QCD saturation predictions in momentum space: heavy quarks at HERA

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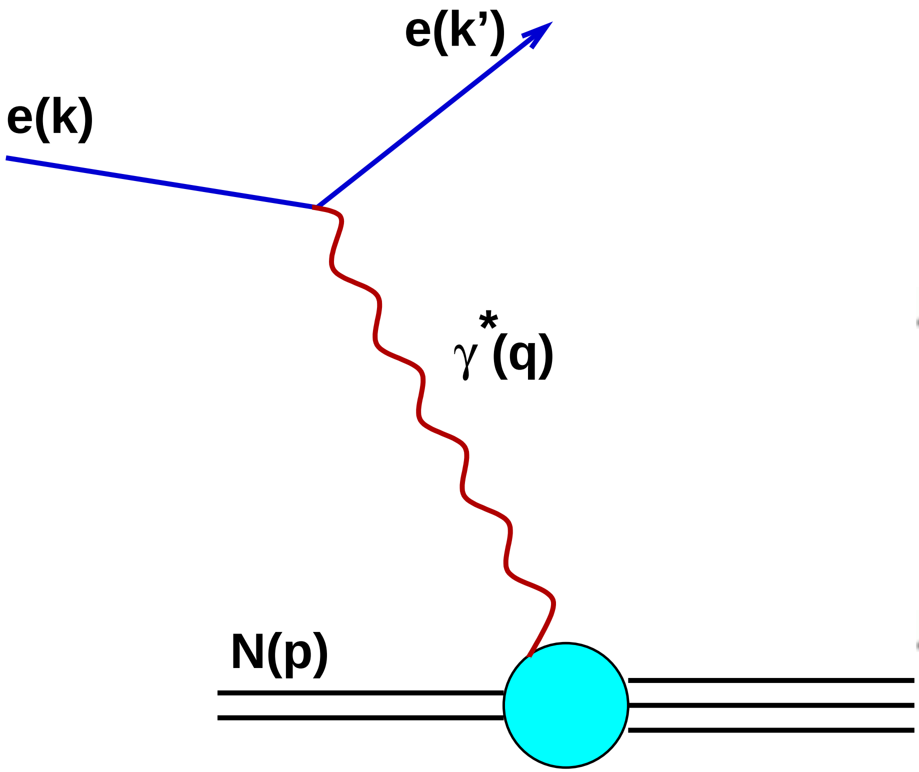
Outline

- DIS and the high energy QCD challenge
- The Dipole Model
- Scattering Amplitudes in High Energy QCD
- Description of the $\gamma^* p$ data
- Discussions
- Improved IIM model
- Perspectives of new predictions

Deep Inelastic Scattering (DIS)

Kinematics and variables



- The total energy squared of the photon-nucleon system

$$s = (p+q)^2$$

- Photon virtuality

$$q^2 = (k - k') = -Q^2 < 0$$

- The Bjorken variable

$$x \equiv x_{Bj} = \frac{Q^2}{2p \cdot q} = \frac{Q^2}{Q^2 + s}$$

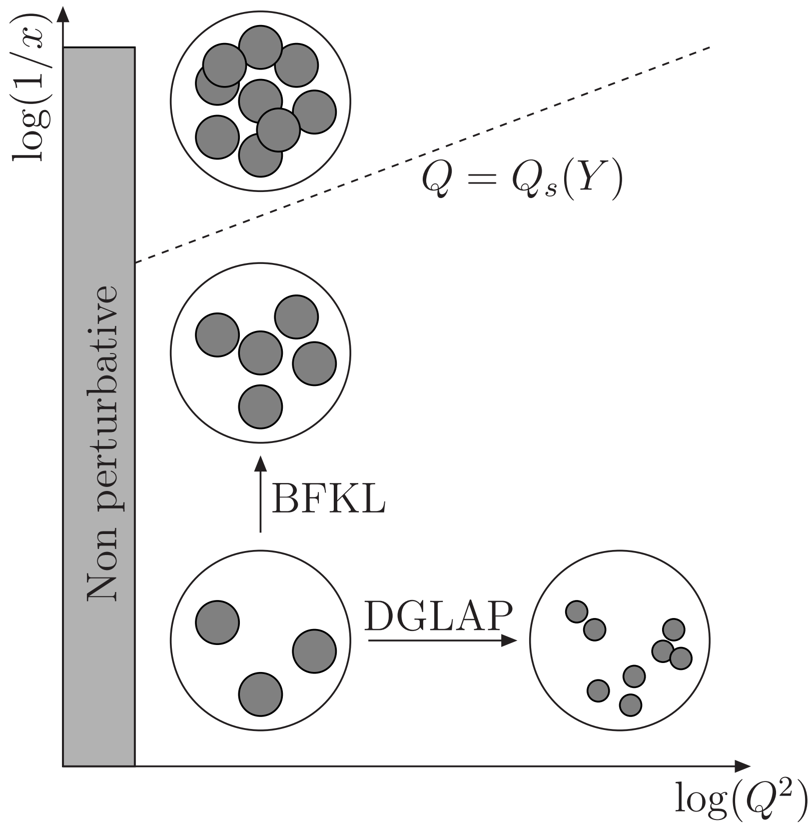
- The high energy limit:

$$s \to \infty, \quad x \approx \frac{Q^2}{s} \to 0$$

QCD at high energies

- As energy increases (with Q fixed) the gluon density grows fast and so does the cross sections for hadronic interactions
  - This is still a challenge in Quantum Chromodynamics

- At this regime gluon recombination and multiple scattering might be important to restore unitarity



- $Q_s(Y)$ is the so called saturation scale

- The nonlinear saturation effects are important for all $Q \lesssim Q_S(Y)$, which is known as saturation region

Towards saturation

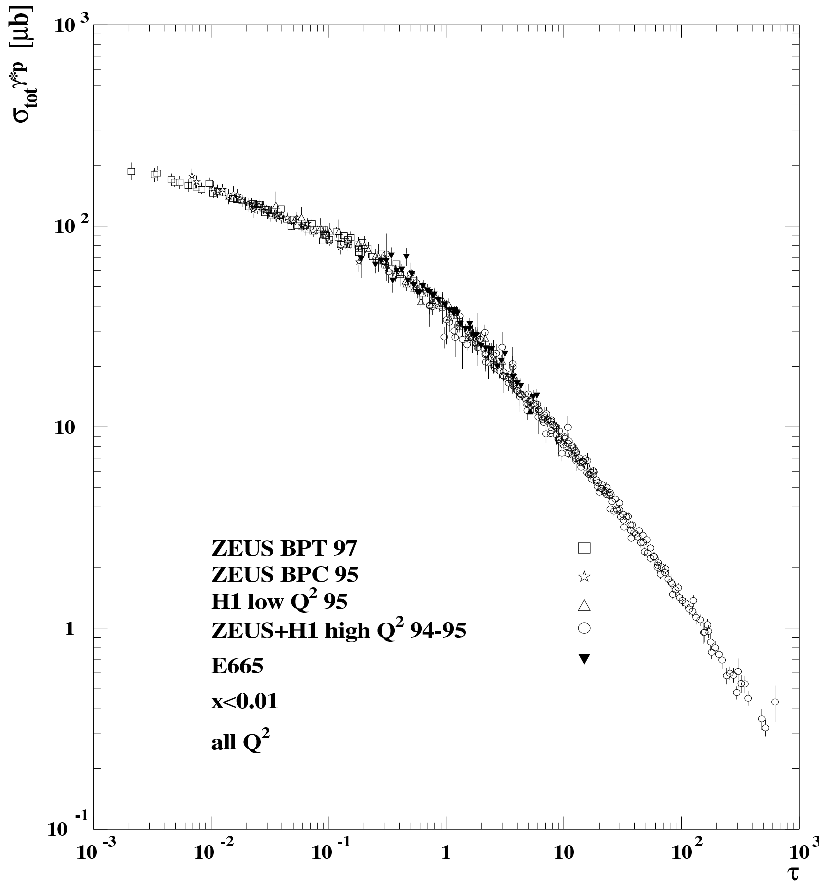
- There has been a large amount of work devoted to the description and understanding of QCD in the high energy limit corresponding to the *saturation*
  - Theory: non-linear QCD equations describing the evolution of scattering amplitudes towards saturation - AGL, BK and JIMWLK equations
  - Phenomenology: discovery of geometric scaling in DIS at HERA

- The Balitsky-Kovchegov (BK) nonlinear equation describes the evolution in rapidity of the scattering amplitude of a dipole off a given target
  - This equation has been shown to lie in the same universality class as the Fisher-Kolmogorov-Petrovsky-Piscounov (FKPP) equation
- Geometric scaling has a natural explanation in terms of the so-called traveling wave solutions of BK equation

- The evolution at intermediate energies is well understood and is described by the linear BFKL equation
- The deep saturation regime can also be evaluated in some models, but the transition between these two regimes is still a challenge

Geometric Scaling

- Geometric scaling is a phenomenological feature of high energy deep inelastic scattering (DIS) which has been observed in the HERA data on inclusive $\gamma^* - p$ scattering, which is expressed as a scaling property of the virtual photon-proton cross section



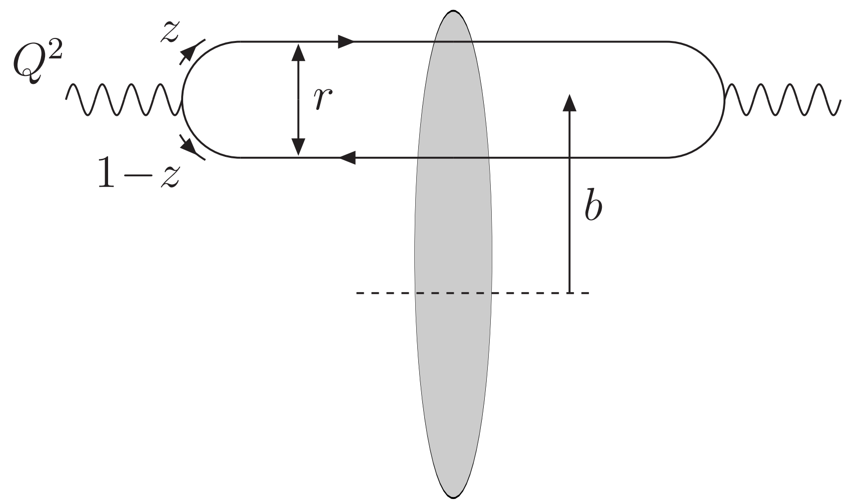
$$\sigma^{\gamma^* p}(Y,Q) = \sigma^{\gamma^* p}(\tau)\,, \quad \tau = \frac{Q^2}{Q_s^2(Y)}$$

where Q is the virtuality of the photon, $Y = \log 1/x$ is the total rapidity and $Q_s(Y)$ is an increasing function of Y called saturation scale

[Stasto, Golec Biernat and Kwiecinsky, 2001]

$\sigma^{\gamma^* p}$ cross section

- Consider the collision between a virtual photon and a proton at high energy; in a frame where the proton carries most of the total energy one can consider that the photon fluctuates into a $q\bar{q}$ pair



- The cross section

$$\sigma_{T,L}^{\gamma^* p}(Y,Q) = \int d^2r \int_0^1 dz \, \left|\Psi_{T,L}(r,z;Q^2)\right|^2 \sigma_{dip}^{\gamma^* p}(r,Y), \qquad (1)$$

$\sigma_{dip}^{\gamma^* p}(Y,r)$ is the dipole-proton cross section, z is the fraction of photon's momentum carried by the quark, r is the size of the dipole and b is the impact parameter

Dipole-proton cross section

- The transverse and longitudinal photon wavefunctions

$$|\Psi_T(r,z;Q^2)|^2 = \frac{2N_c\alpha_{em}}{4\pi^2}\sum_q e_q^2\left\{\left[z^2+(1-z)^2\right]\bar{Q}_q^2K_1^2(\bar{Q}_qr)+m_q^2K_0^2(\bar{Q}_qr)\right\} \quad (2)$$

and

$$|\Psi_L(r,z;Q^2)|^2 = \frac{2N_c\alpha_{em}}{4\pi^2}\sum_q e_q^2\left\{4Q^2z^2(1-z)^2K_0^2(\bar{Q}_qr)\right\} \quad (3)$$

where $\bar{Q}_q = z(1-z)Q^2+m_q^2$, m_q the light quark mass and $K_{0,1}$ are the Mc Donald functions of rank zero and one, respectively

- If one treats the proton as an homogeneous disk of radius R_p, one can write the dipole-proton cross section in terms of the dipole-proton forward scattering amplitude $T(r,Y)$

$$\sigma_{dip}^{\gamma^* p}(r,Y) = 2\pi R_p^2 T(r,Y)$$

F_2 structure function (I)

- The proton structure function F_2 can be obtained from the $\gamma^* p$ cross section through the relation

$$
\begin{aligned}
F_2(x, Q^2) &= \frac{Q^2}{4\pi^2 \alpha_{em}} \left[\sigma_T^{\gamma^* p}(x, Q^2) + \sigma_L^{\gamma^* p}(x, Q^2)\right] \\
&= \frac{Q^2}{4\pi^2 \alpha_{em}} \sigma^{\gamma^* p}(x, Q^2) \qquad (4)
\end{aligned}
$$

- It is possible to express the $\gamma^* p$ cross section in terms of the scattering amplitude in momentum space, $\tilde{T}(k, Y)$, through the Fourier transform

$$
\begin{aligned}
\tilde{T}(k, Y) &= \frac{1}{2\pi} \int \frac{d^2 r}{r^2}\, e^{i\mathbf{k}\cdot\mathbf{r}}\, T(r, Y) \\
&= \int_0^\infty \frac{dr}{r} J_0(kr)\, T(r, Y) \qquad (5)
\end{aligned}
$$

F_2 structure function (II)

- After a bit of algebra one obtains

$$F_2(x,Q^2) = \frac{Q^2 R_p^2 N_c}{4\pi^2} \int_0^\infty \frac{dk}{k} \int_0^1 dz\, |\tilde{\Psi}(k,z;Q^2)|^2 \tilde{T}(k,Y) \tag{6}$$

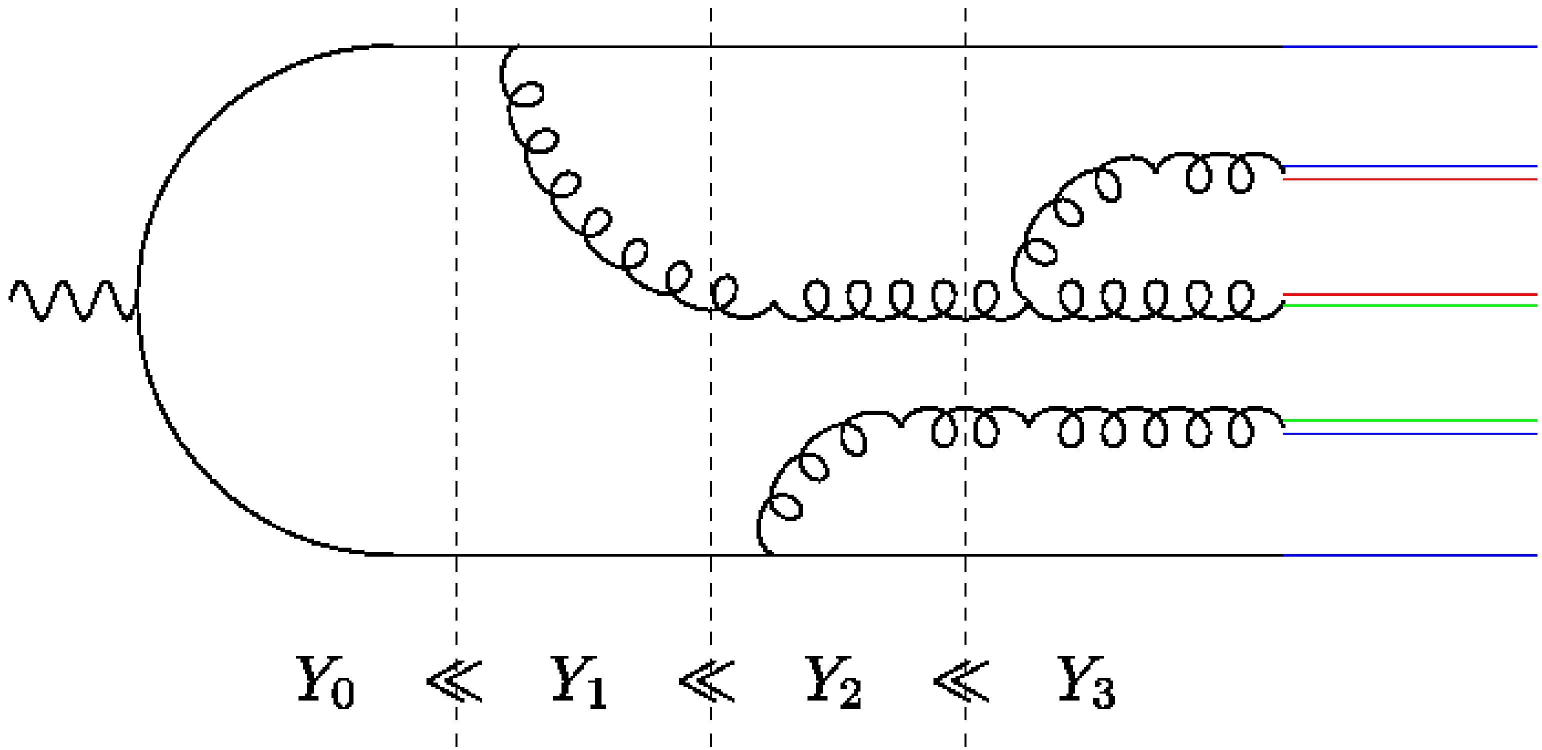
- The wavefunction is now expressed in momentum space

$$|\tilde{\Psi}(k^2,z;Q^2)|^2 = \sum_q \left(\frac{4\bar{Q}_q^2}{k^2+4\bar{Q}_q^2}\right)^2 e_q^2 \Bigg\{ \left[z^2+(1-z)^2\right] \left[\frac{4(k^2+\bar{Q}_q^2)}{\sqrt{k^2(k^2+4\bar{Q}_q^2)}} \text{arcsinh}\left(\frac{k}{2\bar{Q}_q}\right) + \frac{k^2-2\bar{Q}_q^2}{2\bar{Q}_q^2}\right] + \frac{4Q^2z^2(1-z)^2+m_q^2}{\bar{Q}_q^2}\left[\frac{k^2+\bar{Q}_q^2}{\bar{Q}_q^2} - \frac{4\bar{Q}_q^4+2\bar{Q}_q^2k^2+k^4}{\bar{Q}_q^2\sqrt{k^2(k^2+4\bar{Q}_q^2)}}\text{arcsinh}\left(\frac{k}{2\bar{Q}_q}\right)\right]\Bigg\}$$

- The amplitude $\tilde{T}(k,Y)$, as we shall see, obeys the Balitsky-Kovchegov equation in momentum space, where the asymptotic behaviour of its solutions is naturally expressed

Dipole evolution

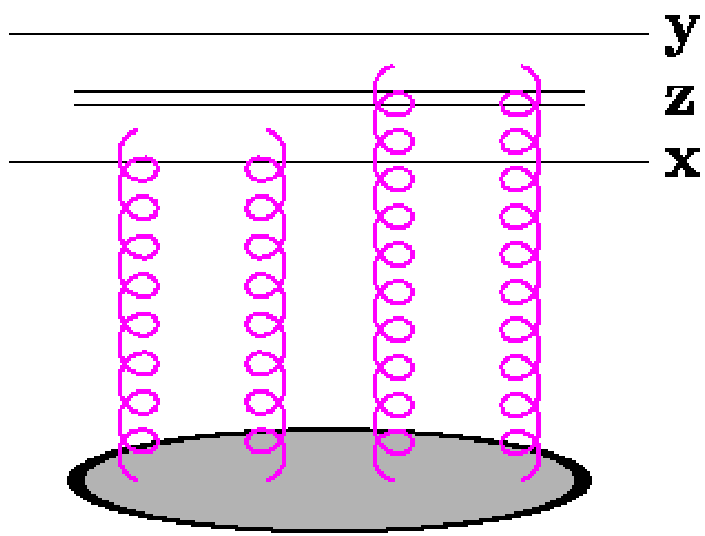
- Consider a fast-moving $q\bar{q}$



- In the large N_c limit the gluons emitted can be replaced by quark-anti-quark pairs, which interact with the target via two gluon exchanges

Balitsky-Kovchegov equation

Multiple scattering



- In the evolution of the scattering amplitude, the multiple scattering appears as a term proportional to T^2 ($\bar{\alpha} = \alpha_s N_c/\pi$)

$$\begin{aligned} \partial_Y T(\mathbf{x},\mathbf{y},Y) &= \bar{\alpha}\int d^2\mathbf{z}\,\frac{(\mathbf{x}-\mathbf{y})^2}{(\mathbf{x}-\mathbf{z})^2(\mathbf{z}-\mathbf{y})^2}\left[T(\mathbf{x},\mathbf{z},Y)+T(\mathbf{z},\mathbf{y},Y)-T(\mathbf{x},\mathbf{y},Y)\right. \\ &\quad \left. -T(\mathbf{x},\mathbf{z},Y)T(\mathbf{z},\mathbf{y},Y)\right] \end{aligned} \tag{7}$$

- This is the Balitsky-Kovchegov (BK) equation in coordinate space

BK equation in momentum space

- If one neglects the dependence on the impact parameter $\mathbf{b} = (\mathbf{x} + \mathbf{y})/2$ and integrates out the remaining angular dependence of $\mathbf{r}$, the BK equation becomes an equation for $T(r, Y)$
- After performing the Fourier transform the equation can be expressed in momentum space

$$\partial_Y \tilde{T} = \bar{\alpha}\chi(-\partial_L)\tilde{T} - \bar{\alpha}\tilde{T}^2 \tag{8}$$

  where

$$\chi(\gamma) = 2\psi(1) - \psi(\gamma) - \psi(1-\gamma) \tag{9}$$

  is the characteristic function of the Balitsky-Fadin-Kuraev-Lipatov (BFKL) kernel and $L = \log(k^2/k_0^2)$, with k_0 some fixed soft scale.
- The kernel χ is an integro-differential operator which may be defined with the help of the formal series expansion

$$\begin{aligned}\chi(-\partial_L) &= \chi(\gamma_0)\mathbf{1} + \chi'(\gamma_0)(-\partial_L - \gamma_0\mathbf{1}) + \frac{1}{2}\chi''(\gamma_0)(-\partial_L - \gamma_0\mathbf{1})^2 \\ &\quad + \frac{1}{6}\chi^{(3)}(\gamma_0)(-\partial_L - \gamma_0\mathbf{1})^3 + \ldots \end{aligned} \tag{10}$$

  for some γ_0 between 0 and 1

BK and FKPP equations

- It has been shown [Munier and Peshcanski, 03] that, after the change of variables

$$t \sim \bar{\alpha} Y, \quad x \sim \log(k^2/k_0^2), \quad u \sim \tilde{T} \tag{11}$$

BK equation reduces to Fisher and Kolmogorov-Petrovsky-Piscounov (FKPP) equation, when its kernel is approximated by the first three terms of the expansion, the so-called diffusive approximation

$$\chi(-\partial_L) \approx \chi(\gamma_c)\mathbf{1} + \chi'(\gamma_c)(-\partial_L - \gamma_c\mathbf{1}) + \frac{1}{2}\chi''(\gamma_c)(-\partial_L - \gamma_c\mathbf{1})^2, \tag{12}$$

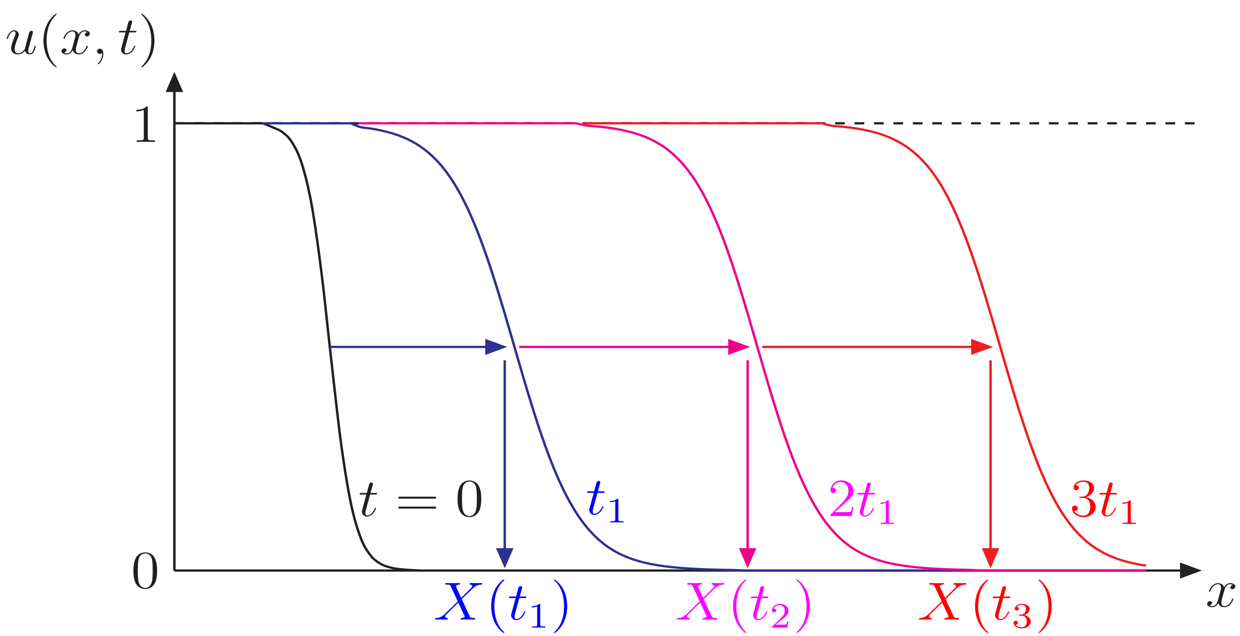
- The FKPP equation is a known equation in non-equilibrium statistical physics, whose dynamics is called reaction-diffusion dynamics,

$$\partial_t u(x,t) = \partial_x^2 u(x,t) + u - u^2, \tag{13}$$

where t is time and x is the coordinate.

Traveling wave solutions

- The FKPP evolution equation admits the so-called traveling wave solutions
  - For a traveling wave solution one can define the position of a wave front $x(t) = v_c(t)t$, irrespective of the details of the nonlinear effects
  - At large times, the shape of a traveling wave is preserved during its propagation, and the solution becomes only a function of the scaling variable $x - v_c t$



Traveling waves and saturation

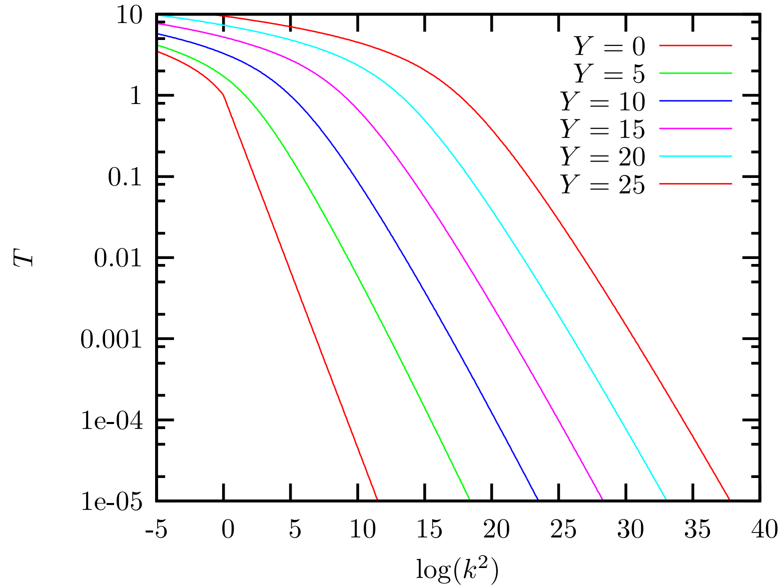
- In the language of saturation physics the position of the wave front is nothing but the saturation scale

$$x(t) \sim \ln Q_s^2(Y)$$

and the scaling corresponds to the geometric scaling

$$x - x(t) \sim \ln k^2/Q_s^2(Y)$$

Numerical solution to BK equation



- Summarizing:

$$\begin{array}{rcl} \text{Time } t & \longrightarrow & Y \\ \text{Space } x & \longrightarrow & L \\ \text{Wave front } u(x - v_c t) & \longrightarrow & \tilde{T}(L - \lambda Y) \\ \text{Traveling Waves} & \longrightarrow & \text{Geometric Scaling} \end{array} \quad (14)$$

Behaviour at $k \gg Q_s$

- This property of the FKPP equation is actually true if one considers the BK equation with the full BFKL kernel
- At asymptotic rapidities, the amplitude $\tilde{T}(k, Y)$, instead of depending separately on k and Y, depends only on the scaling variable $k^2/Q_s^2(Y)$, where we have introduced the *saturation scale* $Q_s^2(Y) = k_0^2 \exp(\lambda Y)$, measuring the position of the wavefront
- A more detailed calculation allows also for the extraction of two additional subleading corrections, resulting into the following expression for the tail of the scattering amplitude

$$\tilde{T}\,(k, Y) \overset{k \gg Q_s}{\approx} \left(\frac{k^2}{Q_s^2(Y)}\right)^{-\gamma_c} \log\left(\frac{k^2}{Q_s^2(Y)}\right) \exp\left[-\frac{\log^2\left(k^2/Q_s^2(Y)\right)}{2\bar{\alpha}\chi''(\gamma_c)Y}\right] \tag{15}$$

where the saturation scale

$$Q_s^2(Y) = k_0^2 \exp\left(\lambda Y - \frac{3}{2\gamma_c}\log(Y) - \frac{3}{\gamma_c^2}\sqrt{\frac{2\pi}{\bar{\alpha}\chi''(\gamma_c)}}\frac{1}{\sqrt{Y}}\right) \tag{16}$$

Critical parameters

- The critical parameters γ_c and λ are obtained from the knowledge of the BFKL kernel alone and correspond to the selection of the slowest possible wave:

$$\lambda = \min_{\gamma} \bar{\alpha} \frac{\chi(\gamma)}{\gamma} = \bar{\alpha} \frac{\chi(\gamma_c)}{\gamma_c} = \bar{\alpha}\chi'(\gamma_c)$$

- For the leading-order BFKL kernel, one finds $\gamma_c = 0.6275...$, and $\lambda = 4.88\bar{\alpha}$
- The geometric scaling expresses the fact that when one moves along the saturation line, the behaviour of the scattering amplitudes remains unchanged
- The last term in the expression for the tail introduces an explicit dependence on the rapidity Y and hence violates geometric scaling. However, this term can be neglected when

$$\frac{\log^2 \left(k^2/Q_s^2(Y)\right)}{2\bar{\alpha}\chi''(\gamma_c)Y} < 1$$

  This means that geometric scaling is obtained for

$$\log \left(k^2/Q_s^2(Y)\right) \lesssim \sqrt{2\chi''(\gamma_c)\bar{\alpha}Y}$$

Dipole scattering amplitude

- In order to complete the description, we also need expressions for $\tilde{T}$ around the saturation scale and at saturation
- In the infrared domain , one can show that the amplitude behaves like

$$\tilde{T}\left(\frac{k}{Q_s(Y)}, Y\right) \stackrel{k \ll Q_s}{=} c - \log\left(\frac{k}{Q_s(Y)}\right) \tag{17}$$

  where c is an unfixed constant

- We are now left with the matching around the saturation scale
  - The easiest way is to use the expression for the tail given previously for $k > Q_s$ and the above expression for $k < Q_s$ and match the constant c to obtain a continuous distribution
- The problem with this definition by parts is that it may introduce oscillations in the coordinate space amplitude $T(r, Y)$ which may even lead to negative amplitudes
- Then the best way to obtain the description of the transition to the saturation region is to perform an analytic interpolation between both asymptotic behaviours.

The model

JTSA, M. B. Gay Ducati, M. A Betemps and G. Soyez, hep-ph/0612091 **Accepted for publication in Physical Review D**

- Our starting point to describe the transition to saturation is an expression which is monotonically decreasing with L and which reproduces (up to the logarithmic factor), the amplitude for geometric scaling (tail)

$$T_{\text{dil}} = \exp\left[-\gamma_c \log\left(\frac{k^2}{Q_s^2(Y)}\right) - \frac{L_{\text{red}}^2 - \log^2(2)}{2\bar{\alpha}\chi''(\gamma_c)Y}\right] \tag{18}$$

with

$$L_{\text{red}} = \log\left[1 + \frac{k^2}{Q_s^2(Y)}\right] \quad \text{and} \quad Q_s^2(Y) = k_0^2\, e^{\lambda Y} \tag{19}$$

- This result is unitarised by an eikonal, $T_{\text{unit}} = 1 - \exp(-T_{\text{dil}})$ and we obtain that the following choice gives good results:

$$T(k,Y) = \left[\log\left(\frac{k}{Q_s} + \frac{Q_s}{k}\right) + 1\right]\left(1 - e^{-T_{\text{dil}}}\right) \tag{20}$$

- The equations above determine our model for the scattering amplitude, to be inserted into the expression for the F_2 structure function

Dataset

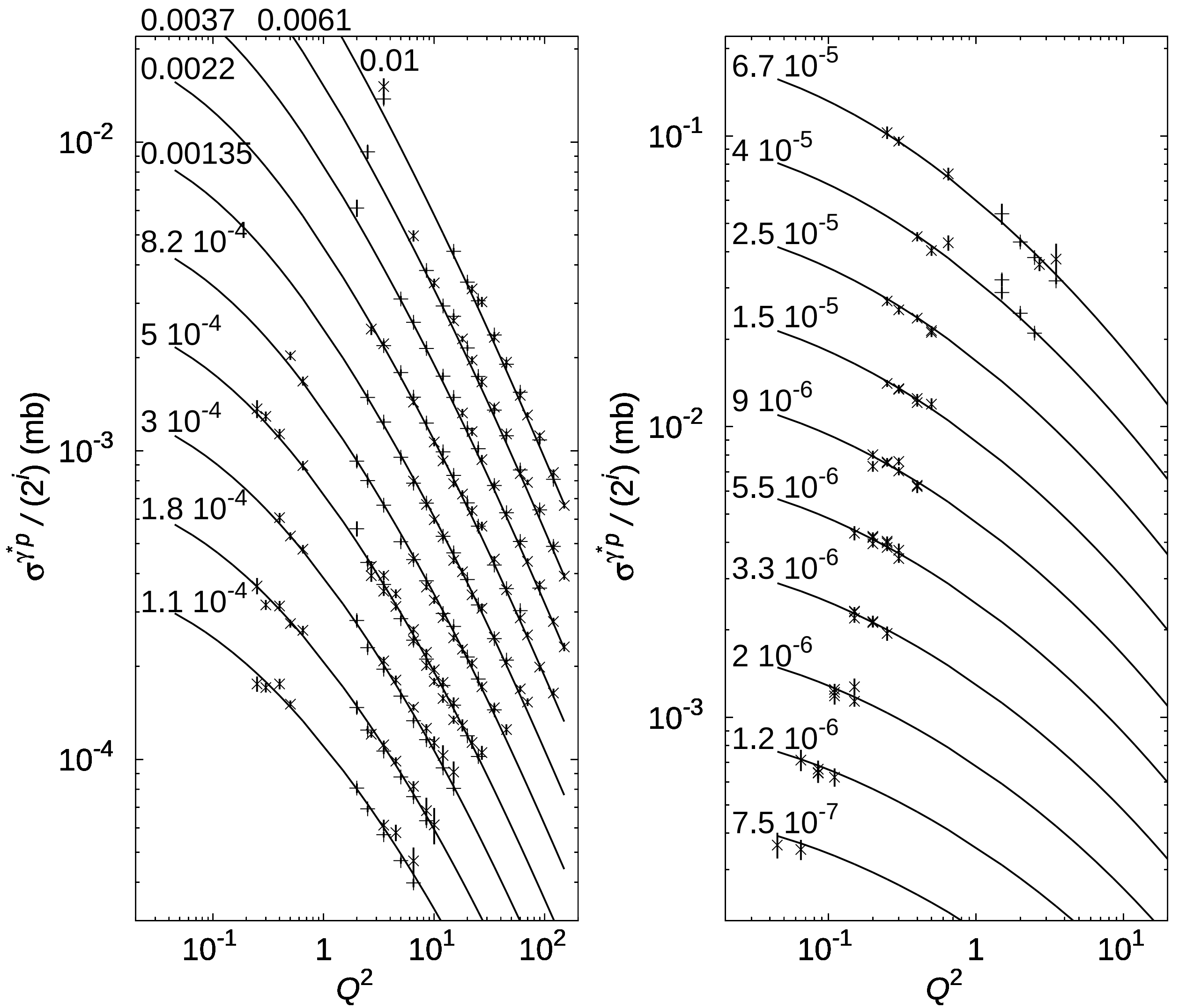
- We fit all the last HERA measurements of the proton structure function from H1 and ZEUS
- Our analysis is restricted to the following kinematic range:

$$\begin{cases} x \leq 0.01, \\ 0.045 \leq Q^2 \leq 150 \text{ GeV}^2 \end{cases}$$

  - The first limit comes from the fact that our approach is meant to describe the high-energy amplitudes *i.e.* the small x behaviour
  - The second cut prevents to reach too high values of Q^2 for which DGLAP corrections need to be included properly
- Total amount of data points: 279; we have allowed for a 5% renormalisation uncertainty on the H1 data
- We have kept $\gamma_c = 0.6275$ and $\bar{\alpha} = 0.2$ fixed assumed different situations for the quarks masses: the light-quarks mass m_q has been set to 50 or 140 MeV while we have used $m_c = m_q$ or $m_c = 1.3$ GeV for the charm mass
- λ, χ_c'' , k_0^2 and R_p are free parameters

Results (I): $\sigma^{\gamma^* p}$ cross section

$m_q = 50$ MeV and $m_c = 1.3$ GeV



H1 [EPJC 21, 2002] and ZEUS [EPJC 12, 2000; EPJC 21, 2001]

Results (II): Parameters

- The parameters obtained from the fit to the experimental data for F_2^p:

Masses	k_0^2 (10^{-3} GeV^2)	λ	χ_c''	R_p (GeV^{-1})
$m_q = 50$ MeV, $m_c = 50$ MeV	3.782 ± 0.293	0.213 ± 0.004	4.691 ± 0.221	2.770 ± 0.045
$m_q = 50$ MeV, $m_c = 1.3$ GeV	7.155 ± 0.624	0.193 ± 0.003	2.196 ± 0.161	3.215 ± 0.065
$m_q = 140$ MeV, $m_c = 1.3$ GeV	3.917 ± 0.577	0.161 ± 0.005	2.960 ± 0.279	4.142 ± 0.167

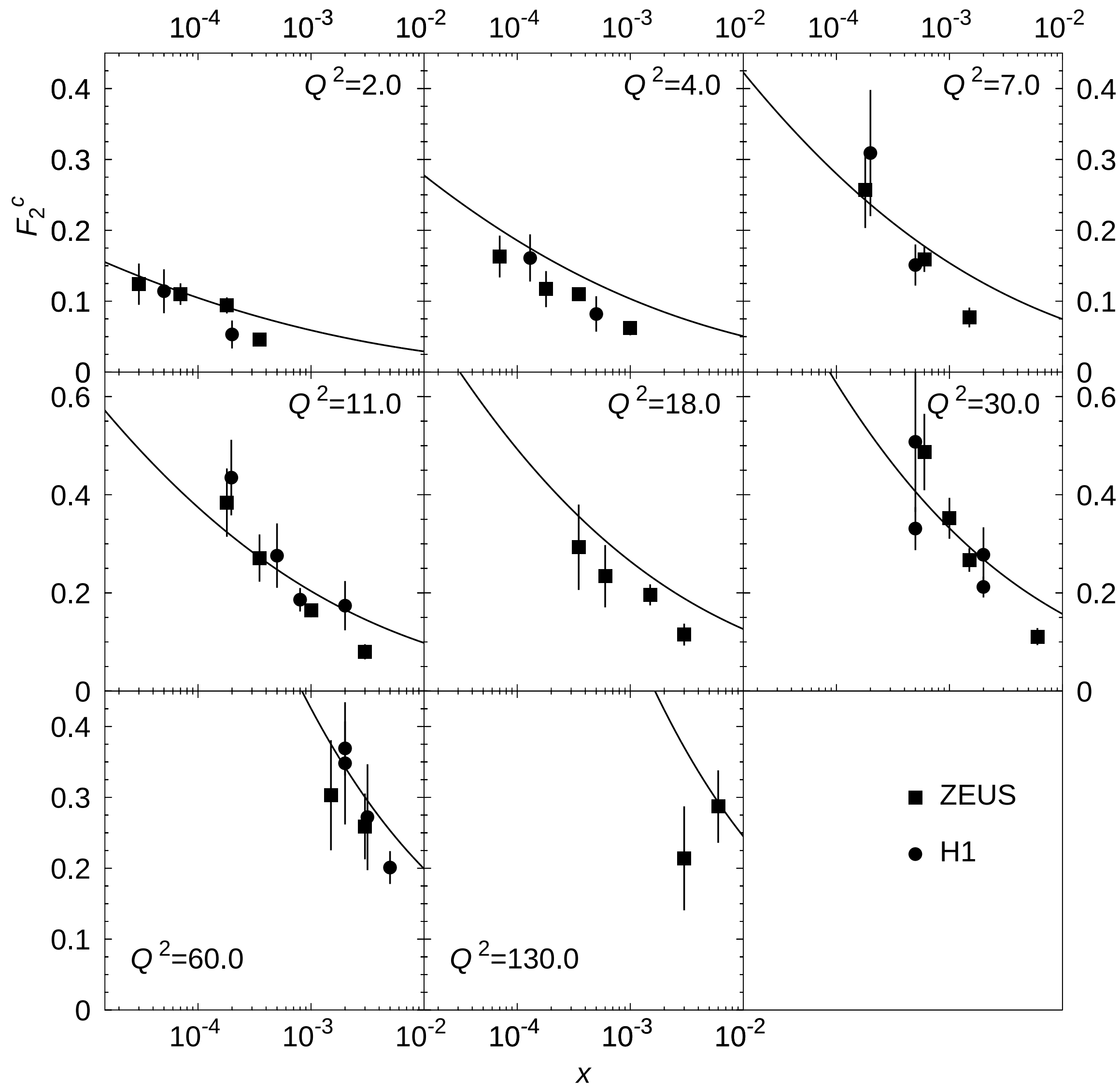
- The χ^2 per point

Masses	χ^2/nop
$m_q = 50$ MeV, $m_c = 50$ MeV	0.960
$m_q = 50$ MeV, $m_c = 1.3$ GeV	0.988
$m_q = 140$ MeV, $m_c = 1.3$ GeV	1.071

- Good agreement with the measurements of F_2^p due to the small χ^2

Results (III): prediction for F_2^c

$m_q = 50$ MeV and $m_c = 1.3$ GeV



H1 [PLB 528, 2002; EPJC 45, 2006] and ZEUS [PRD 69, 2004]

Analysis: previous models

- It is interesting to compare our results with those from previous approaches, namely
  - GBW model [Golec-Biernat and Wüsthoff, 1999]
  - IIM model or CGC fit [Iancu, Itakura and Munier, 2004]
  - recent developments concerning the Bartels-Golec-Biernat-Kowalski (BGK) model [Golec-Biernat and Sapeta, 2006]

Parametrization	Quark masses	nop	χ^2/nop
GBW	$m_q = 140$ MeV, $m_c = 1.5$ GeV	372	1.5
IIM	$m_q = 140$ MeV, no charm	156	0.81
BGK	$m_q = 0$ MeV, $m_c = 1.3$ GeV	288	1.06
This work	$m_q = 50$ MeV, $m_c = 1.3$ GeV	279	0.988

- These three models were developed in coordinate space and not in momentum space
- Our choice is directly motivated by the analysis of the BK equation in momentum space leading to universal asymptotic results on which we heavily rely

Analysis: conclusions

- Our model can be differentiated from the previous ones at two levels:
  - The analysis is based on the BK equation to account for unitarity effects. Thus, we expect it to be more precise, especially in the small-x and low Q^2 domain under study
  - We improved the IIM model by including massive charm
- Moreover, in the case of GBW model, one obtains a Fourier transform of the dipole cross section which presents an unrealistic perturbative behaviour, in the case of IIM it presents non-positive values [Betemps and Gay Ducati, 2004]
- These problems are tamed in our model, where the inverse Fourier transform (scattering amplitude in coordinate space) remains between 0 and 1
- We then conclude that the dipole scattering amplitude proposed in this work should be a good parametrization to investigate the properties of the observables at RHIC and LHC energies, considering the dipole: R_{pA} and fluctuations
- Further studies to see if one can get further analytic insight directly from the BK equation

Improved IIM model

- In the original paper, the fit to HERA data did not include heavy quark contributions
- In this way, this model has been recently improved [Soyez, 2007]
- The model

$$T(r;Y) = \begin{cases} T_0 \exp\left[-\gamma_c(\rho-\rho_s) - \frac{(\rho-\rho_s)^2}{2\kappa\lambda Y}\right] & \text{if } rQ_s \leq 2, \\ 1-\exp\left[-a(\rho-\rho_s-b)^2\right] & \text{if } rQ_s > 2, \end{cases} \qquad (21)$$

  where the parameters a and b are fixed so as to ensure that T and its derivative are continuous at $rQ_s = 2$

  - Here, $\bar{\alpha}\chi_c'' = (\chi_c''/\chi_c')\lambda Y = \kappa\lambda Y$ with $\kappa = \chi_c''/\chi_c'$
- Parameters:
  - The saturation scale $Q_s^2(Y) = \left(\frac{x_0}{x}\right)^\lambda$ GeV^2
  - x_0 free, R_p free, $T_0 = 0.7$ fixed
  - λ free: LO BFKL predicts $\lambda = \bar{\alpha}\chi_c' \approx 0.9$ and NLO BFKL analysis gives $\lambda \sim 0.3$ [Triantafyllopoulos, 2003]
  - The value of κ set from the LO BFKL kernel which gives $\kappa \approx 9.9$ [Triantafyllopoulos, 2003]

The critical slope

- The value of the critical slope γ_c is a **fundamental issue**
  - In the original work, where only the light quarks were considered, the authors have fixed it to the value obtained from the LO BFKL kernel ($\gamma_c \approx 0.6275$)
  - When including the heavy quarks, keeping that value leads to a dramatic decrease of the saturation momentum together as well as to a poor χ^2 for the fit
  - Allowing that parameter to vary, one recovers a similar saturation scale and a good fit
  - In addition, the value for γ_c coming out of the fit is rather close to what one expects from NLO BFKL ($\gamma_c \gtrsim 0.7$)

		γ_c	λ	x_0 (10^{-4})	R_p (GeV^{-1})	χ^2/n.o.p.
light quarks only	(ZEUS only)	0.6275	0.253	0.267	3.25	-
	γ_c fixed	0.6275	0.2574 ± 0.0037	0.2750 ± 0.0240	3.241 ± 0.018	0.959
	γ_c free	0.6194 ± 0.0091	0.2545 ± 0.0051	0.2131 ± 0.0651	3.277 ± 0.044	0.956
light+heavy quarks	γ_c fixed	0.6275	0.1800 ± 0.0026	0.0028 ± 0.0003	3.819 ± 0.017	1.116
	γ_c free	$\mathbf{0.7376 \pm 0.0094}$	$\mathbf{0.2197 \pm 0.0042}$	$\mathbf{0.1632 \pm 0.0471}$	$\mathbf{3.344 \pm 0.041}$	$\mathbf{0.900}$

The idea is...

- Follow a similar procedure with our model in momentum space

$$T(k,Y) = \left[\log\left(\frac{k}{Q_s} + \frac{Q_s}{k}\right) + 1\right]\left(1 - e^{-T_{\text{dil}}}\right) \tag{22}$$

with

$$T_{\text{dil}} = A\exp\left[-\gamma_c \log\left(\frac{k^2}{Q_s^2(Y)}\right) - \frac{L_{\text{red}}^2 - \log^2(2)}{2\kappa\lambda Y}\right] \tag{23}$$

and

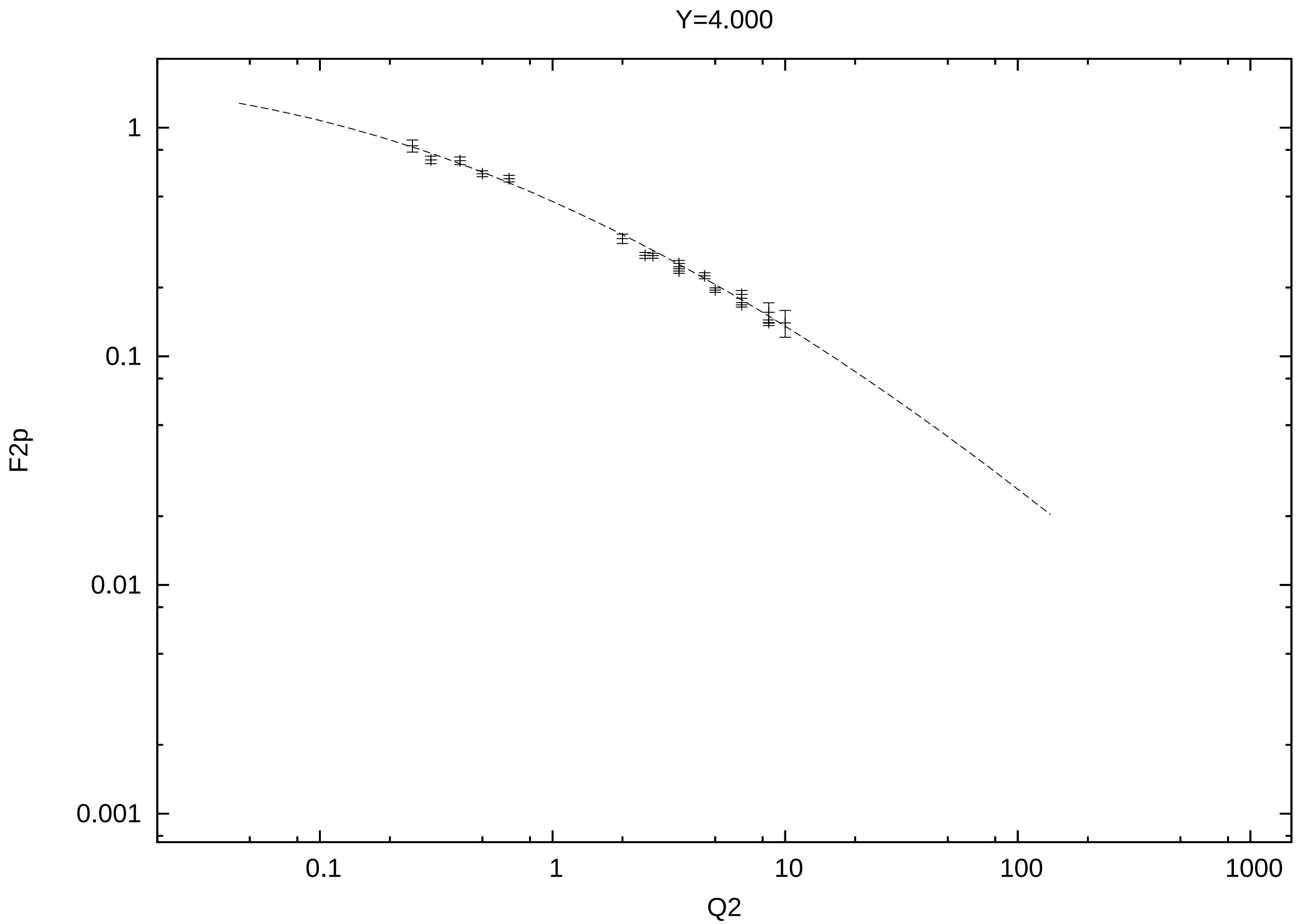
$$L_{\text{red}} = \log\left[1 + \frac{k^2}{Q_s^2(Y)}\right] \quad \text{and} \quad Q_s^2(Y) = k_0^2\, e^{\lambda Y} \tag{24}$$

- Include bottom contribution $m_b = 4.5$ GeV
- Keep κ fixed at its LO (or NLO) value
- Make γ_c freely vary and try to find a value around 0.7, as well as a not so strong decrease in the saturation scale when heavy quarks are included
- Test other (or more general) logarithmic factors

Some tests...

- γ_c fixed at 0.7

k_0^2 (10^{-3} GeV^2)	λ	χ_c''	R_p (GeV^{-1})	$\chi^2/n.o.p.$
9.108 ± 0.063	0.213 ± 0.003	1.869 ± 0.131	2.975 ± 0.045	1.105



Modified Bjorken variable

- Basically, this shift accounts for the threshold for heavy-quark production. In order to produce a $q\bar{q}$ pair (with a quark mass m_q), you need

$$W^2 > 4m_q^2 \tag{25}$$

- Since x is defined as $\left(Q^2/(W^2+Q^2)\right)$ (or, $W^2 = Q^2(1-x)/x$), the production threshold turns into

$$x\left(1+\frac{4m_q^2}{Q^2}\right) < 1 \tag{26}$$

- By using

$$x_{eff} = x\left(1+\frac{4m_q^2}{Q^2}\right) \tag{27}$$

the usual kinematical limit $x_{eff} < 1$ corresponds to the production threshod.

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