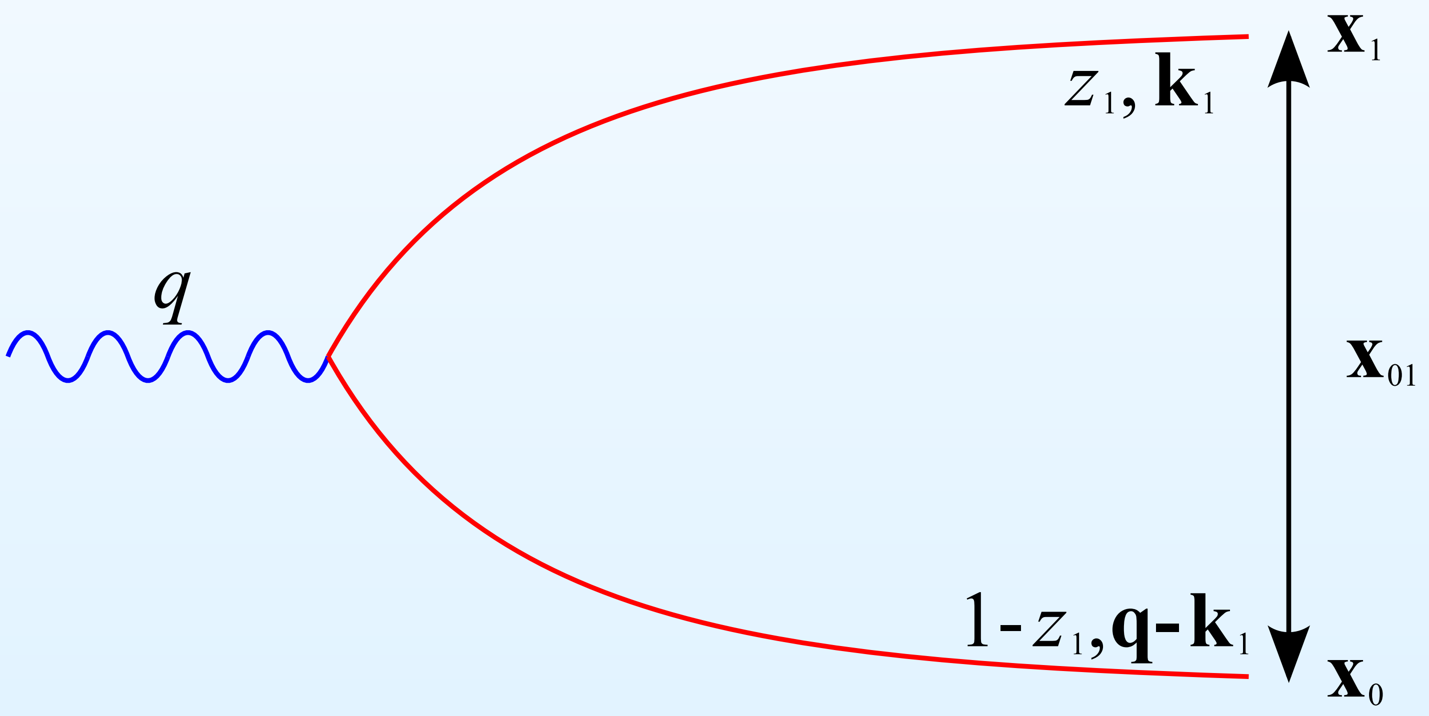
- When the center of mass energy in a collision is much bigger than the fixed hard scale (for example, in DIS, the photon virtuality is the hard scale and $s \gg Q^2$), parton densities increase with growing energy.
- This produces larger scattering amplitudes.
- In this regime of saturation, BFKL equation is not valid anymore.
- Using the $q\bar{q}$-dipole model, Kovchegov was able to derive a new equation.
- Kovchegov equation is not exact, but it is a mean field approximation of a hierarchy of equations derived previously by Balitsky.
- This equation is nonlinear and can be reduced to BFKL equation when the nonlinear term is disregarded.

Dipole picture

- For simplicity, we should consider deep inelastic scattering of a virtual photon on a hadron or nucleus.
- In the dipole picture, the system is studied in the reference frame of the target (the hadron or the nucleus is at rest).
- Therefore, all QCD evolution is included in the wave function of the incoming photon.
- The incoming photon cannot directly strongly interact with the target.
- The splitting of the photon into a quark-antiquark pair is then considered.
- This heavy quark-antiquark pair is called *onium*.
- Quarks have color, but the incident photon is colorless.
- Then, the quark has color and the antiquark the correspondent anticolor.

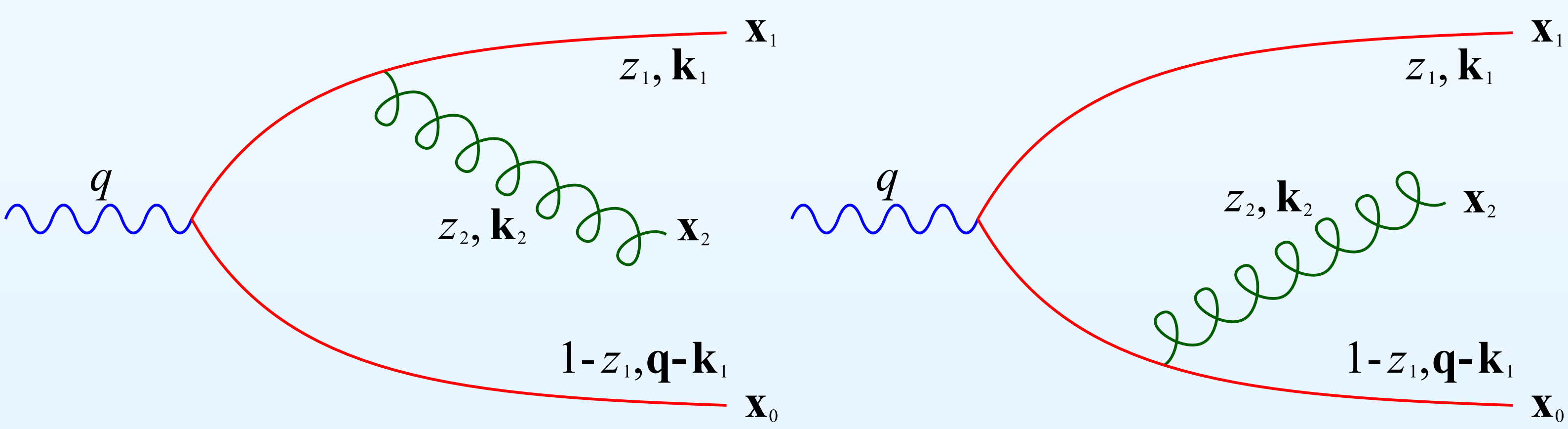
Dipole picture

- In the figure, the photon splitting is shown.
- z_1 is the fraction of light cone momentum.
- $\mathbf{x}_0$ and $\mathbf{x}_1$ are quark and antiquark positions ($\mathbf{x}_{01} = \mathbf{x}_0 - \mathbf{x}_1$).

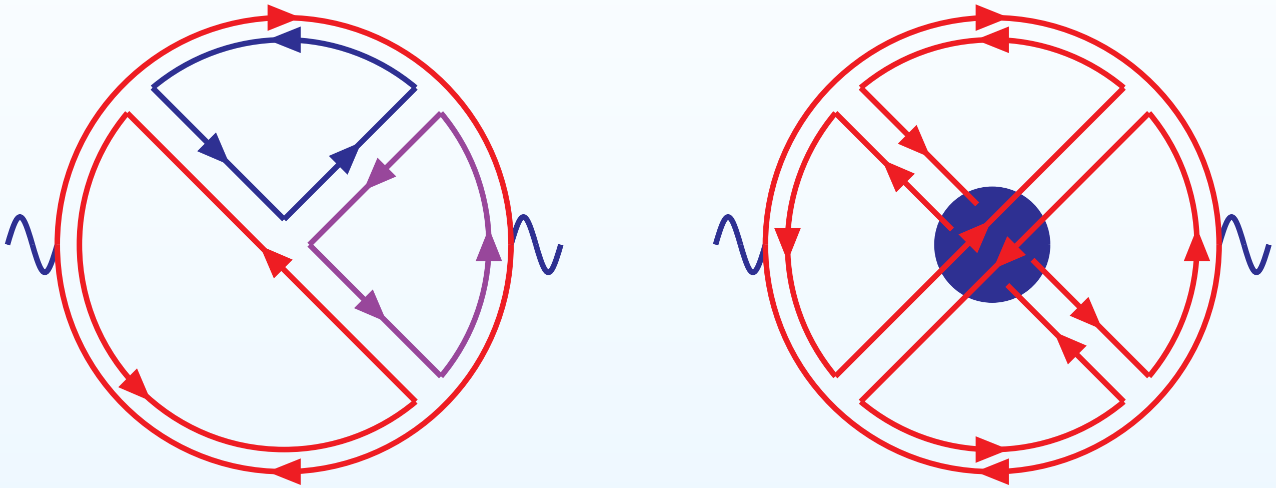


Gluon emission

- The quark or antiquark can emit soft gluons ($z_2 << z_1$).
- These soft gluons can also emit softer gluons.



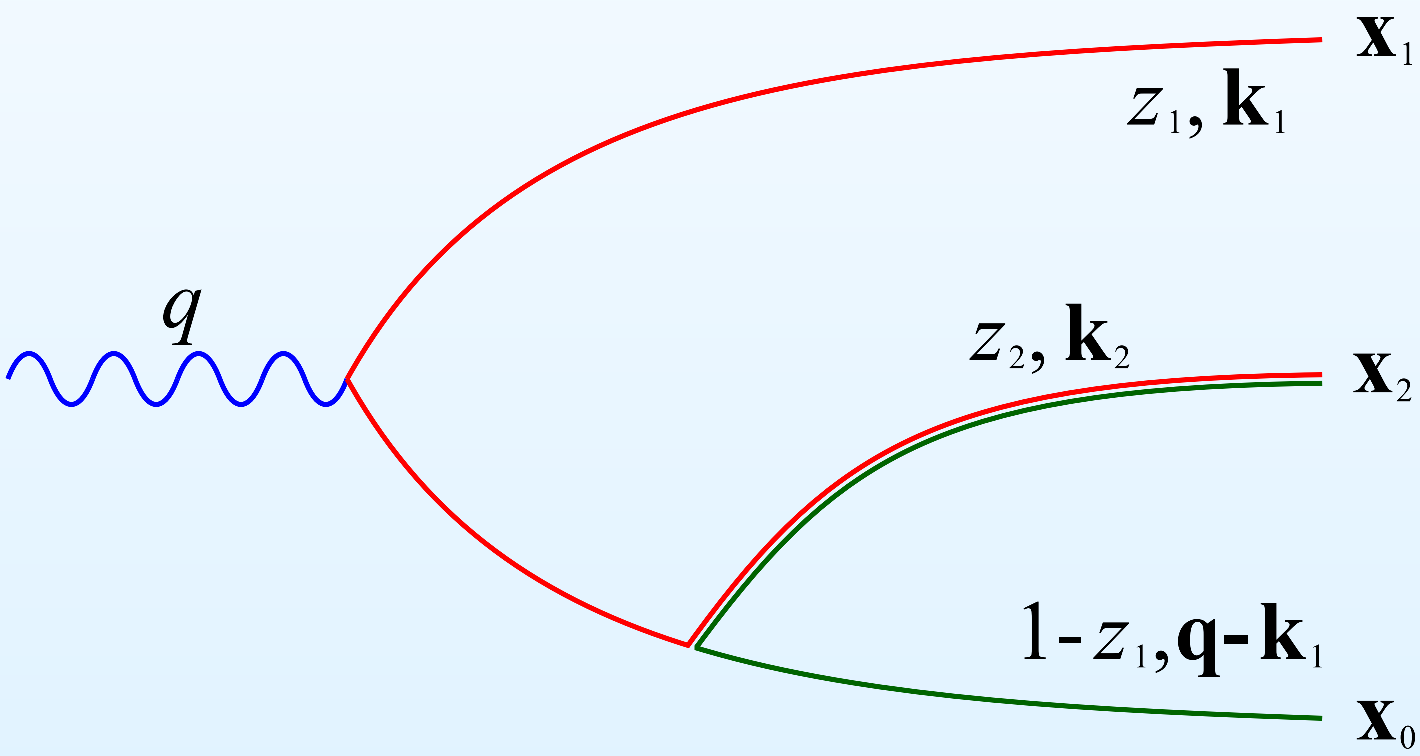
Planar diagrams



- Each loop count as N_c.
- Each vertex count as α_s.
- Therefore, planar diagrams are enhanced by N_c^2 compared to nonplanar diagrams of the same order em α_s.

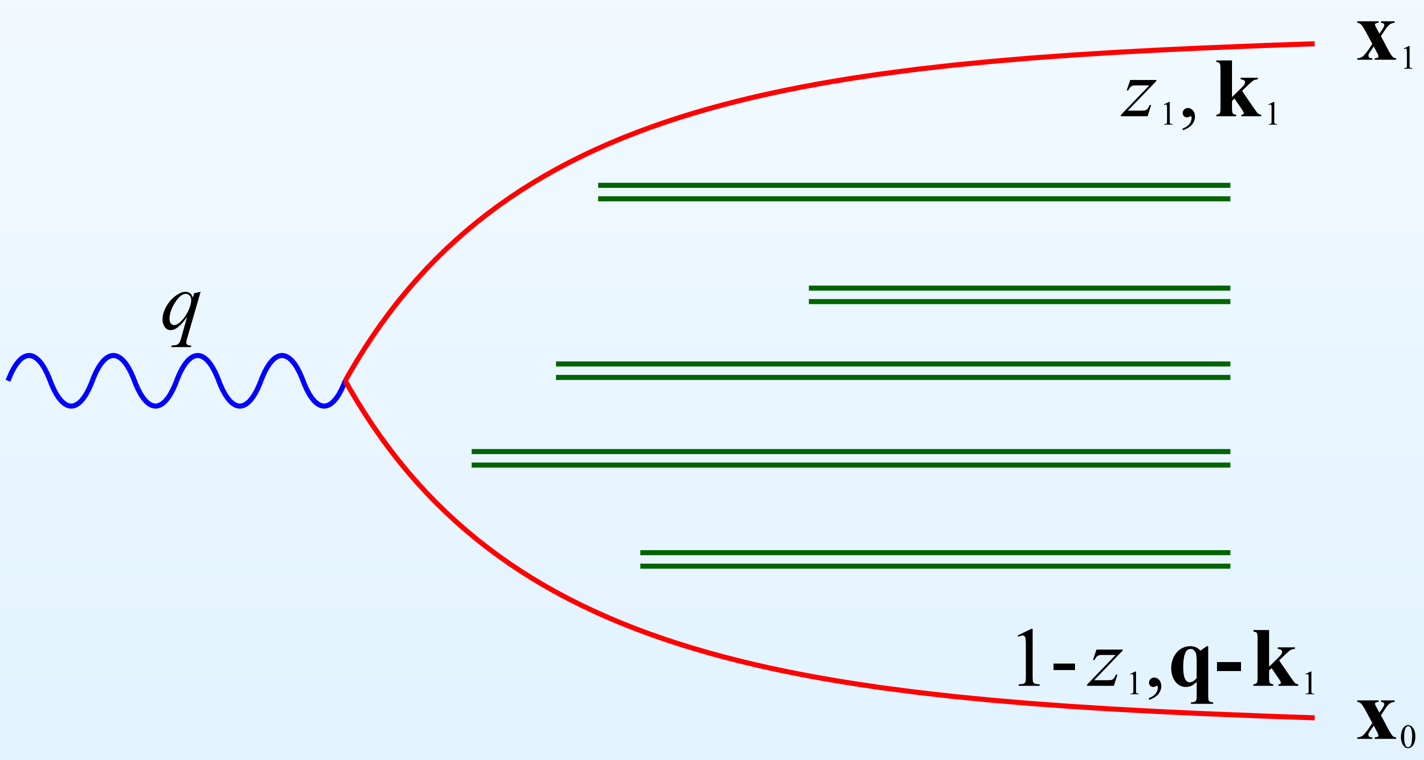
Large-N_c limit

- In the limit of large number of colors ($N_c \to \infty$), these soft gluons can be considered as quark-antiquark pairs.
- Therefore, the original dipole ($\mathbf{x}_{01}$) *splits* into two new dipoles ($\mathbf{x}_{02}$ and $\mathbf{x}_{21}$).



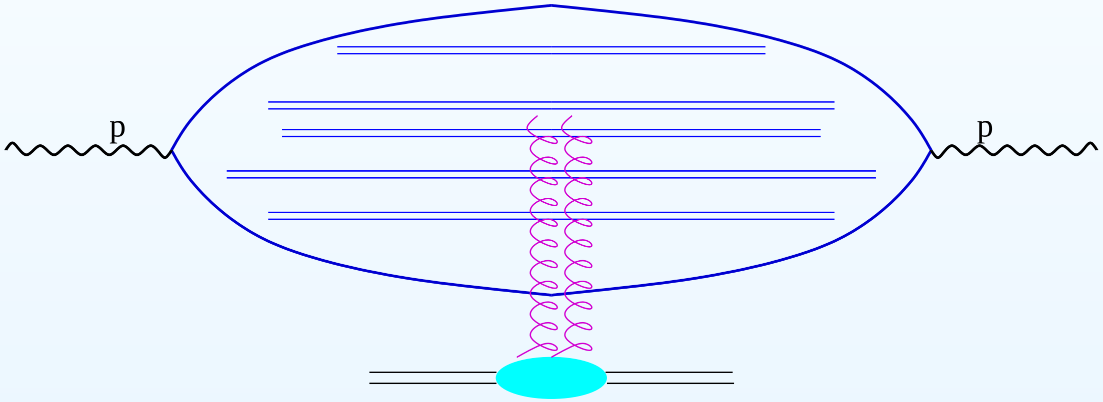
The onium

- The original incoming photon can be represented now by an arbitrary number of dipoles.
- These dipoles do not interact with each other.



Onium-hadron scattering

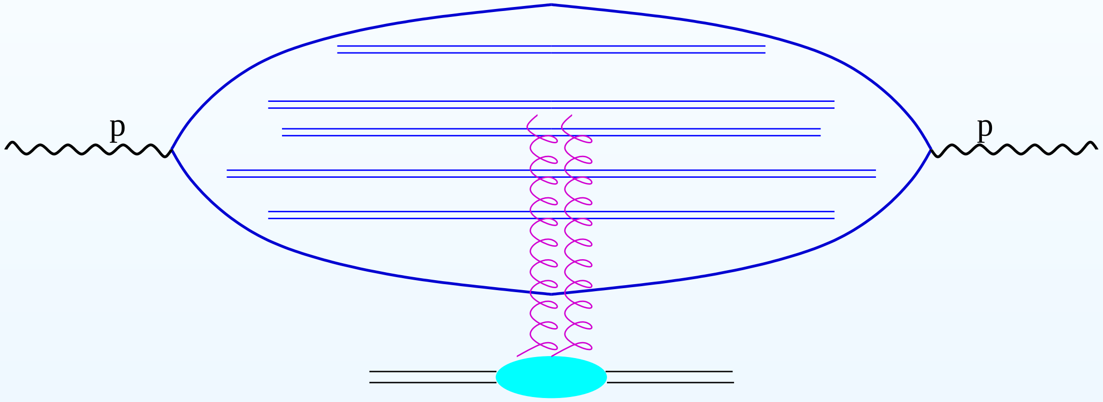
- Now, the nucleus or hadron is not going to scatter off the photon, but off the complex structure of color dipoles:



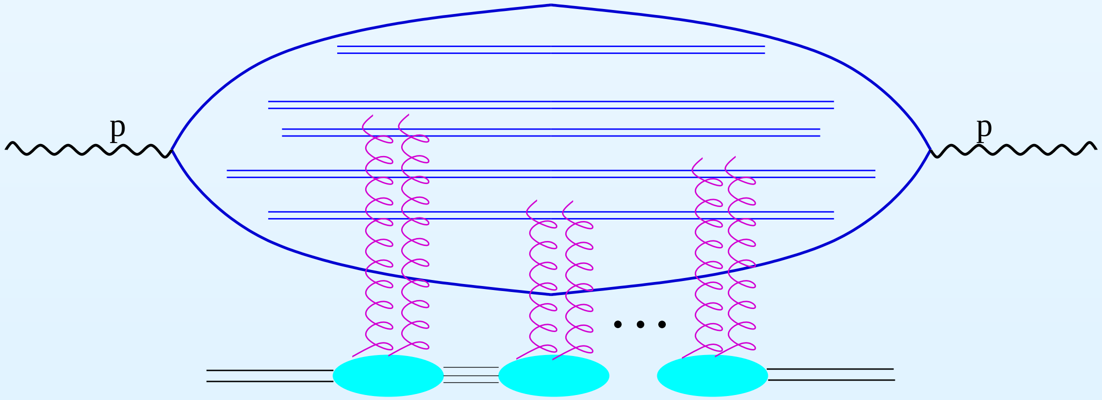
- Two gluons must be exchanged to conserve color.

Single and multiple scattering

- Single dipole scattering leads to linear BFKL equation.

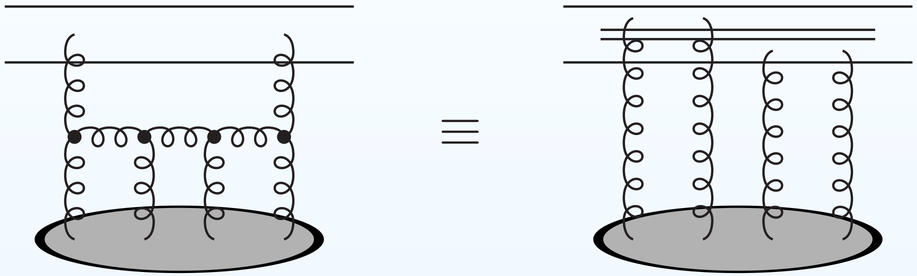


- Multiple dipole scatterings lead to nonlinear BK equation.

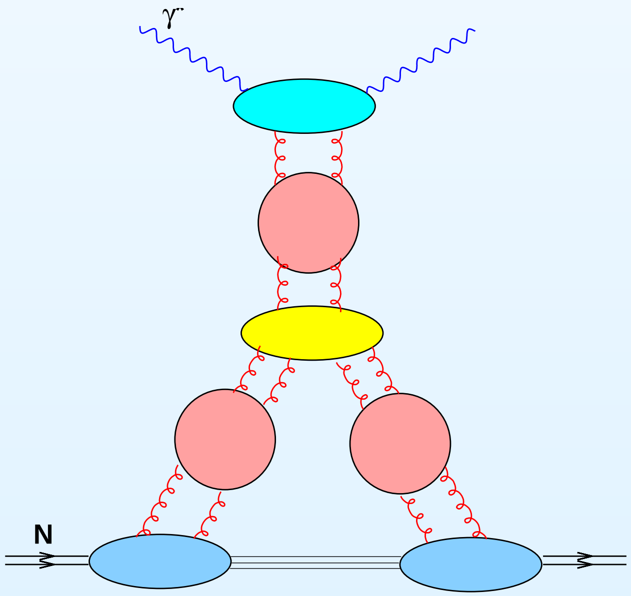


Fan diagrams

- Pomeron merging in the target can be understood as a multiple scattering between target and projectile.

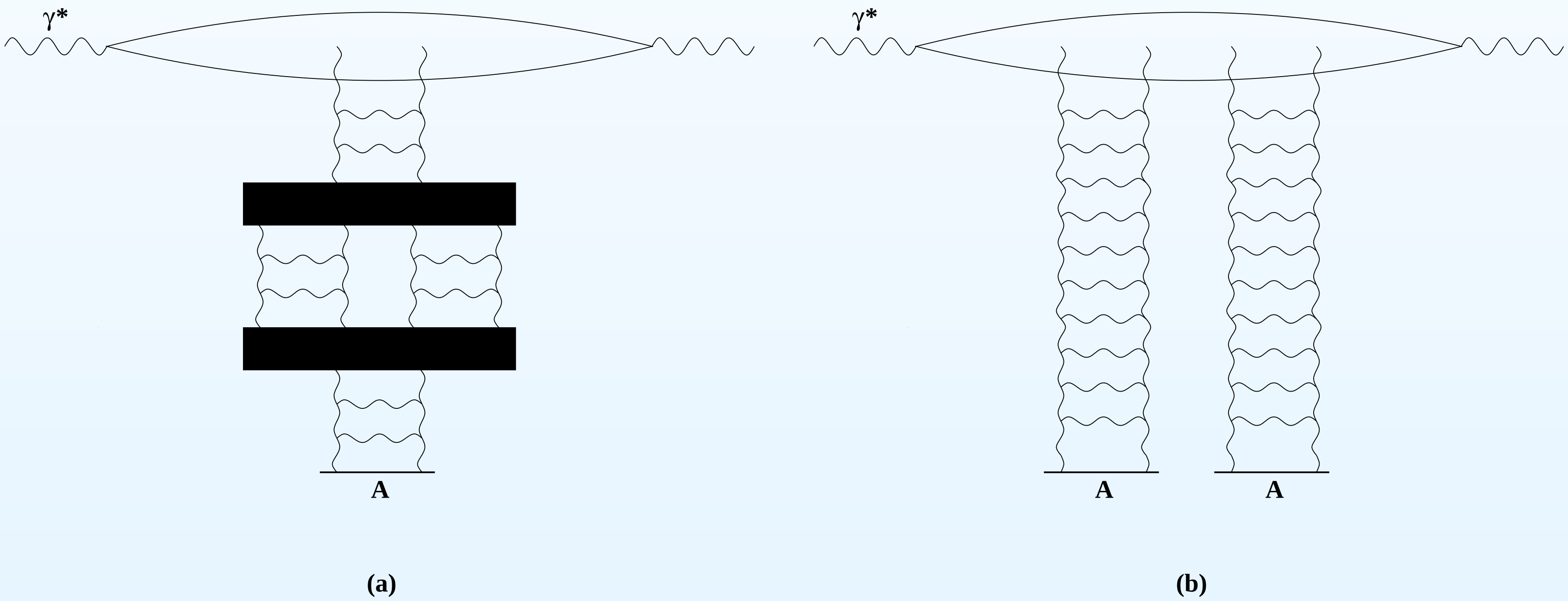


- Therefore, all fan diagrams are included.



Diagrams not included

- Not all diagrams are included, however.
- Diagrams representing Pomeron merging (a) cannot be represented as a multiple scattering (b).



F_2 structure function

- In the dipole picture, the F_2 structure function is given by:

$$F_2(x, Q^2) = \frac{Q^2}{4\pi^2 \alpha_{em}} \int \frac{d^2\mathbf{x}_{01} dz}{2\pi} \Phi(\mathbf{x}_{01}, z) \sigma_{dip}(\mathbf{x}_{01}, Y) \tag{1}$$

where $\Phi(\mathbf{x}_{01}, z) = \Phi_T(\mathbf{x}_{01}, z) + \Phi_L(\mathbf{x}_{01}, z)$ is the sum of the square of the transverse (T) and longitudinal (L) photon wavefunctions (more specifically, the wavefunction for a photon to go into a dipole)

$$\Phi_T(\mathbf{x}_{01}, z) = \frac{2N_c \alpha_{EM}}{\pi} a^2 K_1^2(x_{01} a)[z^2 + (1-z)^2] \tag{2}$$

$$\Phi_L(\mathbf{x}_{01}, z) = 4Q^2 z^2 (1-z)^2 K_0^2(x_{01} a). \tag{3}$$

with $a^2 = Q^2 z(1-z)$.

- σ_{dip} is the dipole-nucleus (or nucleon) cross section

The forward scattering amplitude

- The dipole cross section is given by

$$\sigma_{dip}(\mathbf{x}_{01}, Y) = 2 \int d^2\mathbf{b}_{01} \mathcal{N}(\mathbf{b}_{01}, \mathbf{x}_{01}, Y). \quad (4)$$

- The quantity $\mathbf{b}_{01}$ is the impact parameter given by:

$$\mathbf{b}_{01} = \frac{\mathbf{x}_1 + \mathbf{x}_0}{2} \quad (5)$$

- $\mathcal{N}(\mathbf{b}_{01}, \mathbf{x}_{01}, Y)$ is the propagator of the quark-antiquark pair through the nucleus related to the *forward scattering amplitude* of the quark-antiquark pair with the nucleus.

Balitsky–Kovchegov equation

- The Balitsky–Kovchegov equation is given by:

$$\begin{aligned}\frac{d\mathcal{N}(\mathbf{b}_{01},\mathbf{x}_{01},Y)}{dY} &= \frac{\bar{\alpha}_s}{2\pi}\int d^2\mathbf{x}_2 \frac{\mathbf{x}_{01}^2}{\mathbf{x}_{20}^2\mathbf{x}_{12}^2}\Big[\mathcal{N}\Big(\mathbf{b}_{01}+\frac{\mathbf{x}_{21}}{2},\mathbf{x}_{20},Y\Big) \\ &+\mathcal{N}\Big(\mathbf{b}_{01}+\frac{\mathbf{x}_{20}}{2},\mathbf{x}_{21},Y\Big)-\mathcal{N}(\mathbf{b}_{01},\mathbf{x}_{01},Y) \\ &-\mathcal{N}\Big(\mathbf{b}_{01}+\frac{\mathbf{x}_{12}}{2},\mathbf{x}_{20},Y\Big)\mathcal{N}\Big(\mathbf{b}_{01}+\frac{\mathbf{x}_{20}}{2},\mathbf{x}_{21},Y\Big)\Big] \qquad (6)\end{aligned}$$

- This equation evolves $\mathcal{N}(\mathbf{b}_{01},\mathbf{x}_{01},Y)$.
- The evolution quantity is the rapidity $Y \approx \ln 1/x$.
- Besides the evolution variable, one has 4 degrees of freedom (2 from $\mathbf{b}_{01}$ and 2 from $\mathbf{x}_{01}$).
- It resumms all powers of $\alpha_s \ln 1/x$.
- $\bar{\alpha}_s = \alpha_s N_c/\pi$

Balitsky approach

- A general feature of high-energy scattering is that a fast particle moves along its straight-line classical trajectory and the only quantum effect is the eikonal phase factor acquired along this propagation path.
- In QCD, for the fast quark or gluon scattering off some target, this eikonal phase factor is a Wilson line:
- the infinite gauge link ordered along the straight line collinear to particle's velocity n^μ:

$$U^\eta(x_\perp) = \mathrm{Pexp}\Big\{ig\int_{-\infty}^{\infty} du\, n_\mu\, A^\mu(un + x_\perp)\Big\}, \tag{7}$$

- Here A_μ is the gluon field of the target, $x_\perp$ is the transverse position of the particle which remains unchanged throughout the collision, and the index η labels the rapidity of the particle.
- Repeating the argument for the target (moving fast in the spectator's frame) we see that particles with very different rapidities perceive each other as Wilson lines.
- Therefore, Wilson-line operators form the convenient effective degrees of freedom in high-energy QCD.

Balitsky approach

- At small $x_B = Q^2/(2p \cdot q)$, the virtual photon decomposes into a pair of fast quarks moving along straight lines separated by some transverse distance.
- The propagation of this quark-antiquark pair reduces to the “propagator of the color dipole” $U(x_\perp)U^\dagger(y_\perp)$ - two Wilson lines ordered along the direction collinear to quarks’ velocity.
- The structure function of a hadron is proportional to a matrix element of this color dipole operator

$$\hat{\mathcal{U}}^\eta(x_\perp, y_\perp) = 1 - \frac{1}{N_c}\mathrm{Tr}\{\hat{U}^\eta(x_\perp)\hat{U}^{\dagger\eta}(y_\perp)\} \tag{8}$$

- sandwiched by target’s states.
- The gluon parton density is approximately

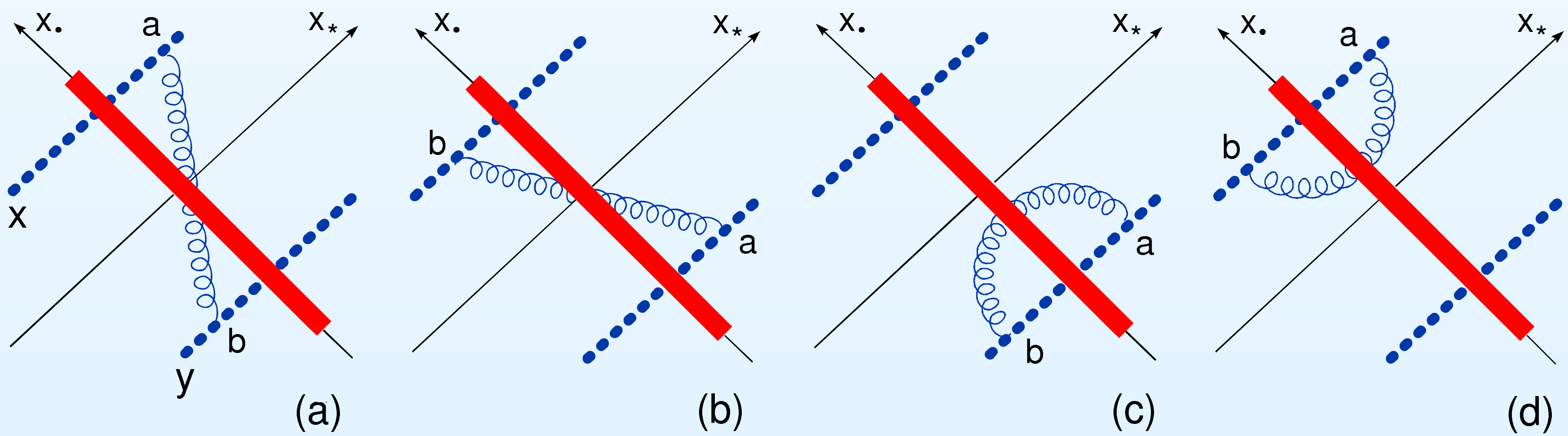
$$x_B G(x_B, \mu^2 = Q^2) \ \simeq \ \langle p|\,\hat{\mathcal{U}}^\eta(x_\perp, 0)|p\rangle\Big|_{x_\perp^2 = Q^{-2}} \tag{9}$$

where $\eta = \ln\frac{1}{x_B}$.

Balitsky approach

- The energy dependence of the structure function is translated then into the dependence of the color dipole on the slope of the Wilson lines determined by the rapidity η.
- At relatively high energies and for sufficiently small dipoles we can use the leading logarithmic approximation (LLA) where $\alpha_s \ll 1,\ \alpha_s \ln x_B \sim 1$ and get the non-linear BK evolution equation for the color dipoles:

$$\frac{d}{d\eta}\,\hat{\mathcal{U}}(x,y) = \frac{\alpha_s N_c}{2\pi^2}\int d^2z\, \frac{(x-y)^2}{(x-z)^2(z-y)^2}[\hat{\mathcal{U}}(x,z)+\hat{\mathcal{U}}(y,z)-\hat{\mathcal{U}}(x,y)-\hat{\mathcal{U}}(x,z)\hat{\mathcal{U}}(z,y)] \tag{10}$$



Balitsky hierarchy

- The operator equation creates an hierarchy of equations for the observables:

$$\frac{d}{d\eta}\left\langle \hat{\mathcal{U}}(x,y)\right\rangle = \frac{\alpha_s N_c}{2\pi^2}\int d^2z\,\frac{(x-y)^2}{(x-z)^2(z-y)^2}\left[\left\langle \hat{\mathcal{U}}(x,z)\right\rangle + \left\langle \hat{\mathcal{U}}(y,z)\right\rangle - \left\langle \hat{\mathcal{U}}(x,y)\right\rangle - \left\langle \hat{\mathcal{U}}(x,z)\hat{\mathcal{U}}(z,y)\right\rangle\right] \quad (11)$$

$$\frac{d}{d\eta}\langle T_{xy}\rangle = \frac{\bar{\alpha}_s}{2\pi}\int d^2z\,\mathcal{M}(x,y,z)\left[\langle T_{xz}\rangle + \langle T_{yz}\rangle - \langle T_{xy}\rangle - \langle T_{xz}T_{zy}\rangle\right] \quad (12)$$

- In the mean field approximation, $\langle T_{xz}T_{zy}\rangle = \langle T_{xz}\rangle\langle T_{zy}\rangle$, and BK equation is recovered.

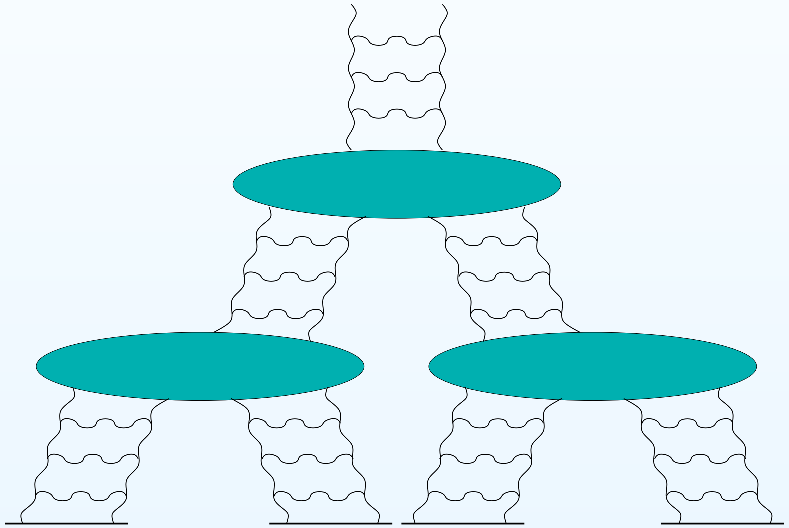
The BFKL limit

- The last (and nonlinear) term is a direct consequence of taking into account multiple scatterings.
- BK equation is a closed equation, instead of Balitsky original hierarchy of equations.
- The mean field approximation of Balitsky first equation gives the BK equation.
- If we disregard the last term, we obtain the following linear equation that can be shown equivalent to BFKL equation:

$$
\begin{aligned}
\frac{d\mathcal{N}(\mathbf{b}_{01},\mathbf{x}_{01},Y)}{dY} \quad = \quad & \frac{\bar{\alpha}_s}{2\pi}\int d^2\mathbf{x}_2\,\frac{\mathbf{x}_{01}^2}{\mathbf{x}_{20}^2\mathbf{x}_{12}^2}\left[\mathcal{N}\left(\mathbf{b}_{01}+\frac{\mathbf{x}_{21}}{2},\mathbf{x}_{20},Y\right)\right. \\
& \left.+\mathcal{N}\left(\mathbf{b}_{01}+\frac{\mathbf{x}_{21}}{2},\mathbf{x}_{20},Y\right)-\mathcal{N}\left(\mathbf{b}_{01},\mathbf{x}_{01},Y\right)\right] \qquad (13)
\end{aligned}
$$

BK *versus* AGL equation

- BK, AGL and GLR equations sum the same diagrams (for example):



- All three use the eikonal approximation, i. e., incoming photon momentum has large q_+.
- However, BK equation does not consider large Q^2. It resums all terms $1/(2q_+q_-) \approx 1/Q^2$.

BK in 0+1 dimensions

- If one neglects the spatial dependence ($\mathcal{N}(\mathbf{x}_{02}, \mathbf{b}_0, Y)$), BK equation is given by the logistic equation:

$$\frac{d\mathcal{N}}{dY} = \omega(\mathcal{N} - \mathcal{N}^2) \qquad \omega > 0. \tag{14}$$

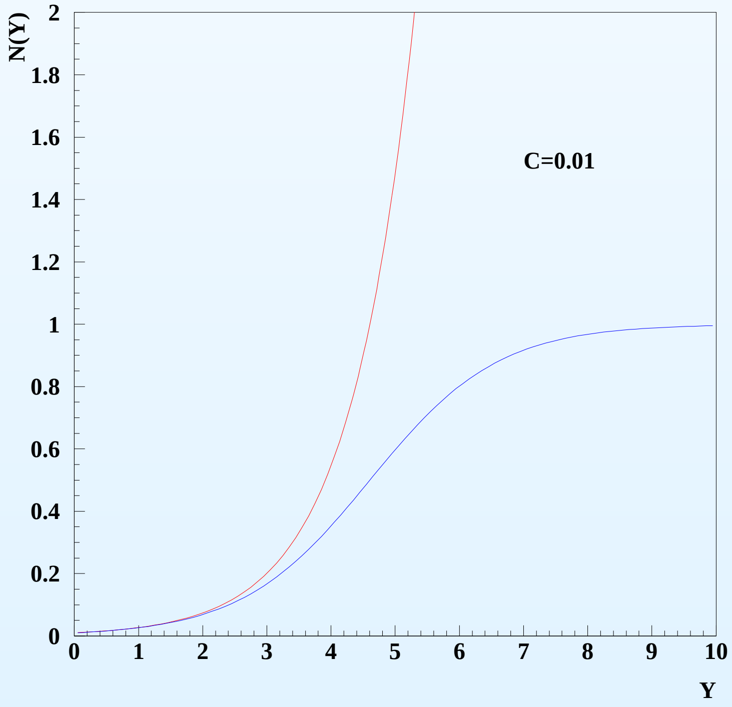
- This equation has two fixed points, one unstable ($\mathcal{N} = 0$) and another stable ($\mathcal{N} = 1$).
- For every initial condition in $0 < \mathcal{N} \leq 1$, for $Y \to \infty$, $\mathcal{N} \to 1$.
- In other words, there is saturation for high$Y$ (small x).
- If the $\mathcal{N}^2$ term is neglected, the fixed point at $\mathcal{N} = 1$ disappears and $\mathcal{N} \to \infty$ for $Y \to \infty$.

BK in 0+1 dimensions

- The logistic equation complete solution is ($C = \mathcal{N}(Y=0)$):

$$\mathcal{N}(Y) = \frac{C\mathrm{e}^{\omega Y}}{1 + C(\mathrm{e}^{\omega Y} - 1)}. \quad (15)$$

- Solutions: linear (red) and nonlinear (blue) equations.



BK in 1+1 dimensions

- If impact parameter is disregarded and only sizes of dipoles are considered:

$$\mathcal{N}(\mathbf{b},\mathbf{x},Y) \to \mathcal{N}(r,Y). \tag{16}$$

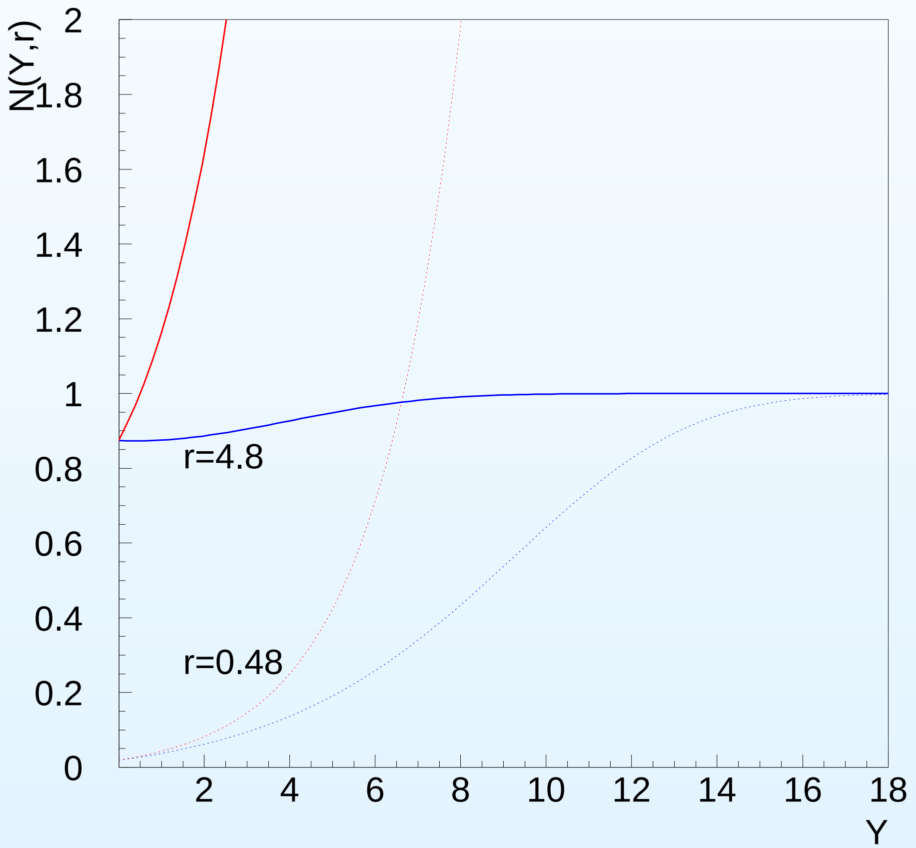
- BK equation is then simplified:

$$\begin{aligned} \frac{d\mathcal{N}(|\mathbf{x}_{01}|,Y)}{dY} &= \frac{\bar{\alpha}_s}{2\pi}\int d^2\mathbf{x}_2 \frac{\mathbf{x}_{01}^2}{\mathbf{x}_{20}^2\mathbf{x}_{12}^2}\left[\mathcal{N}\left(|\mathbf{x}_{20}|,Y\right)+\mathcal{N}\left(|\mathbf{x}_{21}|,Y\right)\right. \\ & \left. -\mathcal{N}\left(|\mathbf{x}_{01}|,Y\right)-\mathcal{N}\left(|\mathbf{x}_{21}|,Y\right)\mathcal{N}\left(|\mathbf{x}_{20}|,Y\right)\right] \end{aligned} \tag{17}$$

- Physically speaking, this represents the scattering on infinite and uniform nucleus.
- We note that the kernel already depends only on dipole sizes.

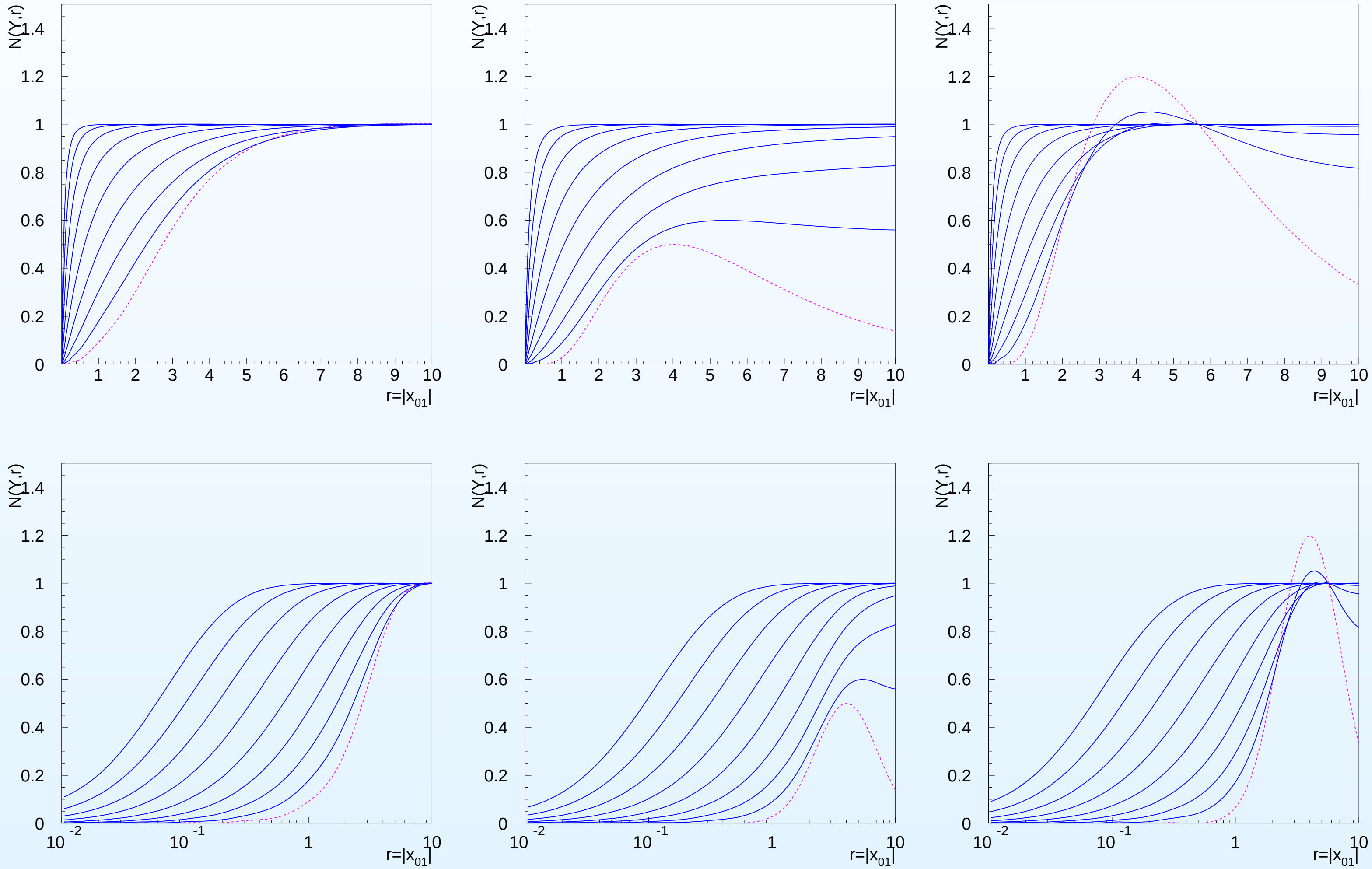
BK in 1+1 dimensions

- The same kind of solutions of BK equation in (0+1) dimensions is found for BK equation in (1+1) dimensions for fixed r.
- Solutions: linear (red) and nonlinear (blue) equations.



BK in 1+1 dimensions

- Solutions for different initial conditions in linear (above) and log (below) scales.



Saturation scale

- The figures shown previously can be divided into three regions in r:
  - The small r region, where nonlinear corrections are negligible;
  - The large r region, where nonlinear corrections dominate and $\mathcal{N} \approx 1$;
  - The region between the first two.
- One can introduce the saturation scale:

$$\begin{aligned} r &< \frac{1}{Q_s(Y)} \rightarrow \mathcal{N} << 1, \\ r &> \frac{1}{Q_s(Y)} \rightarrow \mathcal{N} \approx 1. \end{aligned}$$

Saturation scale

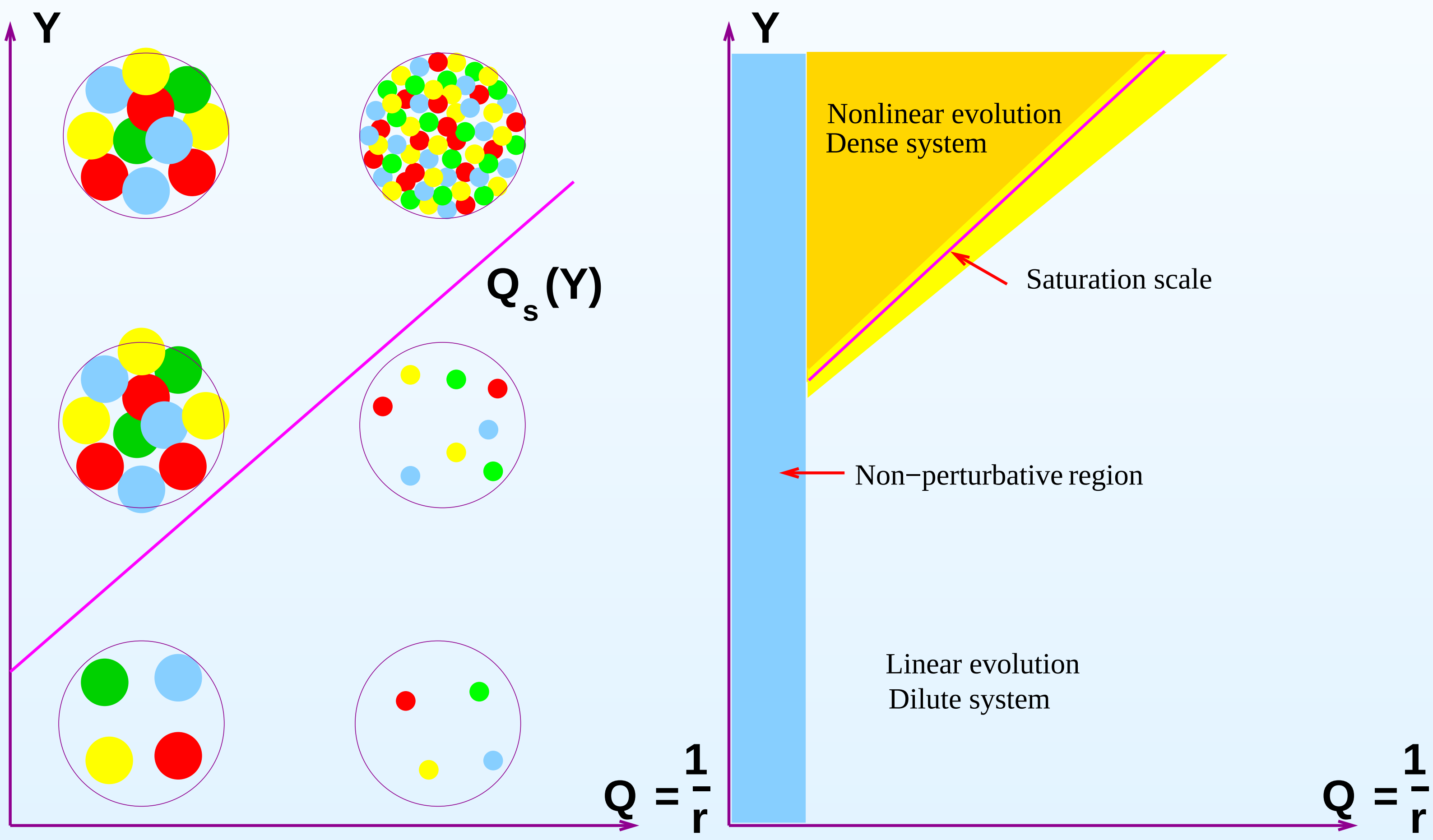
- One can get a saturation scale from BK solutions.
- Neglecting some corrections, it is given by:

$$Q_s(Y) = Q_0 \exp(\bar{\alpha}_s \lambda Y) Y^{-\beta} \qquad \lambda \approx 2.4$$

- The saturation scale is the dividing line between dense and dilute regions.
- The higher the rapidity the denser the system gets and partons start to re-interact.
- Also, saturation occurs earlier if the size of partons is bigger.

Saturation scale

- Dilute and saturated regimes can be represented in 2-D graphs ($Y = \ln(1/x)$):



GBW model

- Linear growing and saturation properties of BK equation are roughly similarly found in the Golec-Biernat and Wusthoff (GBW) *saturation model*.
- One of the key ideas to GBW model is the assumption of a x-dependent radius:

$$R_0(x) = \frac{1}{Q_0}\left(\frac{x}{x_0}\right)^{\frac{\lambda}{2}}, \tag{18}$$

which scales the quark-antiquark separation r in the dipole cross section:

$$\hat{r} = \frac{r}{2R_0(x)}. \tag{19}$$

- Then, the dipole cross section can be written as:

$$\sigma_{dip}(x, r^2) = \sigma_0 g(\hat{r}^2). \tag{20}$$

- $Q_0 = 1$ GeV sets the dimension.

GBW model: dipole cross section

- The function $g(\hat{r}^2)$ chosen is:

$$g(\hat{r}^2) = 1 - \exp[-\hat{r}^2]. \tag{21}$$

- The dipole cross section is then:

$$\begin{aligned} \sigma_{dip}(x,r) &= \sigma_0 \left(1 - \exp\left[-\left(\frac{r}{2R_0}\right)^2\right]\right) \\ &= \sigma_0 \left(1 - \exp\left[-r^2 \frac{Q_0^2}{4}\left(\frac{x_0}{x}\right)^\lambda\right]\right) \end{aligned} \tag{22}$$

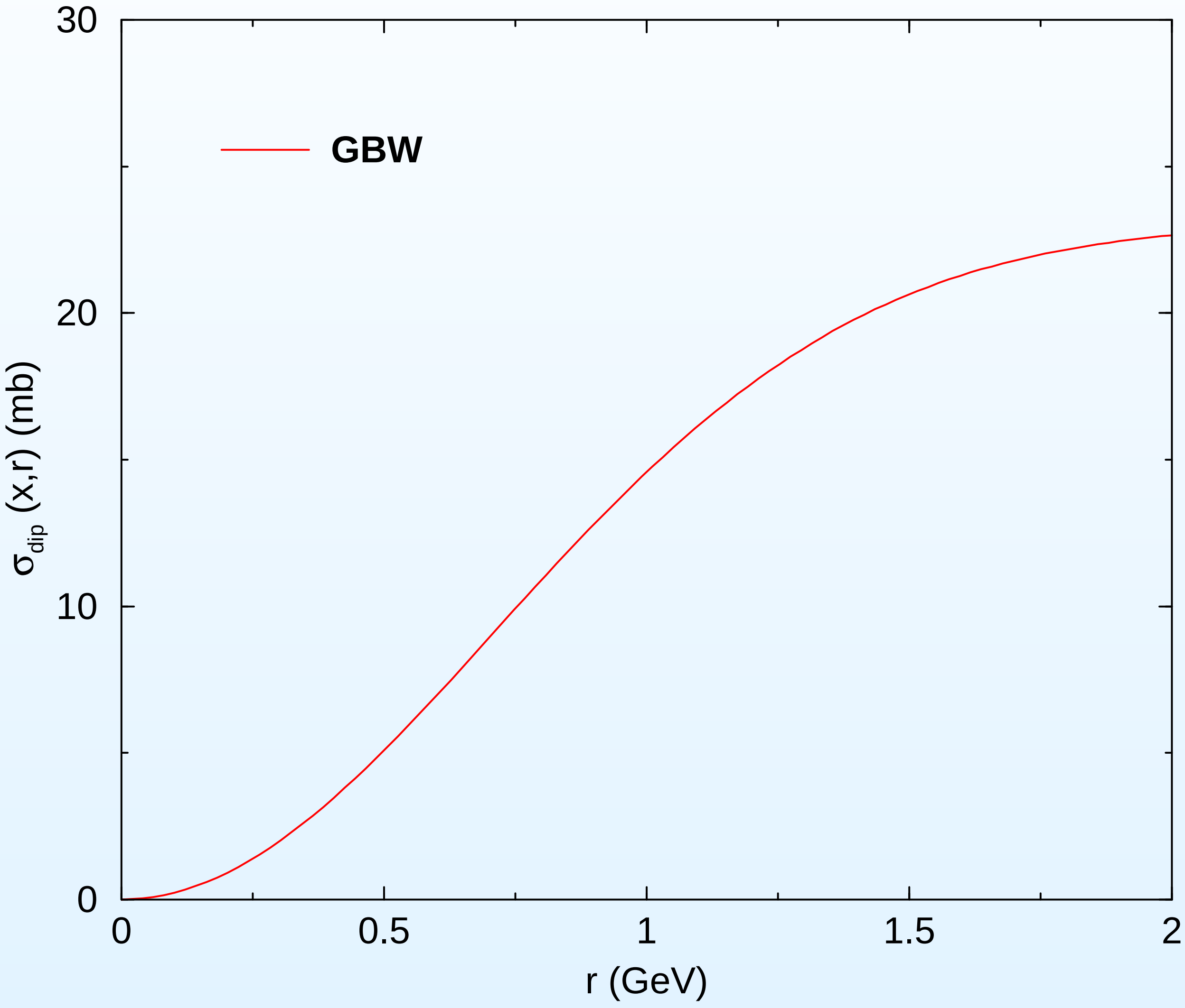
- The three parameters were fitted to $\sigma_0 = 23mb$, $\lambda = 0.288$ and $x_0 = 3\dot{1}0^{-4}$.
- The cross section can be written as a function of Y also:

$$\sigma_{dip}(Y,r) = \int d^2\mathbf{b} N(\mathbf{b},r,Y) = \sigma_0 \left[1 - \exp\left(-\frac{r^2 Q_s^2(Y)}{4}\right)\right]$$

with $Q_s^2(Y) = \exp(0.28(Y - Y_0))$.

GBW model: σ_{dip} behavior

The cross section behavior is shown in the figure:



Saturation and color transparency

- In GBW model, when $r << 1/Q_s^2(Y)$, the phenomenon of color transparency appears (small dipoles have small chance of interaction):

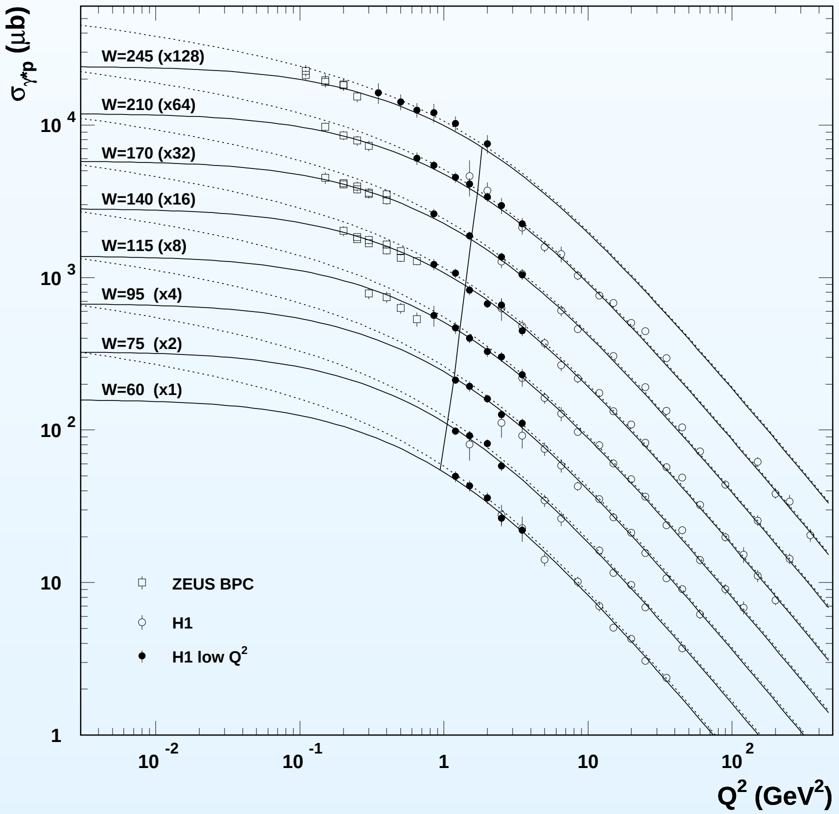
$$\frac{\sigma(Y,r)}{\sigma_0} = \frac{r^2 Q_s^2(Y)}{4}.$$

- When $r >> 1/Q_s^2(Y)$, the cross section saturates:

$$\frac{\sigma(Y,r)}{\sigma_0} \approx 1.$$

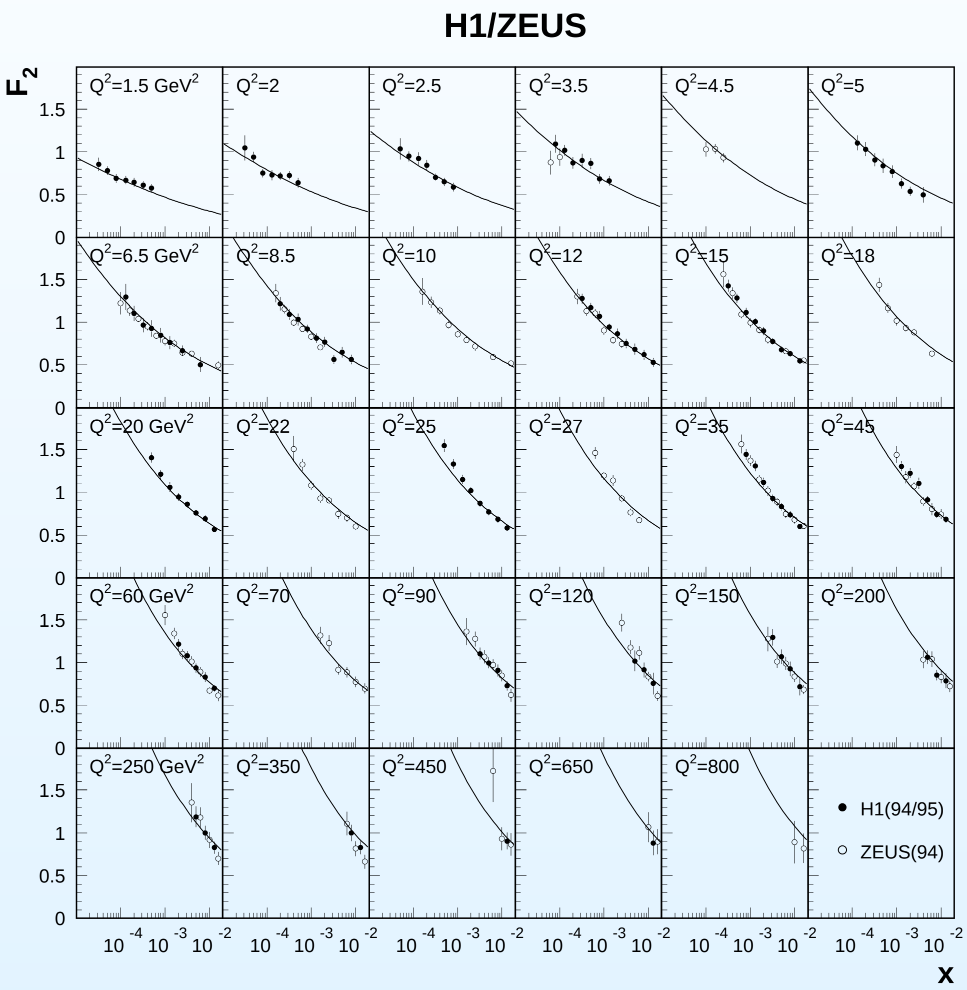
GBW Results

The $\gamma^* p$-cross section for various energies (solid lines: quark mass of 140MeV; dotted lines: zero quark mass).



GBW Results

The results for the fit to the inclusive HERA data on F_2 for different values of the virtuality.



Saturation without unitarization

- Consider impact parameter dependence.
- Total inelastic cross section, in the saturation regime ($t = ln(s/s0)$):

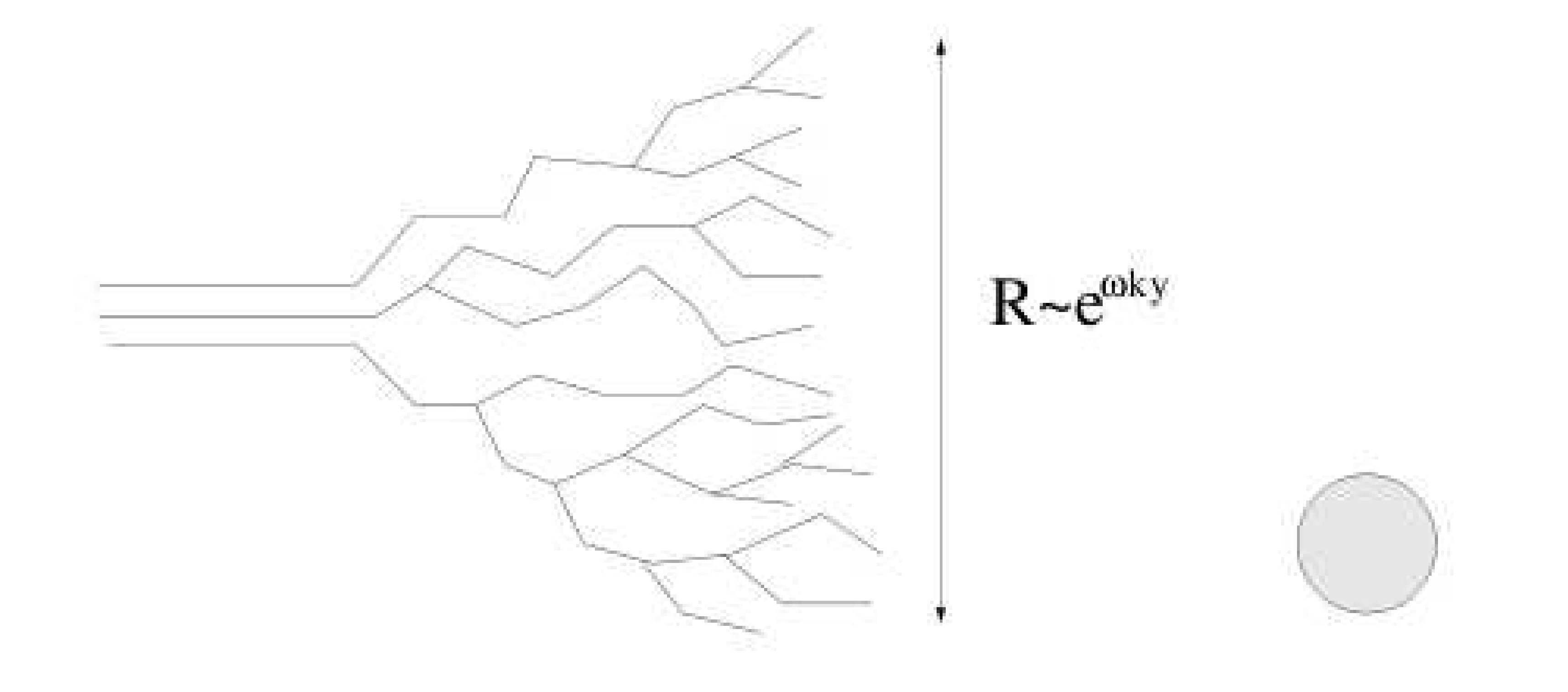
$$\sigma = \pi R^2(t)\,. \tag{23}$$

- To satisfy the Froissart bound the radius $R(t)$ should grow at most linearly with t.
- However, at large rapidities the radius of the saturated region is exponentially large

$$R(t) = R(t_0)\exp\Big[\frac{\alpha_s N_c}{2\pi}\epsilon(t - t_0)\Big]\,. \tag{24}$$

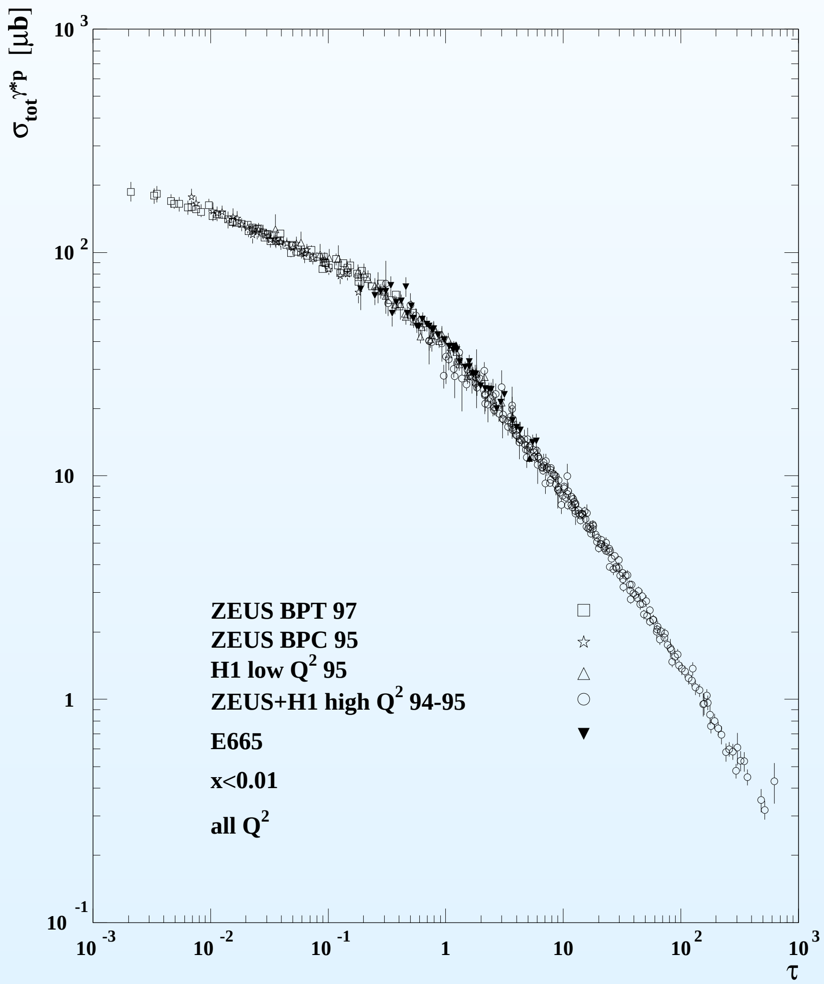
[A. Kovner, U. A. Wiedemann; Phys.Rev. D66 (2002) 051502.]

Saturation without unitarization



Geometric scaling

- *Geometric scaling* is a phenomenological feature of DIS which has been observed in the HERA data on inclusive $\gamma^* - p$ scattering, which is expressed as a scaling property of the virtual photon-proton cross section



$$\sigma^{\gamma^* p}(Y,Q) = \sigma^{\gamma^* p}(\tau)\,, \qquad \tau = \frac{Q^2}{Q_s^2(Y)}$$

where Q is the virtuality of the photon, $Y = \log 1/x$ is the total rapidity and $Q_s(Y)$ is the saturation scale

[Stasto, Golec Biernat and Kwiecinsky, 2001]

Geometric scaling

- Looking to the solutions of BK equation, one sees that they reach a universal shape independently of the initial condition.
- The solutions even look similar when calculated for different Y (they appear to be shifted when Y is changed).
- In terms of the scattering amplitude, the geometric scaling means that BK solution depends only on $rQ_s(Y)$ for asymptotic values of rapidity.

$$\mathcal{N}(r,Y) = \mathcal{N}(rQ_s(Y)).$$

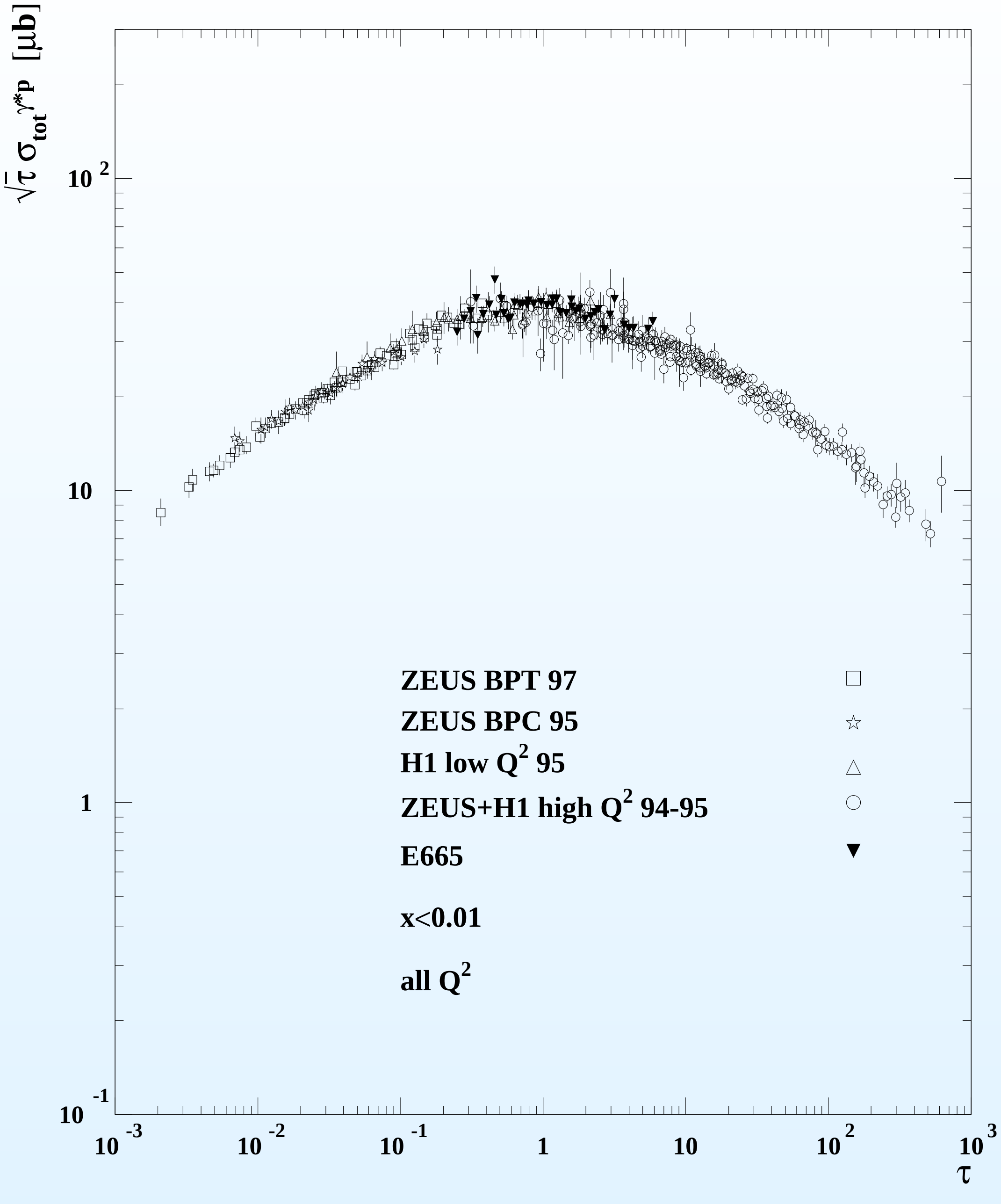
- Using $Q_s(Y) \approx Q_0 \exp(\bar{\alpha}_s \lambda Y)$;

$$\mathcal{N}(r,Y) = \mathcal{N}(Q_0 \exp(\ln r + \bar{\alpha}_s \lambda Y))$$

and the geometric scaling relation resembles as a traveling wave with velocity $\bar{\alpha}_s \lambda$, time Y and spatial coordinate $\ln r$.

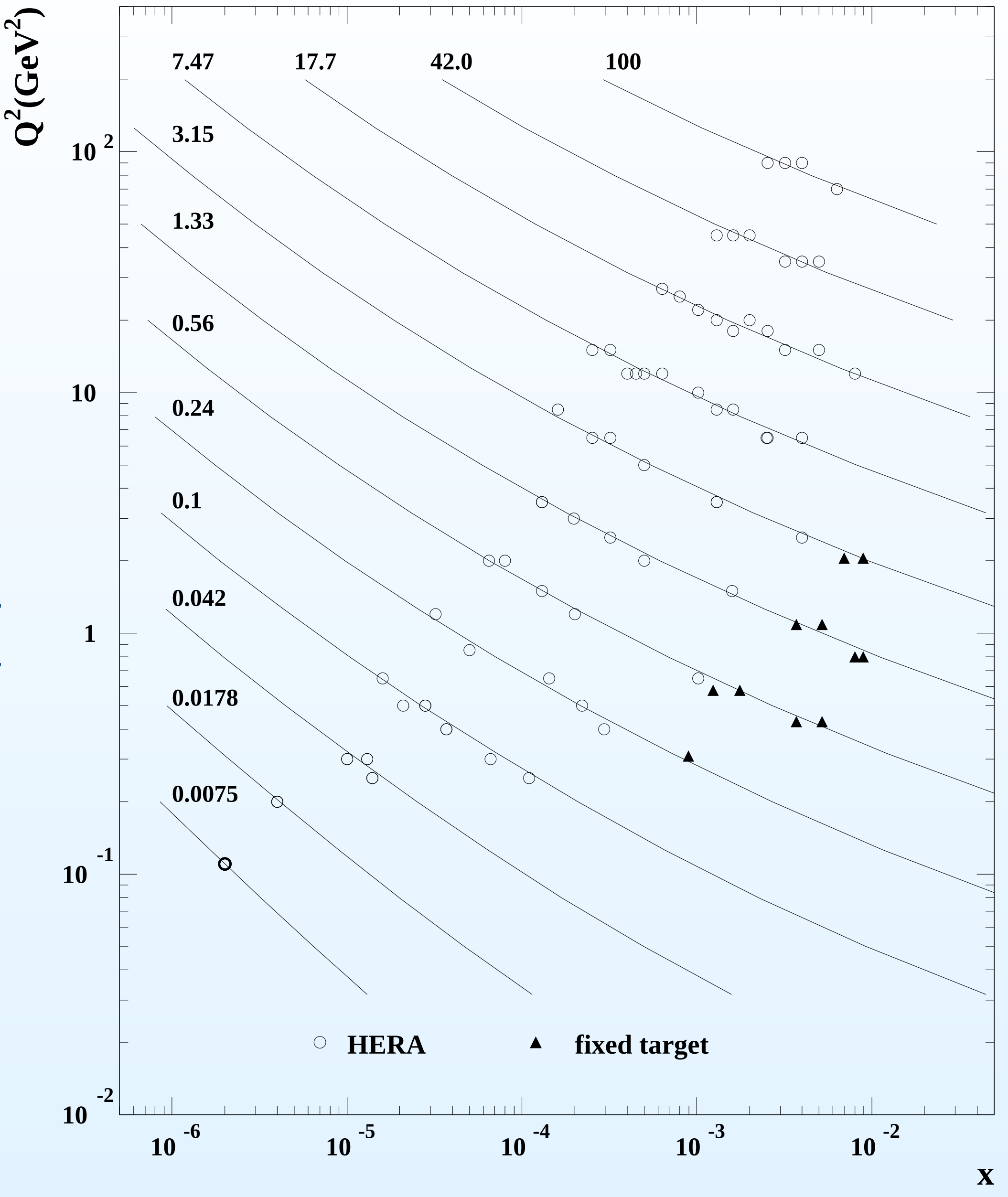
Geometric scaling

$\sqrt{\tau}\sigma_{\gamma^* p}$ plotted versus the scaling variable τ.



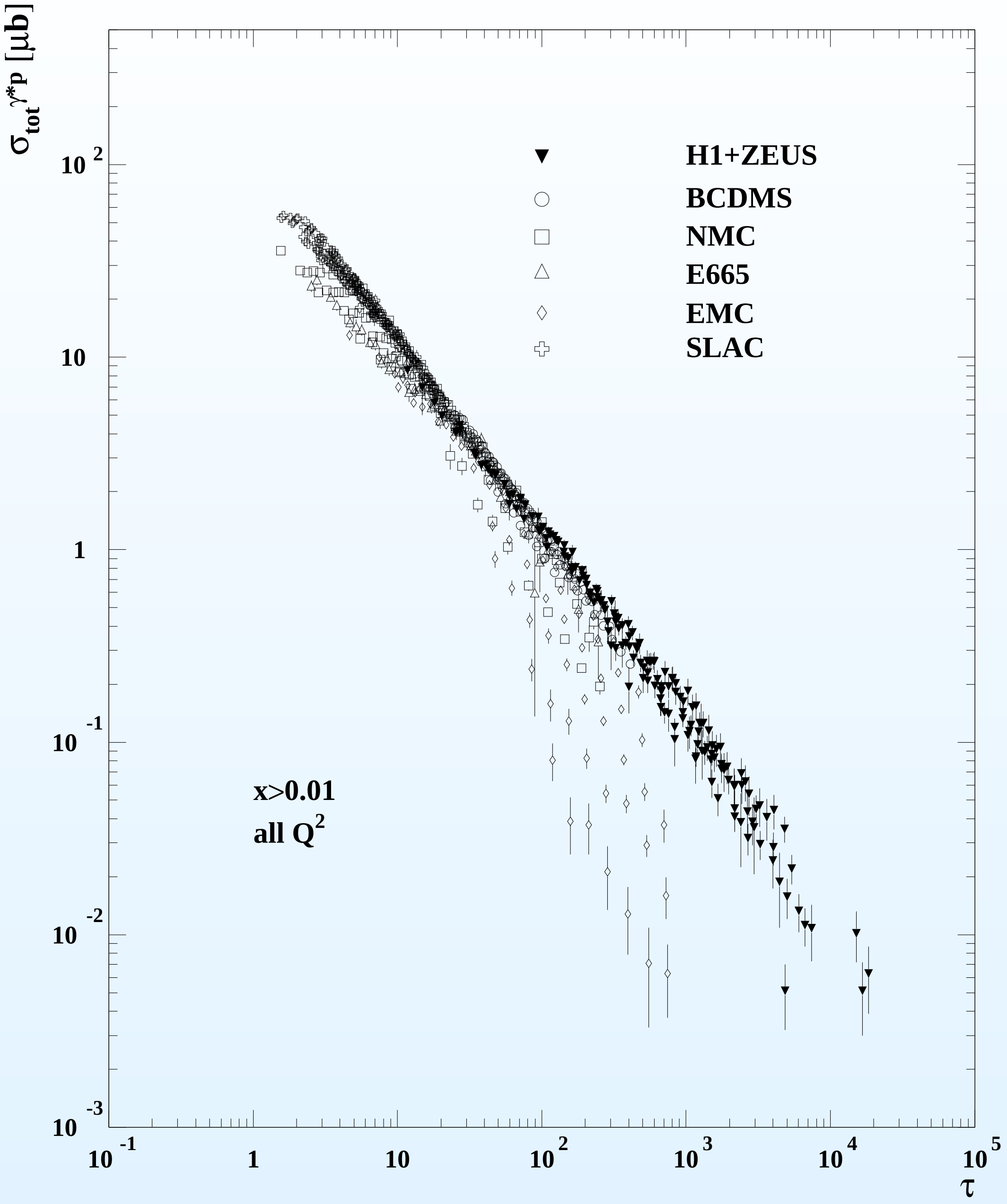
Geometric scaling

Different values of scaling variable *versus* experimental data.



Geometric scaling

Experimental data on $\sigma_{\gamma^* p}$ from the region $x > 0.01$ plotted versus the scaling variable $\tau = Q^2 R_0^2(x)$.



BK equation in momentum space

- Performing the Fourier transform:

$$\tilde{\mathcal{N}}(k,Y) = \int \frac{d^2r}{2\pi} e^{-i\vec{k}\cdot\vec{r}} \frac{\mathcal{N}(r,Y)}{r^2}, \quad (25)$$

BK equation is given by:

$$\frac{d\tilde{\mathcal{N}}(k,Y)}{dY} = \bar{\alpha}_s \int \frac{dk'}{k'} \mathcal{K}(k,k')\tilde{\mathcal{N}}(k',Y) - \bar{\alpha}_s \tilde{\mathcal{N}}^2(k,Y). \quad (26)$$

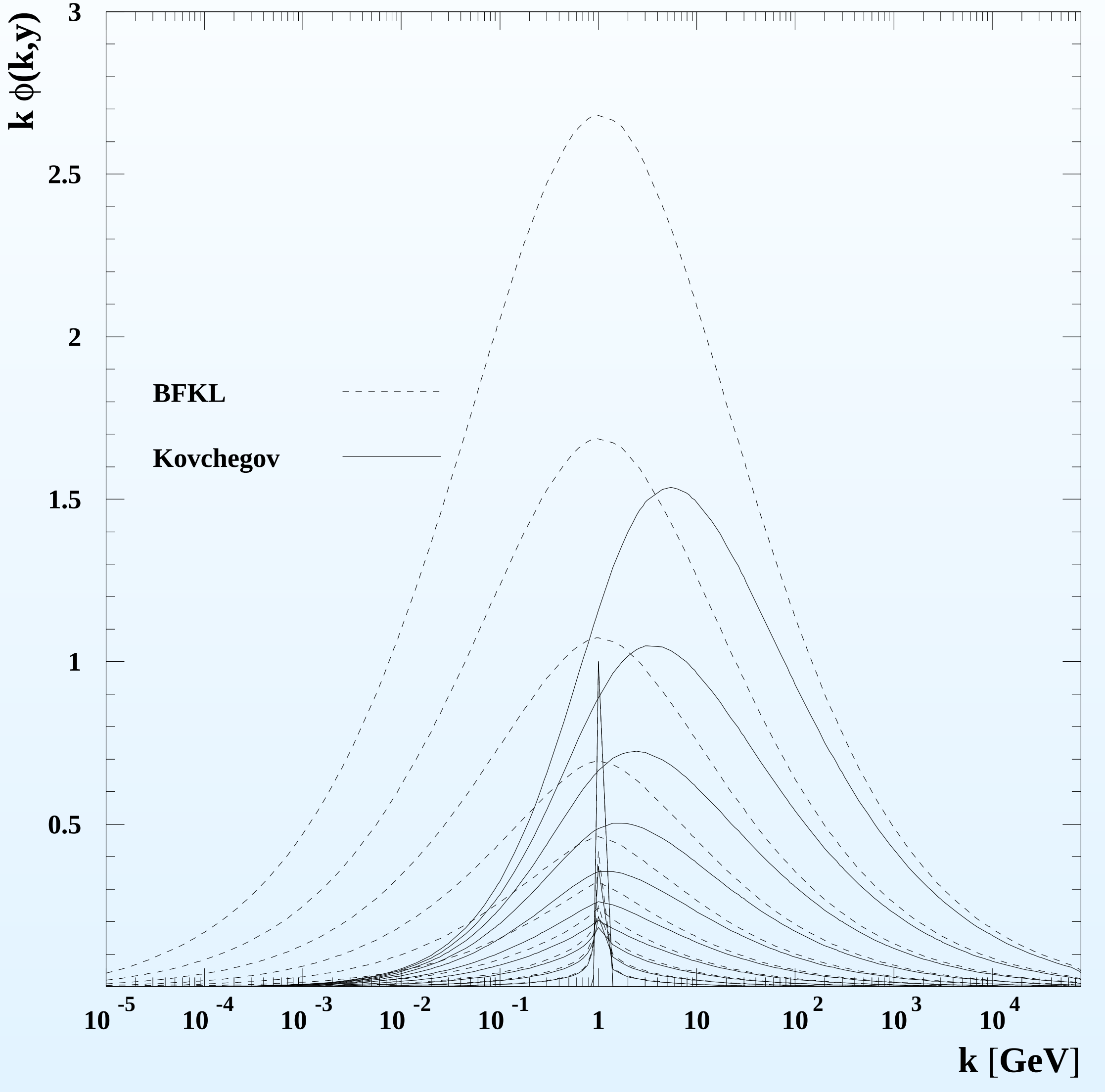
- The solution to the linear part (BFKL) in saddle point approximation is:

$$k\tilde{\mathcal{N}}(k,Y) = \frac{1}{\sqrt{\pi\bar{\alpha}_s\chi''(0)Y}} e^{\bar{\alpha}_s\chi(0)Y} \exp\left(-\frac{\ln^2(k^2/k_0^2)}{2\bar{\alpha}_s\chi''(0)Y}\right) \quad (27)$$

with $\chi(0) = 4\ln 2$ and $\chi''(0) = 28\zeta(3)$.

- The last term in the expression above presents the diffusion.

BK equation in momentum space



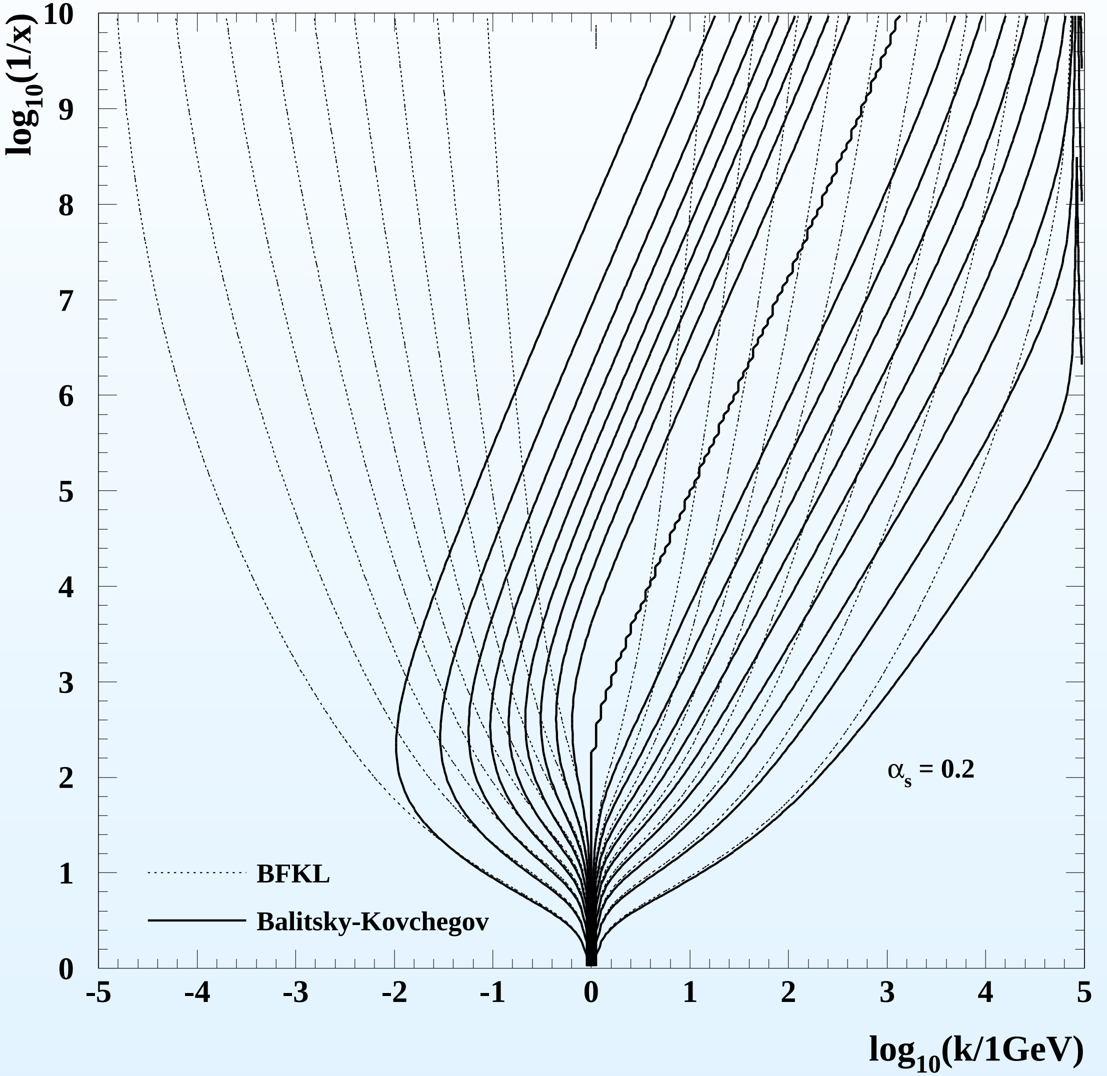
BK equation in momentum space

- BFKL presents strong diffusion and can be interpreted as a random walk in $\ln k$ with rapidity as time.
- The Gaussian shapes of last figure are then expected, as well as the width increase of distributions.
- The initial condition used was

$$k\tilde{\mathcal{N}}(k, Y = 0) = \delta(k) \tag{28}$$

- One sees that BK equation leads to a suppression of the diffusion into infrared.
- Therefore, one defines the saturation scale as the distribution peak position $Q_s(Y) = k_{\max}(Y)$.
- Other way to see the same effects is using the distribution $\Psi(k, Y) = \frac{k\tilde{\mathcal{N}}(k,Y)}{k_{\max}(Y)\tilde{\mathcal{N}}(k_{\max}(Y),Y)}$:

BK equation in momentum space



BK equation in momentum space

- In the last figure, we see that BK equation shifts the contour plot towards higher values of transverse momenta.
- Two regions are found: in the first, there is still diffusion, particularly for large k.
- In the second, diffusion is suppressed and the contour lines are parallel.
- This is an indication of geometric scaling, since straight lines can be parametrized by $\xi = \ln k/k_0 - \lambda Y + \xi_0$.
- One sees also that for large k BK and BFKL equations present similar solutions, but BFKL presents an unlimited increase with energy. On the other side, BK solutions are bounded.

Traveling waves

- The BK equation can be compactly written in momentum space also as

$$\partial_Y \tilde{\mathcal{N}}(\rho, Y) = \bar{\alpha}\chi(-\partial_\rho)\tilde{\mathcal{N}} - \bar{\alpha}\tilde{\mathcal{N}}^2. \tag{29}$$

- The Mellin–transformed BFKL kernel $\chi(\gamma)$ is given by:

$$\chi(\gamma) = 2\psi(1) - \psi(\gamma) - \psi(1-\gamma), \qquad \text{with} \qquad \psi(\gamma) = \frac{\Gamma'(\gamma)}{\Gamma(\gamma)}. \tag{30}$$

- $\chi(-\partial_\rho)$ is an integro-differential operator defined with the help of the formal series expansion:

$$\begin{aligned} \chi(-\partial_\rho) \quad &= \quad \chi(\gamma_0)\mathbf{1} + \chi'(\gamma_0)(-\partial_\rho - \gamma_0\mathbf{1}) + \frac{1}{2}\chi''(\gamma_0)(-\partial_\rho - \gamma_0\mathbf{1})^2 \\ &\quad + \frac{1}{6}\chi^{(3)}(\gamma_0)(-\partial_\rho - \gamma_0\mathbf{1})^3 + \dots \end{aligned} \tag{31}$$

for some γ_0 between 0 and 1, where we used the identity operator $\mathbf{1}$.

BK and FKPP equations

- Munier and Peschanski showed that after the change of variables

$$t \sim \bar{\alpha} Y, \quad x \sim \log(k^2/k_0^2), \quad u \sim \phi, \tag{32}$$

- The approximation of BFKL kernel by the first three terms of the expansion:

$$\chi(-\partial_\rho) = \chi(\gamma_0)\mathbf{1} + \chi'(\gamma_0)(-\partial_\rho - \gamma_0\mathbf{1}) + \frac{1}{2}\chi''(\gamma_0)(-\partial_\rho - \gamma_0\mathbf{1})^2,$$

- And a saddle point approximation:

$$\chi(-\partial_\rho) = -\chi'(\gamma_c)\partial_\rho + \frac{1}{2}\chi''(\gamma_c)(-\partial_L - \gamma_c\mathbf{1})^2,$$

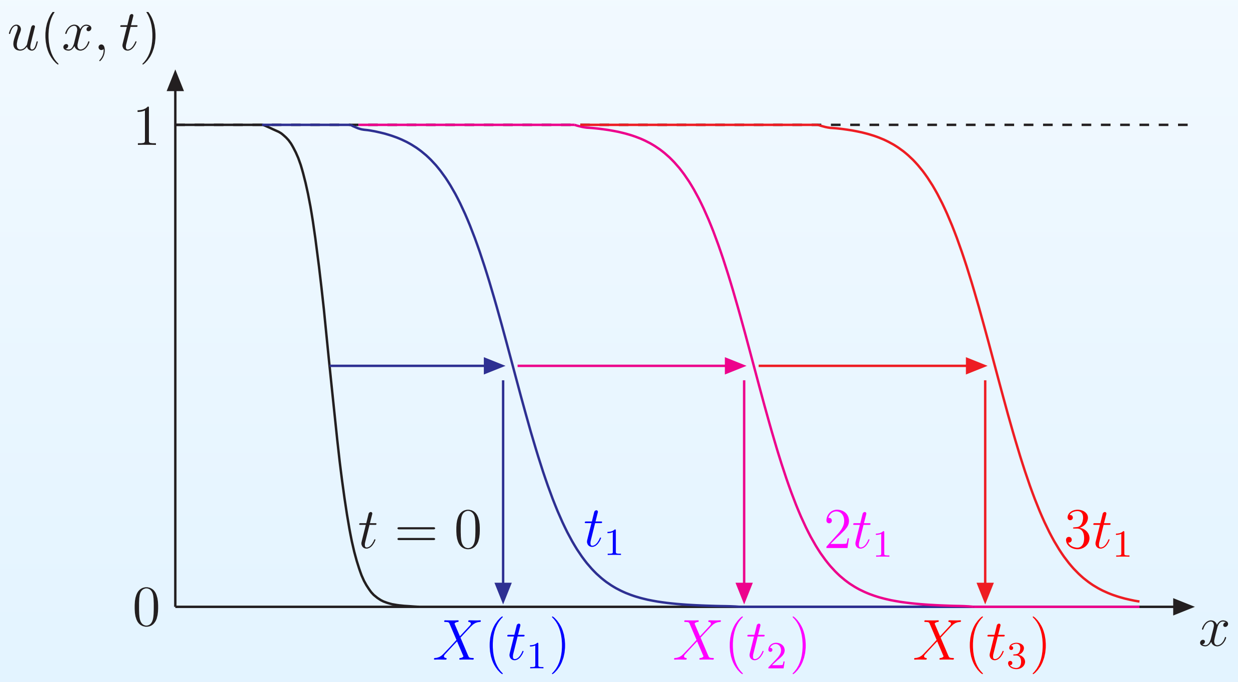
- BK equation is reduced to the Fisher and Kolmogorov-Petrovsky-Piscounov (FKPP) equation from non-equilibrium statistical physics:

$$\partial_t u(x,t) = \partial_x^2 u(x,t) + u(x,t) - u^2(x,t), \tag{33}$$

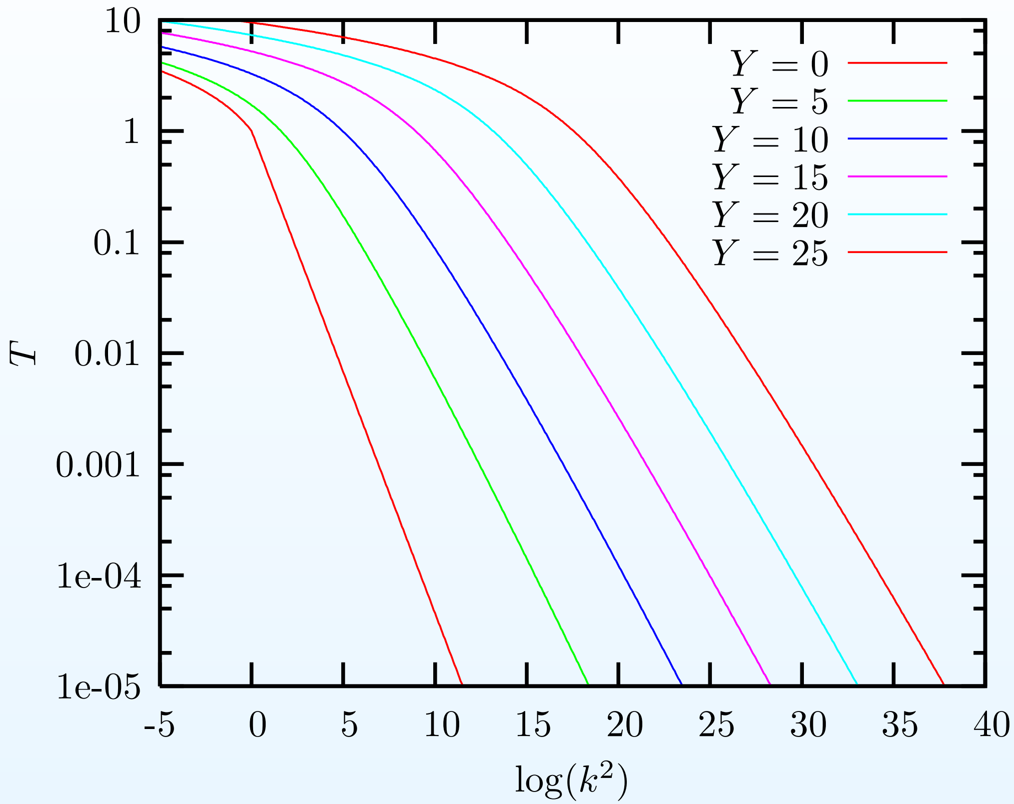
where t is time and x is the coordinate. FKPP dynamics is called reaction-diffusion dynamics.

Traveling waves

- FKPP equation admits so-called traveling wave solutions.
- The position of a wave front $x(t) = v(t)t$ for a traveling wave solution does not depend on details of nonlinear effects.
- At large times, the shape of a traveling wave is preserved during its propagation, and the solution becomes only a function of the scaling variable $x - vt$.



Traveling waves and saturation



- In the language of saturation physics the position of the wave front is nothing but the saturation scale $x(t) \sim \ln Q_s^2(Y)$ and the scaling corresponds to the geometric scaling $x - x(t) \sim \ln k^2/Q_s^2(Y)$.
- Summarizing: time $t \to Y$; space $x \to L$; wave front $u(x - vt) \to \tilde{N}(L - vY)$ and traveling waves $\to$ geometric scaling.

Traveling waves and saturation

- In the dilute regime, in which one has

$$\tilde{T}\,(k,Y) \stackrel{k\gg Q_s}{\approx} \left(\frac{k^2}{Q_s^2(Y)}\right)^{-\gamma_c} \log\left(\frac{k^2}{Q_s^2(Y)}\right) \exp\left[-\frac{\log^2\left(k^2/Q_s^2(Y)\right)}{2\bar{\alpha}\chi''(\gamma_c)Y}\right], \quad (34)$$

where

$$\lambda = \min\, \bar{\alpha}\frac{\chi(\gamma)}{\gamma} = \bar{\alpha}\frac{\chi(\gamma_c)}{\gamma_c} = \bar{\alpha}\chi'(\gamma_c), \quad \bar{\alpha} \equiv \frac{\alpha_s N_c}{\pi}. \quad (35)$$

- Geometric scaling is obtained within the window

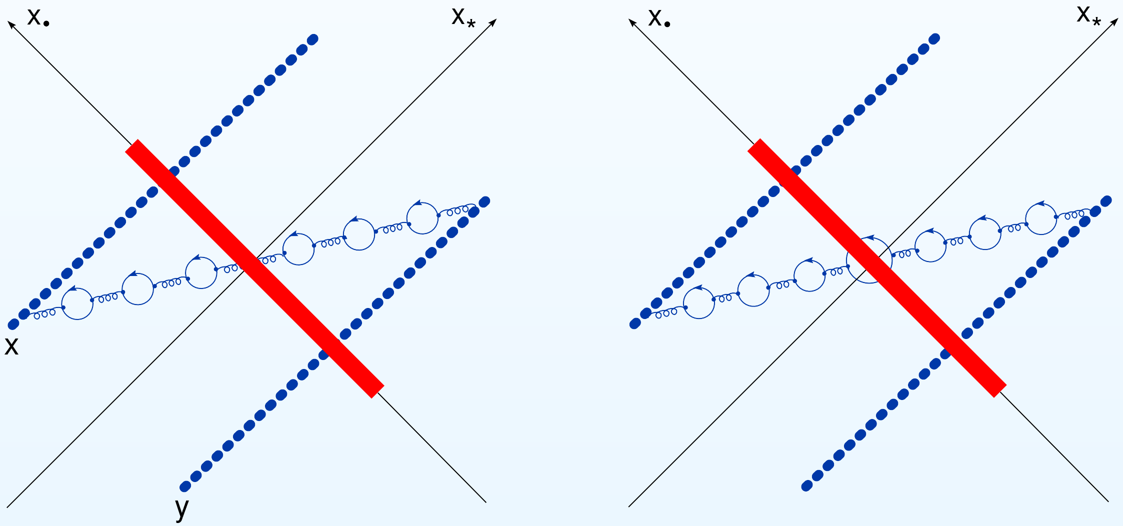
$$\log\left(k^2/Q_s^2(Y)\right) \lesssim \sqrt{2\chi''(\gamma_c)\bar{\alpha}Y}. \quad (36)$$

Next-to-leading order BK

- As usual, to get the region of application of the leading-order evolution equation one needs to find the next-to-leading order (NLO) corrections.
- Unlike the DGLAP evolution, the argument of the coupling constant in LO BK equation is left undetermined in the LLA.
- Careful analysis of this argument is very important from both theoretical and experimental points of view.
- Balitsky calculated the quark contribution to NLO B equation Phys.Rev. D75, 014001 (2007).
- Balitsky and Chirilli calculated the gluon contribution to NLO B equation Phys.Rev. D77, 014019 (2008).

Next-to-leading order BK

- NLO result does not lead automatically to the argument of coupling constant in front of the leading term.
- In order to get this argument, it was used the renormalon-based approach.



- The running coupling was roughly found to be $\alpha_s(|x-y|)$, but:

$$\begin{array}{ll} \frac{\alpha_s((x-y)^2)}{2\pi^2}\frac{(x-y)^2}{X^2Y^2} & |x-y| \ll |x-z|, |y-z| \\ \frac{\alpha_s(X^2)}{2\pi^2 X^2} & |x-z| \ll |x-y|, |y-z| \\ \frac{\alpha_s(Y^2)}{2\pi^2 Y^2} & |y-z| \ll |x-y|, |x-z| \end{array} \qquad (37)$$

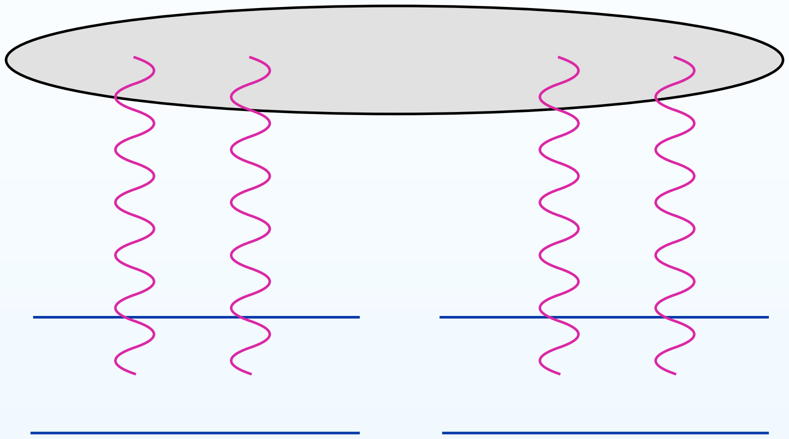
Next-to-leading order BK

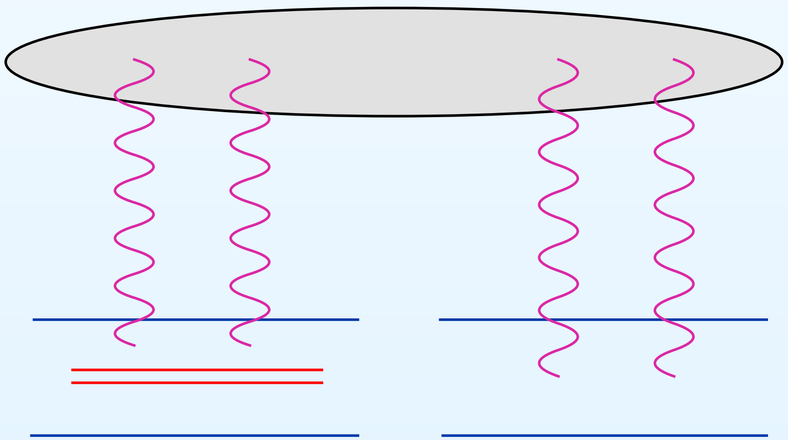
- Next-to-leading order BK:

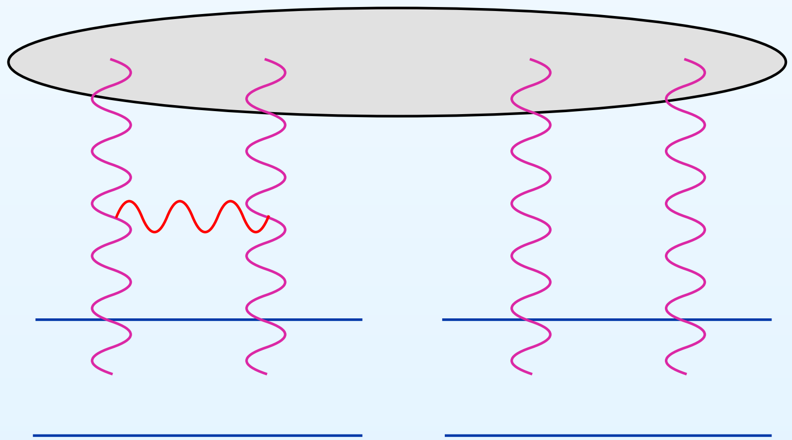
$$\frac{d}{d\eta}N(x,y) = \frac{\alpha_s N_c}{2\pi^2}\int d^2z \frac{(x-y)^2}{X^2Y^2}\left\{1 + \frac{\alpha_s N_c}{4\pi}\left[\frac{11}{3}\ln(x-y)^2\mu^2 - \frac{11}{3}\frac{X^2-Y^2}{(x-y)^2}\ln\frac{X^2}{Y^2} + \frac{67}{9} - \frac{\pi^2}{3} - 2\ln\frac{X^2}{(x-y)^2}\ln\frac{Y^2}{(x-y)^2}\right]\right\}$$
$$\times [N(x,z) + N(z,y) - N(x,y) - N(x,z)N(z,y)]$$
$$+ \frac{\alpha_s^2 N_c^2}{8\pi^4}\int d^2z d^2z' \left\{ -\frac{2}{(z-z')^4} + \left[\frac{X^2Y'^2 + X'^2Y^2 - 4(x-y)^2(z-z')^2}{(z-z')^4(X^2Y'^2 - X'^2Y^2)} + \frac{(x-y)^4}{X^2Y'^2(X^2Y'^2 - X'^2Y^2)}\right.\right.$$
$$\left.\left. + \frac{(x-y)^2}{X^2Y'^2(z-z')^2}\right]\ln\frac{X^2Y'^2}{X'^2Y^2}\right\}[N(z,z') - N(x,z)N(z,z') - N(z,z')N(z',y) - N(x,z)N(z',y) + N(x,z)N(z,y)$$
$$+ N(x,z)N(z,z')N(z',y)]. \qquad (136)$$

- $X = x - z$, $Y = y - z$, $X = x - z'$, $Y = y - z'$.
- As we can infer, in the complete hierarchy, the single dipole evolution is related to the sextupole terms.

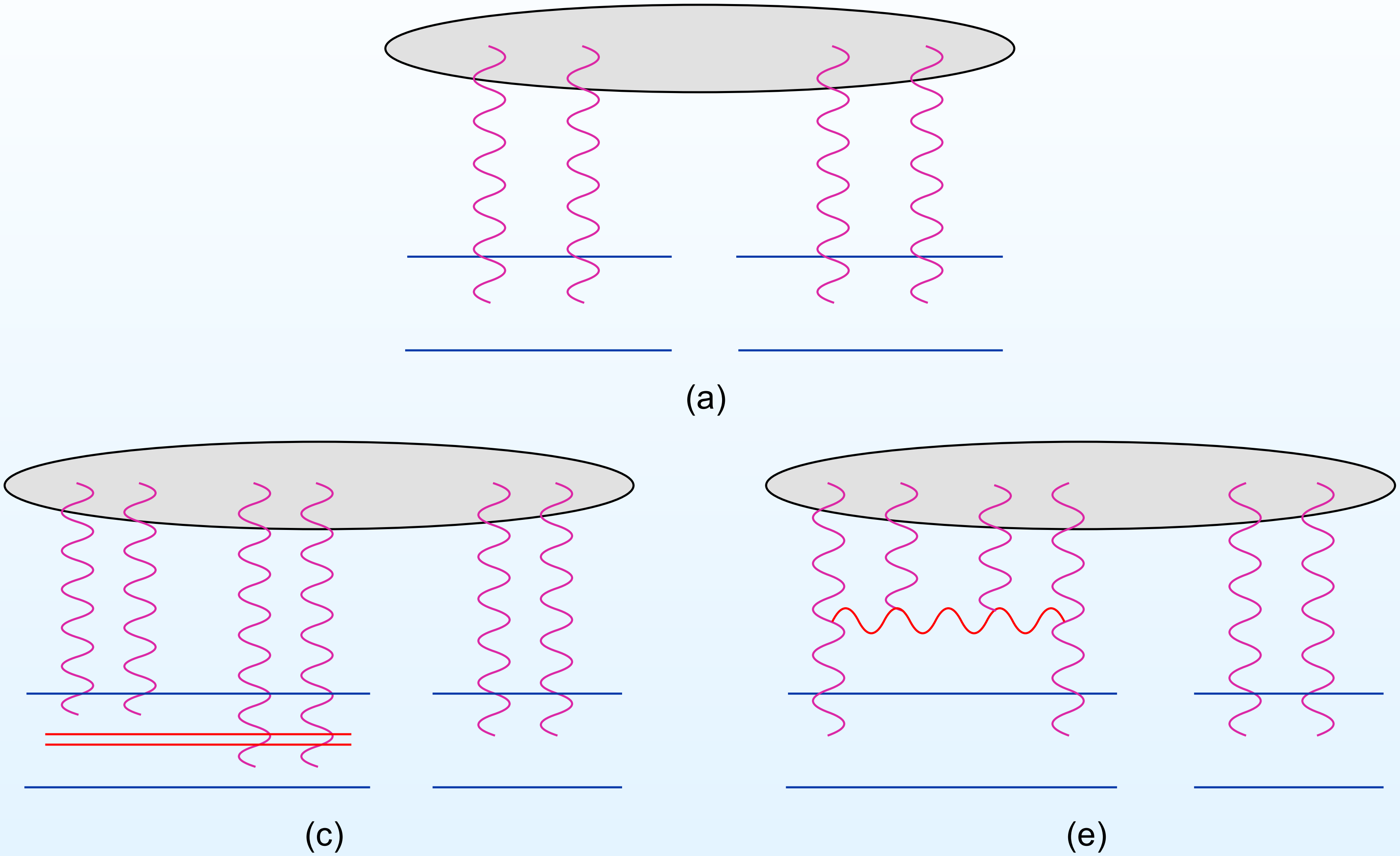
Beyond BK equation: Pomeron loops


(a)

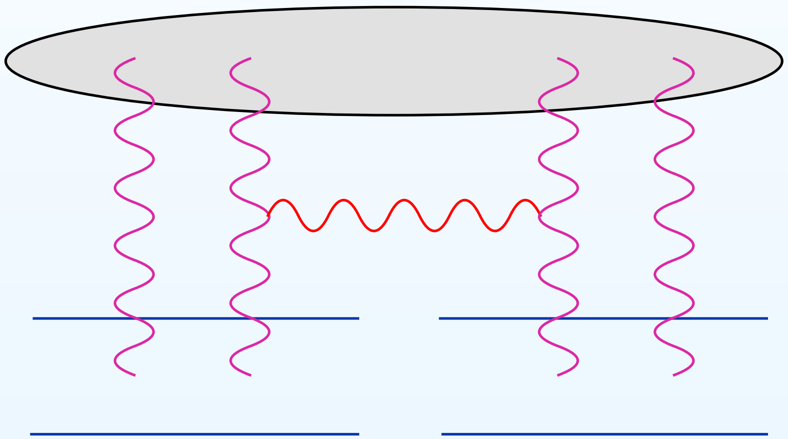

(b)


(d)

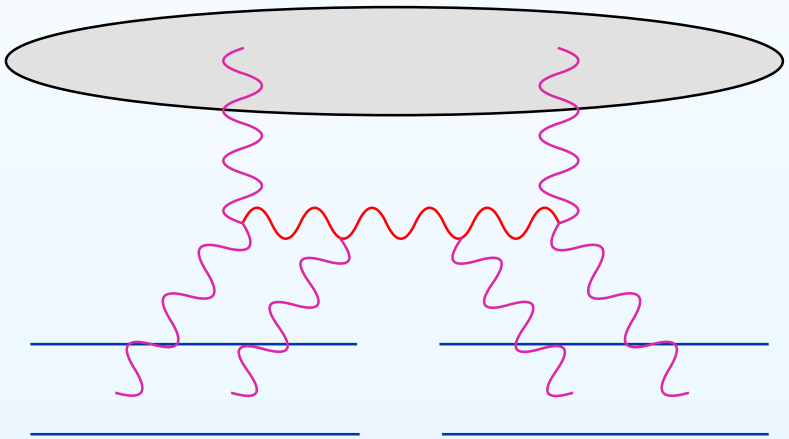
Beyond BK equation: Pomeron loops



Beyond BK equation: Pomeron loops



(f)



(g)

Beyond BK equation: Fluctuations

- BK equation completely misses the effects of fluctuations in the gluon (dipole) number, related to discreteness of the evolution.
- Also, since BK equation uses a mean field approach, $< T^2 > - < T >^2 = 0$.
- After an approximation related to the impact parameter dependence, it can be shown that a Langevin equation for the event-by-event amplitude can rebuild the Pomeron loop hierarchy.
- This is formally the BK equation with a Gaussian white noise term.
- It lies in the same universality class as the stochastic FKPP equation (sFKPP):

$$\partial_t u(x,t) = \partial_x^2 u(x,t) + u(x,t) - u^2(x,t) + \sqrt{\frac{2}{N} u(x,t)(u(x,t)-1)}\,\nu(x,t). \quad (38)$$

- White noise is defined as:

$$< \nu(x,t) >= 0 \qquad\qquad < \nu(x,t)\nu(x',t') >= \delta(t-t')\delta(x-x'). \quad (39)$$

[E. Iancu, D.N. Triantafyllopoulos; Nucl.Phys.A 756, 419 (2005)]

Fluctuations

- Each realization of the noise means a single realization of the target in the evolution, which is stochastic, and this leads to an amplitude for a single event.
- Different realizations lead to a dispersion in the solutions and also in the saturation momentum $\rho_s \equiv \ln(Q_s^2(Y)/k_0^2)$.
- For each single event, the evolved amplitude shows a traveling-wave pattern, with speed of the wave smaller than the value predicted by BK equation:

$$\lambda^* \simeq \lambda - \frac{\pi^2 \gamma_c \chi''(\gamma_c)}{2\ln(1/\alpha_s^2)} \qquad (40)$$

- The saturation scale $Q_s(Y)$ is now a random variable whose average value is given by

$$\langle Q_s^2(Y) \rangle = \exp\left[\lambda^* Y\right]. \qquad (41)$$

- The dispersion in the position of the individual fronts is given by

$$\sigma^2 = \langle \rho_s^2 \rangle - \langle \rho_s \rangle^2 = D\bar{\alpha}Y, \qquad (42)$$

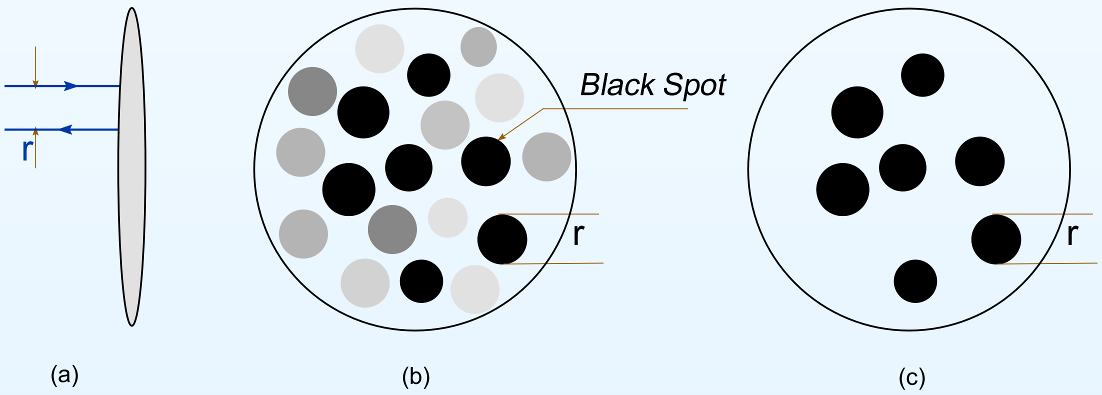
Diffusive scaling

- Geometric scaling:

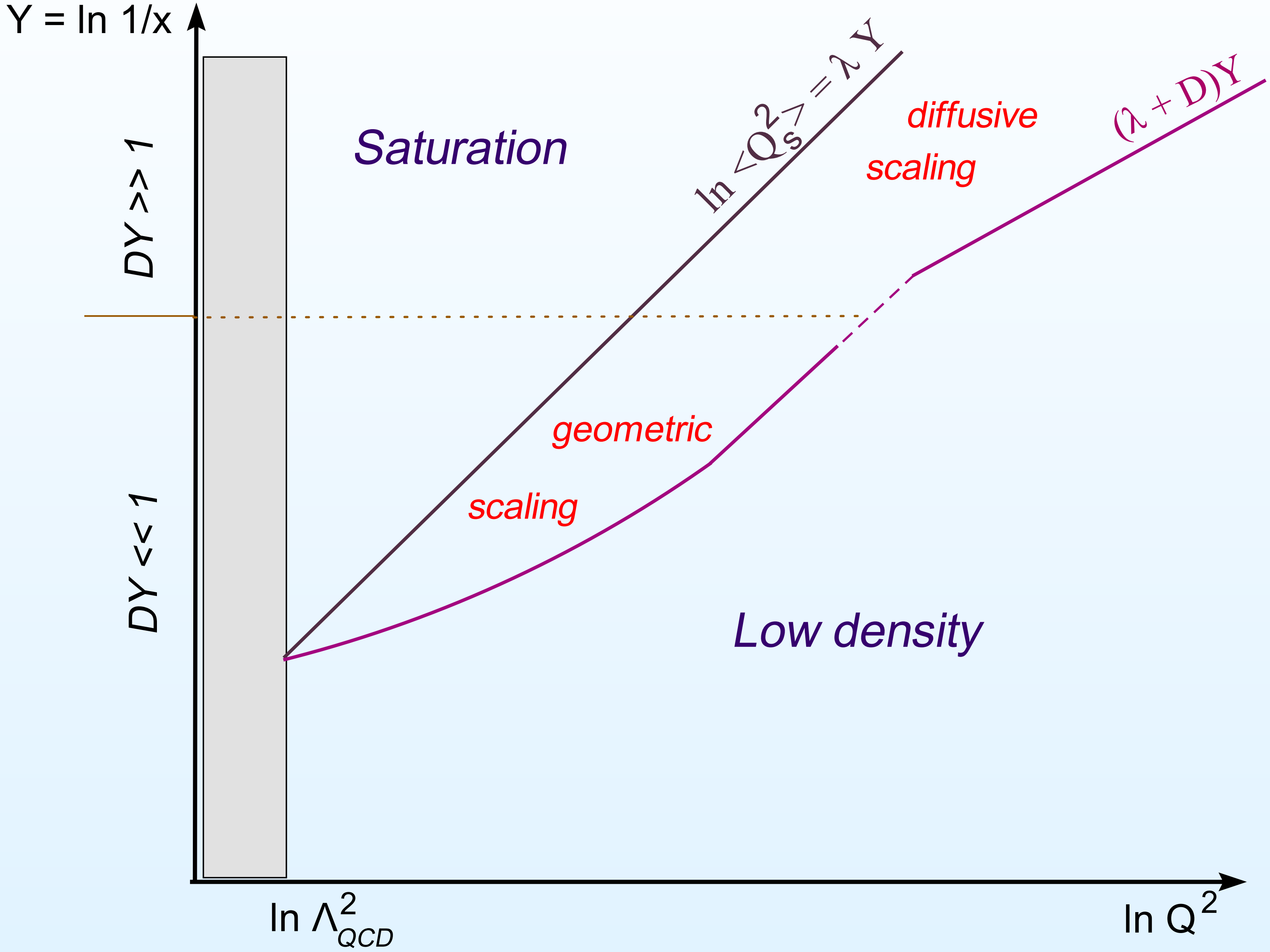
$$\langle T(\rho, \rho_s)\rangle = \mathcal{T}\left(\rho - \langle \rho_s\rangle\right). \quad (43)$$

is replaced by *diffusive scaling*

$$\langle T(\rho, \rho_s)\rangle = \mathcal{T}\left(\frac{\rho - \langle \rho_s\rangle}{\sqrt{\bar{\alpha} D Y}}\right). \quad (44)$$

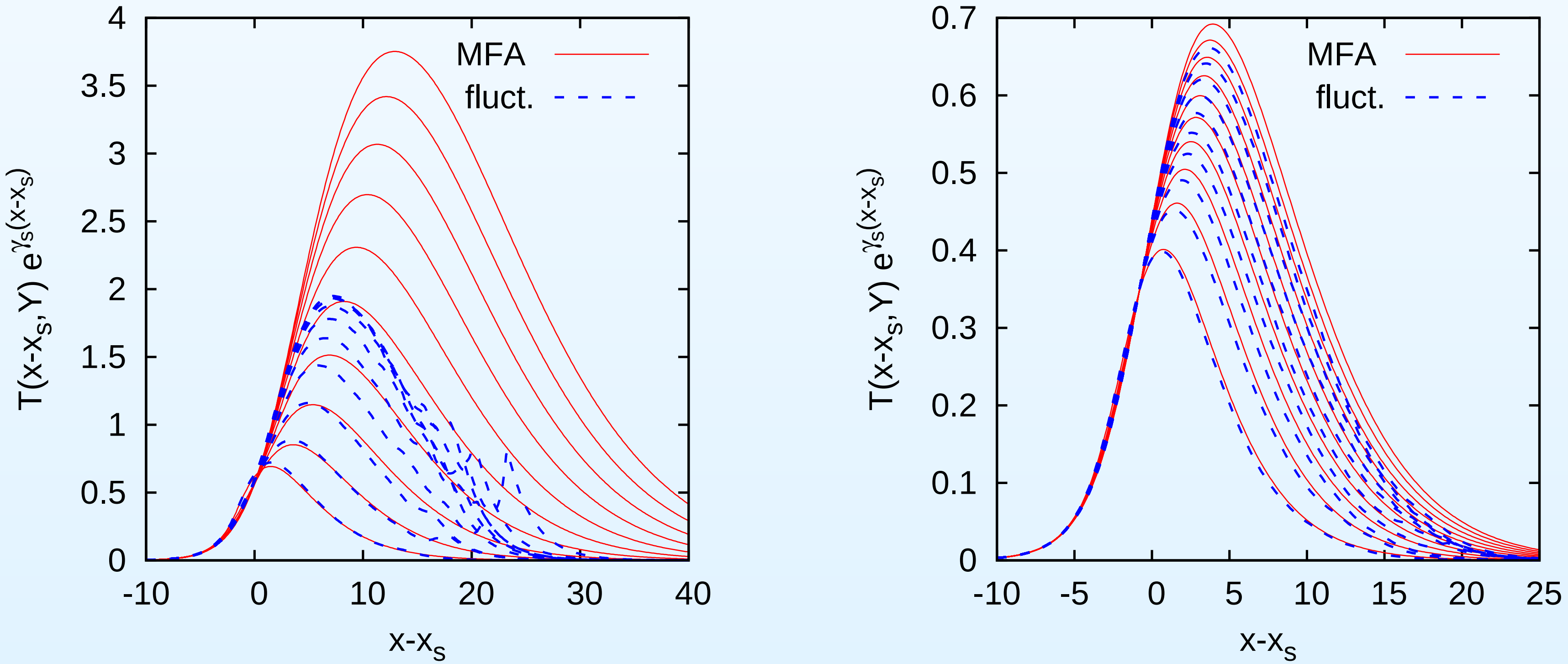


Fluctuations



Toy model

- Are fluctuations really important (in the context of a toy model)?
- At fixed coupling they are [E. Iancu, J.T. de Santana Amaral, G. Soyez, D.N. Triantafyllopoulos; Nucl.Phys. A786, 131 (2007)].
- However, at running coupling they are not [A. Dumitru, E. Iancu, L. Portugal, G. Soyez, D.N. Triantafyllopoulos; JHEP 0708:062 (2007)].
- This reflects the slowing down of the evolution by running coupling effects, in particular, the large rapidity evolution which is required for the formation of the saturation front via diffusion.



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AGL equation from BK equation

- AGL equation can be derived from BK equation. Taking the equation derived by Kovchegov:

$$\mathcal{N}(\mathbf{x}_{01},\mathbf{b}_0,Y) = -\gamma(\mathbf{x}_{01},\mathbf{b}_0)\exp\left[-\frac{4\alpha_s C_F}{\pi}\ln\left(\frac{x_{01}}{\rho}\right)Y\right] + \frac{\alpha_s C_F}{\pi}\int_0^Y dy\,\exp\left[-\frac{4\alpha_s C_F}{\pi}\ln\left(\frac{x_{01}}{\rho}\right)(y-Y)\right] \times \int_\rho d^2x_2\frac{x_{01}^2}{x_{02}^2x_{12}^2}[2\mathcal{N}(\mathbf{x}_{02},\mathbf{b}_0,Y-\mathcal{N}(\mathbf{x}_{02},\mathbf{b}_0,Y)\mathcal{N}(\mathbf{x}_{12},\mathbf{b}_0,Y)] \quad (45)$$

- In the double logarithmic limit, in which the momentum scale of the photon Q^2 is larger than the momentum scale of the nucleus $\Lambda_{\rm QCD}$, the large Q^2 limit of last equation reduces to:

$$\frac{\partial\mathcal{N}(\mathbf{x}_{01},\mathbf{b}_0,Y)}{\partial Y} = \frac{\alpha_s C_F}{\pi}x_{01}^2\int_{x_{01}^2}^{1/\Lambda^2_{\rm QCD}}\frac{d^2x_{02}}{x_{02}^2x_{12}^2}[2\mathcal{N}(\mathbf{x}_{02},\mathbf{b}_0,Y -\mathcal{N}(\mathbf{x}_{02},\mathbf{b}_0,Y)\mathcal{N}(\mathbf{x}_{02},\mathbf{b}_0,Y)]. \quad (46)$$

AGL equation from BK equation

- The last equation can be derived again with respect to $\ln(1/x_{01}^2\Lambda_{\rm QCD}^2)$:

$$\frac{\partial \mathcal{N}(\mathbf{x}_{01},\mathbf{b}_0,Y)}{\partial Y \partial \ln(1/x_{01}^2\Lambda_{\rm QCD}^2)} = \frac{\alpha_s C_F}{\pi}[2-\mathcal{N}(\mathbf{x}_{01},\mathbf{b}_0,Y)]\mathcal{N}(\mathbf{x}_{01},\mathbf{b}_0,Y). \quad (47)$$

- In the physical picture for the dipole evolution in the DLA limit, the produced dipoles at each step of the evolution have much greater transverse dimensions than the parent dipoles.
- The connection in this approximation between $\mathcal{N}$ and the nuclear gluon function is:

$$\mathcal{N}(\mathbf{x}_{01},\mathbf{b}_0=0,Y) = 2\left\{1-\exp\left[-\frac{\alpha_s C_F \pi^2}{N_c^2 S_\perp} x_{01}^2 AxG(x,1/x_{01}^2)\right]\right\}. \quad (48)$$

- From the above equations and using $x_{01}\approx 2/Q$, one obtains:

$$\frac{\partial \mathcal{N}(\mathbf{x}_{01},\mathbf{b}_0,Y)}{\partial Y} = \frac{\alpha_s C_F}{\pi} x_{01}^2 \int_{x_{01}^2}^{1/\Lambda_{\rm QCD}^2} \frac{d^2x_{02}}{x_{02}^2 x_{12}^2}[2\mathcal{N}(\mathbf{x}_{02},\mathbf{b}_0,Y) - \mathcal{N}(\mathbf{x}_{02},\mathbf{b}_0,Y)\mathcal{N}(\mathbf{x}_{02},\mathbf{b}_0,Y)]. \quad (49)$$

- This equation is precisely the AGL equation.

Light-cone kinematics

- Let z be the longitudinal axis of the collision. For an arbitrary 4-vector $v^\mu = (v^0, v^1, v^2, v^3)$, the light-cone (LC) coordinates are defined as

$$v^+ \equiv \frac{1}{\sqrt{2}}(v^0 + v^3), \qquad v^- \equiv \frac{1}{\sqrt{2}}(v^0 - v^3), \qquad \mathbf{v} \equiv (v^1, v^2) \tag{50}$$

- One usually writes $v_\perp \equiv |\mathbf{v}| = \sqrt{(v^1)^2 + (v^2)^2}$
- In these coordinates, $x^+ \equiv \frac{1}{\sqrt{2}}(t+z)$ is the LC time and $x^- \equiv \frac{1}{\sqrt{2}}(t-z)$ is the LC longitudinal coordinate.
- The invariant scalar product of two 4-vectors reads

$$\begin{aligned} p \cdot x &= p^0x^0 - p^1x^1 - p^2x^2 - p^3x^3 \\ &= \frac{1}{2}(p^+ + p^-)(x^+ + x^-) - \frac{1}{2}(p^+ - p^-)(x^+ - x^-) - \mathbf{p}\cdot\mathbf{x} \\ &= p^-x^+ + p^+x^- - \mathbf{p}\cdot\mathbf{x} \end{aligned} \tag{51}$$

- This form of the scalar product suggests that p^- should be interpreted as the LC energy and p^+ as the LC longitudinal momentum.

Light-cone kinematics

- For particles on the mass-shell, $k^{\pm} = (E \pm k_z)/\sqrt{2}$, with $E^2 = (m^2 + k_z^2 + \mathbf{k}^2)$

$$k^+ k^- = \frac{1}{2}(E^2 - k_z^2) = \frac{1}{2}(\mathbf{k}^2 + m^2) \equiv m_\perp^2 \tag{52}$$

- One needs also the *rapidity*

$$y \equiv \frac{1}{2} \ln \frac{k^+}{k^-} = \frac{1}{2} \ln \frac{2k^{+2}}{m_\perp^2} \tag{53}$$

- Under a longitudinal Lorentz boost ($k^+ \to \beta k^+$, $k^- \to (1/\beta) k^-$, with constant β), the rapidity is shifted only by a constant, $y \to y + \beta$
- For a parton inside a right-moving (in the positive z direction) hadron, we introduce the boost-invariant longitudinal momentum fraction x

$$x \equiv \frac{k^+}{P^+} \tag{54}$$