Some aspects of high energy hadronic collisions in the Color Glass Condensate framework

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Introduction
Infrared and collinear divergences

- Calculation of some process at LO:

\[
\begin{align*}
M_\perp, Y \\
x_1 &= M_\perp e^{+Y} / \sqrt{s} \\
x_2 &= M_\perp e^{-Y} / \sqrt{s}
\end{align*}
\]
**Infrared and collinear divergences**

- Calculation of some process at LO:

\[
(\mathbb{M}_\perp, Y) \quad \left\{ \begin{array}{l}
x_1 = M_\perp e^{+Y}/\sqrt{s} \\
x_2 = M_\perp e^{-Y}/\sqrt{s}
\end{array} \right.
\]

- Radiation of an extra gluon:

\[
\alpha_s \int_{x_1} \frac{dz}{z} \int_{M_\perp} \frac{d^2 k_\perp}{k_\perp^2}
\]
Infrared and collinear divergences

- Large logs: $\log(M_\perp)$ or $\log(1/x_1)$, under certain conditions
  - these logs can compensate the additional $\alpha_s$, and void the naive application of perturbation theory
  - resummations are necessary

- Logs of $M_\perp \rightarrow$ DGLAP. Important when:
  - $M_\perp \gg \Lambda_{QCD}$
  - $x_1, x_2$ are rather large

- Logs of $1/x \rightarrow$ BFKL. Important when:
  - $M_\perp$ remains moderate
  - $x_1$ or $x_2$ (or both) are small

- Physical interpretation:
  - The physical process can resolve the gluon splitting if $M_\perp \gg k_\perp$
  - If $x_1 \ll 1$, the gluon that initiates the process is likely to result from bremsstrahlung from another parent gluon
Multiple scatterings

- Single scattering:

- ▶ 2-point function in the projectile ▶ gluon number
Multiple scatterings

- **Single scattering**:  
  - 2-point function in the projectile ⊳ gluon number

- **Multiple scatterings**:  
  - 4-point function in the projectile ⊳ higher correlation
  - multiple scatterings in the projectile
Multiple scatterings

- **Power counting**: rescattering corrections are suppressed by inverse powers of the typical mass scale in the process:
  
  \[ \left( \frac{\mu^2}{M^2} \right)^n \]

- The parameter \( \mu^2 \) has a factor of \( \alpha_s \), and a factor proportional to the gluon density \( \alpha_s \) rescatterings are important at high density.

- Relative order of magnitude:
  
  \[
  \frac{2 \text{ scatterings}}{1 \text{ scattering}} \sim \frac{Q_s^2}{M^2} \quad \text{with} \quad Q_s^2 \sim \alpha_s \frac{xG(x, Q_s^2)}{\pi R^2}
  \]

- When this ratio becomes \( \sim 1 \), all the rescattering corrections become important.

- These effects are not accounted for in DGLAP or BFKL.
99% of the multiplicity below $p_{\perp} \sim 2$ GeV

$Q_s^2$ might be as large as 10 GeV$^2$ at the LHC ($\sqrt{s} = 5.5$ TeV)

- both the logs of $1/x$ and the multiple scatterings are important
Heavy Ion Collisions

- calculate the initial production of semi-hard particles
- prepare the stage for kinetic theory or hydrodynamics

- strong fields $\rightarrow$ classical EOMs
- gluons & quarks out of eq. $\rightarrow$ kinetic theory
- gluons & quarks in eq. $\rightarrow$ hydrodynamics
- hadrons in eq. $\rightarrow$ hydrodynamics
- freeze out

$t$ (time)

$z$ (beam axis)
Goals

- Develop a framework for resumming all the 
  \[ \alpha_s \ln(1/x)^m [ Q/\sqrt{M} ]^n \] corrections

- Generalize the concept of “parton distribution”
  - Due to the high density of partons, observables depend on higher correlations (beyond the usual parton distributions, which are 2-point correlation functions)

- These distributions should be universal, with non-perturbative information relegated into the initial condition of some evolution equation

- Develop techniques for describing the early stages of heavy ion collisions in this framework

- What can be said about exclusive quantities, diffraction?
Outline

- Color Glass Condensate
- Basic principles and bookkeeping
- AGK identities
- Inclusive gluon spectrum at leading order
- Less inclusive quantities
- Loop corrections, factorization, instabilities

- FG, Venugopalan, hep-ph/0601209, 0605246
- Fukushima, FG, McLerran, hep-ph/0610416
  + work in progress with Lappi, Venugopalan
Color Glass Condensate
A nucleon at rest is a very complicated object...

- Contains fluctuations at all space-time scales smaller than its own size
- Only the fluctuations that are longer lived than the external probe participate in the interaction process
- The only role of short lived fluctuations is to renormalize the masses and couplings
- Interactions are very complicated if the constituents of the nucleon have a non trivial dynamics over time-scales comparable to those of the probe
Nucleon at high energy

- **Dilation** of all internal time-scales for a high energy nucleon
- Interactions among constituents now take place over time-scales that are longer than the characteristic time-scale of the probe
  - the constituents behave as if they were free
- Many fluctuations live long enough to be seen by the probe. The nucleon appears **denser at high energy** (it contains more gluons)
assume that the projectile is big, e.g. a nucleus, and has many valence quarks (only two are represented)
on the contrary, consider a small probe, with few partons
at low energy, only valence quarks are present in the hadron wave function
Parton evolution

- when energy increases, new partons are emitted
- the emission probability is $\alpha_s \int \frac{dx}{x} \sim \alpha_s \ln(\frac{1}{x})$, with $x$ the longitudinal momentum fraction of the gluon
- at small-$x$ (i.e. high energy), these logs need to be resummed
as long as the density of constituents remains small, the evolution is **linear**: the number of partons produced at a given step is proportional to the number of partons at the previous step (BFKL)
eventually, the partons start overlapping in phase-space

parton recombination becomes favorable

after this point, the evolution is non-linear:
the number of partons created at a given step depends non-linearly on the number of partons present previously
Saturation criterion

Gribov, Levin, Ryskin (1983)

■ Number of gluons per unit area:

\[ \rho \sim \frac{xG_A(x, Q^2)}{\pi R_A^2} \]

■ Recombination cross-section:

\[ \sigma_{gg \rightarrow g} \sim \frac{\alpha_s}{Q^2} \]

■ Recombination happens if \( \rho \sigma_{gg \rightarrow g} \gtrsim 1 \), i.e. \( Q^2 \lesssim Q_s^2 \), with:

\[ Q_s^2 \sim \frac{\alpha_s xG_A(x, Q_s^2)}{\pi R_A^2} \sim A^{1/3} \frac{1}{x^{0.3}} \]

■ At saturation, the phase-space density is:

\[ \frac{dN_g}{d^2 \vec{x}_\perp d^2 \vec{p}_\perp} \sim \frac{\rho}{Q^2} \sim \frac{1}{\alpha_s} \]
Saturation domain

\[ \log(x^{-1}) \]

\[ \Lambda_{\text{QCD}} \]

\[ \log(Q^2) \]
Color Glass Condensate

- Soft modes have a large occupation number
  - they are described by a classical color field $A^\mu$ that obeys Yang-Mills’s equation:

$$[D_\nu, F^{\nu\mu}]_a = J^\mu_a$$

- The source term $J^\mu_a$ comes from the faster partons. The hard modes, slowed down by time dilation, are described as frozen color sources $\rho_a$. Hence:

$$J^\mu_a = \delta^{\mu+}\delta(x^-)\rho_a(\vec{x}_\perp) \quad (x^- \equiv (t - z)/\sqrt{2})$$

- The color sources $\rho_a$ are random, and described by a distribution functional $W_Y[\rho]$, with $Y$ the rapidity that separates “soft” and “hard”. Evolution equation (JIMWLK):

$$\frac{\partial W_Y[\rho]}{\partial Y} = \mathcal{H}[\rho] \ W_Y[\rho]$$
DIS and other elementary reactions

- Color Glass Condensate
  - Parton model
  - Parton saturation
  - Color Glass Condensate
- Deep Inelastic Scattering
  - Hadron-hadron collisions
- Main issues
- AGK identities
- Bookkeeping
- Inclusive gluon spectrum
- Less inclusive quantities
- Loop corrections
- Summary
DIS and other elementary reactions
DIS and other elementary reactions

Introduction

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Summary
DIS and other elementary reactions
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10 configurations
DIS and other elementary reactions

100 configurations
DIS and other elementary reactions

1000 configurations
Reactions involving a hadron or nucleus and an “elementary” projectile are fairly straightforward to study.

The archetype is the forward DIS amplitude:

\[
\langle T(\vec{x}_\perp, \vec{y}_\perp) \rangle = \int [D\rho] \ W_Y[\rho] \left[ 1 - \frac{1}{N_c} \text{tr}(U(\vec{x}_\perp)U^\dagger(\vec{y}_\perp)) \right]
\]
Reactions involving a hadron or nucleus and an “elementary” projectile are fairly straightforward to study.

Many other reactions have been considered. $qA \rightarrow qX$:
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Many other reactions have been considered. $gA \rightarrow gX$:
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Many other reactions have been considered. $qA \rightarrow q\gamma X$: 
Reactions involving a hadron or nucleus and an “elementary” projectile are fairly straightforward to study.

Many other reactions have been considered. \( qA \rightarrow qgX \):

- all these calculations are relevant for the case of the collision between a saturated projectile and a dilute projectile.
Description of hadronic collisions

- In the case of symmetric collisions (especially nucleus-nucleus collisions), the two projectiles should be treated on the same footing.

- For hadron-hadron collisions, there are two strong sources that contribute to the color current:

\[
J^\mu \equiv \delta^{\mu+}\delta(x^-)\rho_1(\vec{x}_\perp) + \delta^{\mu-}\delta(x^+)\rho_2(\vec{x}_\perp)
\]

- Average over the sources \(\rho_1, \rho_2\)

\[
\langle O_Y \rangle = \int [D\rho_1] [D\rho_2] W_{Y_{beam}-Y}[\rho_1] W_{Y+Y_{beam}}[\rho_2] O[\rho_1, \rho_2]
\]

- Can this procedure – and in particular the above factorization formula – be justified?
Description of hadronic collisions

\[ \mathcal{L} = -\frac{1}{2} \text{tr} F_{\mu\nu} F^{\mu\nu} + \left( J_1^\mu + J_2^\mu \right) A_\mu \]
Main issues

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- Some of them may not produce anything (vacuum diagrams)
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- **Dilute regime**: one source in each projectile interact
- **Dense regime**: non linearities are important
- There can be many simultaneous disconnected diagrams
- Some of them may not produce anything (**vacuum diagrams**)
- All these diagrams can have loops (not at LO though)
Abramovsky-Gribov-Kancheli identities
Generating function

- Let $P_n$ be the probability of producing $n$ particles.
- Define the generating function:

$$F(z) \equiv \sum_{n=0}^{\infty} P_n z^n$$

- From unitarity, $F(1) = \sum_{n=0}^{\infty} P_n = 1$. Thus, we can write

$$\ln(F(z)) \equiv \sum_{r=1}^{\infty} b_r (z^r - 1)$$

- At the moment, we need to know only very little about the $b_r$:
  - $F(z)$ is a sum of diagrams that may or may not be connected.
  - $\ln(F(z))$ involves only connected diagrams. Hence, the $b_r$’s are given by certain sums of connected diagrams.
  - Every diagram in $b_r$ produces $r$ particles.
Example: typical term in the coefficient of $z^{11}$, with contributions from $b_5$ and $b_6$:
Distribution of connected subdiagrams

From this form of the generating function, one gets:

\[ P_n = \sum_{p=0}^{n} e^{-\sum_r b_r} \frac{1}{p!} \sum_{\alpha_1 + \cdots + \alpha_p = n} b_{\alpha_1} \cdots b_{\alpha_n} \]

probability of producing \( n \) particles in \( p \) cut subdiagrams

Summing on \( n \), we get the probability of \( p \) cut subdiagrams:

\[ R_p = \frac{1}{p!} \left( \sum_{r=1}^{\infty} b_r \right)^p e^{-\sum_r b_r} \]

Note: Poisson distribution of average \( \langle N_{\text{subdiagrams}} \rangle = \sum_r b_r \)

By expanding the exponential, we get the probability of having \( p \) cut subdiagrams out of a total of \( m \):

\[ R_{p,m} = \frac{(-1)^{m-p}}{(m-p)!p!} \left( \sum_{r=1}^{\infty} b_r \right)^m \]
AGK identities

- The quantities $R_{p,m}$ obey the following relations:

  \[ \forall m \geq 2, \quad \sum_{p=1}^{m} p R_{p,m} = 0, \]

  \[ \forall m \geq 3, \quad \sum_{p=1}^{m} p(p-1) R_{p,m} = 0, \cdots \]

- Interpretation: contributions with more than 1 subdiagram cancel in the average number of cut subdiagrams, etc...

- Correspondence with the original relations by Abramovsky-Gribov-Kancheli:
  - The original derivation is formulated in the framework of reggeon effective theories
  - Dictionary: reggeon $\longrightarrow$ subdiagram
  - These identities are more general than “reggeons”, and are valid for any kind of subdiagrams
The AGK relations, obtained by “integrating out” the number of produced particles, describe the combinatorics of connected diagrams.

- by doing that, a lot of information has been discarded

For instance, to compute the average number of produced particles, one would write:

\[
\langle n \rangle = \langle N_{\text{subdiagrams}} \rangle \times \langle \# \text{ of particles per diagram} \rangle \sum_r b_r
\]

requires a more detailed description
Power counting and bookkeeping
In the saturated regime, the sources are of order $1/g$ (because $\langle \rho \rho \rangle \sim$ occupation number $\sim 1/\alpha_s$).

The order of each disconnected diagram is given by:

$$\frac{1}{g^2} g^{\# \text{ produced gluons}} g^{2(\# \text{ loops})}$$

The total order of a graph is the product of the orders of its disconnected subdiagrams $\triangleright$ quite messy...
Bookkeeping
Consider squared amplitudes (including interference terms) rather than the amplitudes themselves.
Bookkeeping

- Consider \textit{squared amplitudes} (including interference terms) rather than the amplitudes themselves
- See them as \textit{cuts through vacuum diagrams}
Consider squared amplitudes (including interference terms) rather than the amplitudes themselves.

See them as cuts through vacuum diagrams.

Consider only the simply connected ones, thanks to:

\[ \sum \left( \text{all the vacuum diagrams} \right) = \exp \left\{ \sum \left( \text{simply connected vacuum diagrams} \right) \right\} = e^{iV[j]} \]

Simpler power counting for connected vacuum diagrams:

\[ \frac{1}{g^2} g^2 (# \text{ loops}) \]
The probability of producing exactly \( n \) particles in the collision of the two hadrons is given by:

\[
P_n = \frac{1}{n!} \int \frac{d^3 \vec{p}_1}{(2\pi)^3 2E_1} \cdots \frac{d^3 \vec{p}_n}{(2\pi)^3 2E_n} |\langle \vec{p}_1 \cdots \vec{p}_n_{out} | 0_{in} \rangle|^2
\]

The reduction formula can be written as:

\[
\langle \vec{p}_1 \cdots \vec{p}_n_{out} | 0_{in} \rangle = \frac{1}{Z^{n/2}} \int \left[ \prod_{i=1}^{n} d^4 x_i \ e^{ip_i \cdot x_i} \ (\Box_i + m^2) \frac{\delta}{i\delta j(x_i)} \right] e^{iV[j]}
\]

and we have

\[
P_n = \frac{1}{n!} \ D^n \ e^{iV[j_+]} \ e^{-iV^*[j_-]} \bigg|_{j_+=j_-=j}
\]

with

\[
\left\{
\begin{align*}
D &\equiv \frac{1}{Z} \int_{x,y} G^0_{+-} (x,y) \ (\Box_x + m^2) (\Box_y + m^2) \ \frac{\delta}{\delta j_+(x)} \ \frac{\delta}{\delta j_-(y)} \\
G^0_{+-} (x,y) &\equiv \int \frac{d^3 \vec{p}}{(2\pi)^3 2E_p} \ e^{ip \cdot (x-y)}
\end{align*}
\right.
\]
The operator $D$ acts on a pair of vacuum diagrams by removing two sources and attaching a cut propagator instead:
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$\mathcal{D}$ can also act directly on single diagram, if it is already cut.

The sum of all the cut vacuum diagrams, with sources $j_+$ on one side of the cut and $j_-$ on the other side, can be written as:

$$\sum \left( \text{all the cut vacuum diagrams} \right) = e^{\mathcal{D}} e^{iV[j_+]} e^{-iV^*[j_-]}$$

Note: if we set $j_+ = j_- = j$, then this is $\sum_n P_n = 1$
The operator $\mathcal{D}$ can be used to derive many useful formulas:

$$F(z) = \sum_{n=0}^{+\infty} z^n P_n = e^{z\mathcal{D}} e^{iV[j_+]} e^{-iV^*[j_-]} \bigg|_{j_+ = j_- = j}$$

▷ sum of all cut vacuum graphs, where each cut is weighted by $z$

$$\overline{N} = F'(1) = \mathcal{D} e^{\mathcal{D}} e^{iV[j_+]} e^{-iV^*[j_-]} \bigg|_{j_+ = j_- = j}$$

$$\overline{N(N - 1)} = F''(1) = \mathcal{D}^2 e^{\mathcal{D}} e^{iV[j_+]} e^{-iV^*[j_-]} \bigg|_{j_+ = j_- = j}$$

Note: $\ln F(z) = \sum_{r=1}^{+\infty} b_r (z^r - 1)$ is the same sum, with only connected graphs

▷ $b_r = \text{sum of all connected cut vacuum graphs with exactly } r \text{ cuts}$

All the moments of the particle distribution are obtained by the action of powers of $\mathcal{D}$ on the sum of cut vacuum graphs

▷ these formulas tell us how to construct the various moments in terms of cut vacuum diagrams
Inclusive gluon spectrum
Poisson or not Poisson ?

- A Poisson distribution has a generating function of the form:
  \[ F(z) = e^{N(z-1)} \]

- In our case, the generating function can be written as:
  \[ F(z) = e^{\sum_r b_r (z^r - 1)} \]

  therefore, in order to have a Poisson distribution, one must have
  \[ b_1 = N \]
  \[ b_2, b_3, \cdots = 0 \]

- However, the numbers \( b_r \) are all of order \( 1/g^2 \)
  ⇒ not a Poisson distribution

- Note: \( b_r \) is the sum of connected vacuum graphs with \( r \) cuts
  ⇒ to have a Poisson distribution, each graph must produce only one particle
  ⇒ no correlations between the particles produced by a given configuration of sources
First moment of the distribution

- It is easy to express the average multiplicity as:

\[
\overline{N} = \sum_n n P_n = D \left\{ e^D e^{iV[j_+]} e^{-iV^*[j_-]} \right\}_{j+ = j- = j}
\]

- \( \overline{N} \) is obtained by the action of \( D \) on the sum of all the cut vacuum diagrams. There are two kind of terms:
  - \( D \) picks two sources in two distinct connected cut diagrams
  - \( D \) picks two sources in the same connected cut diagram
Gluon multiplicity at LO

- At LO, only tree diagrams contribute ▷ the second type of topologies can be neglected (it starts at 1-loop)

- In each blob, we must sum over all the tree diagrams, and over all the possible cuts:

  \[ \overline{N}_{LO} = \sum_{\text{trees}} \sum_{\text{cuts}} \]

- A major simplification comes from the following property:

  \[ \sim\sim\sim\sim\sim - \sim\times\sim\sim = \text{retarded propagator} \]

- The sum of all the tree diagrams constructed with retarded propagators is the retarded solution of Yang-Mills equations:

  \[ [D_{\mu}, F^{\mu\nu}] = J^\nu \quad \text{with} \quad A^\mu(x_0 = -\infty) = 0 \]
Gluon multiplicity at LO


\[ \frac{d\bar{N}^{LO}}{dY d^2\vec{p}_\perp} = \frac{1}{16\pi^3} \int_{x,y} e^{ip\cdot(x-y)} \Box_x \Box_y \sum_{\lambda} \epsilon_{\lambda}^\mu \epsilon_{\lambda}^\nu A_\mu(x) A_\nu(y) \]

- \( A^\mu(x) = \) retarded solution of Yang-Mills equations

\[ \int x, y \]$
Gluon multiplicity at LO


\[
\frac{dN_{LO}}{dY d^2 p_\perp} = \frac{1}{16\pi^3} \int_{x,y} e^{i p \cdot (x-y)} \square_x \square_y \sum_\lambda \epsilon_\lambda^\mu \epsilon_\lambda^\nu A_\mu(x) A_\nu(y)
\]

\[A^\mu(x) = \text{retarded solution of Yang-Mills equations}\]

\[\Rightarrow \text{can be cast into an initial value problem on the light-cone}\]
Gluon multiplicity at LO

- Lattice artefacts at large momentum (they do not affect much the overall number of gluons)
- Important softening at small $k_\perp$ compared to pQCD (saturation)
**Gauge condition**: \( x^+ A^- + x^- A^+ = 0 \)

\[
\begin{align*}
A^i(x) &= \alpha^i(\tau, \eta, \vec{x}_\perp) \\
A^\pm(x) &= \pm x^\pm \beta(\tau, \eta, \vec{x}_\perp)
\end{align*}
\]

**Initial values at** \( \tau = 0^+ \): \( \alpha^i(0^+, \eta, \vec{x}_\perp) \) and \( \beta(0^+, \eta, \vec{x}_\perp) \) do not depend on the rapidity \( \eta \)

\( \alpha^i \) and \( \beta \) remain independent of \( \eta \) at all times (invariance under boosts in the \( z \) direction)

\( \uparrow \) numerical resolution performed in \( 1 + 2 \) dimensions
Less inclusive quantities
Definition

- One can encode the information about all the probabilities $P_n$ in a generating function defined as:

$$F(z) \equiv \sum_{n=0}^{\infty} P_n z^n$$

- From the expression of $P_n$ in terms of the operator $D$, we can write:

$$F(z) = e^{zD} e^{iV[j+]} e^{-iV^*[j-]} \bigg|_{j+ = j- = j}$$

- Reminder:
  - $e^{D} e^{iV} e^{-iV^*}$ is the sum of all the cut vacuum diagrams
  - The cuts are produced by the action of $D$

- Therefore, $F(z)$ is the sum of all the cut vacuum diagrams in which each cut line is weighted by a factor $z$
What would it be good for ?

Let us pretend that we know the generating function $F(z)$. We could get the probability distribution as follows:

$$P_n = \frac{1}{2\pi} \int_0^{2\pi} d\theta \ e^{-in\theta} \ F(e^{i\theta})$$

Note: this is trivial to evaluate numerically:

![Graph showing probability distribution $P_n$ versus $n$ for $F_1(z)$ and $F_2(z)$]
F(z) at Leading Order

- We have:  \[ F'(z) = D \left\{ e^{zD} e^{iV} e^{-iV^*} \right\} \]

- By the same arguments as in the case of \( \overline{N} \), we get:

\[ \frac{F'(z)}{F(z)} = \frac{z}{F(z)} + \frac{z}{F(z)} \]

- The major difference is that the cut graphs that must be evaluated have a factor \( z \) attached to each cut line.

- At tree level (LO), we can write \( F'(z)/F(z) \) in terms of solutions of the classical Yang-Mills equations, but these solutions are not retarded anymore, because:

\[ \sim \sim \sim - z \sim \sim \sim \neq \text{retarded propagator} \]
F(z) at Leading Order

- The derivative $F'/F$ has an expression which is formally identical to that of $\overline{N}$,

$$\frac{F'(z)}{F(z)} = \int \frac{d^3\vec{p}}{(2\pi)^3 2E_p} \int_{x,y} e^{i\vec{p} \cdot (x-y)} \Box x \Box y \sum_{\lambda} \epsilon^\mu_\lambda \epsilon^\nu_\lambda A^{(+)}(x) A^{(-)}(y) ,$$

with $A^{(\pm)}(x)$ two solutions of the Yang-Mills equations

- If one decomposes these fields into plane-waves,

$$A^{(\varepsilon)}(x) \equiv \int \frac{d^3\vec{p}}{(2\pi)^3 2E_p} \left\{ f^{(\varepsilon)}(x^0, \vec{p}) e^{-i\vec{p} \cdot x} + f^{(\varepsilon)}(x^0, \vec{p}) e^{i\vec{p} \cdot x} \right\}$$

the boundary conditions are:

$$f^{(+)}(-\infty, \vec{p}) = f^{(-)}(-\infty, \vec{p}) = 0$$

$$f^{(-)}(+\infty, \vec{p}) = z f^{(+)}(+\infty, \vec{p}) , \quad f^{(+)}(+\infty, \vec{p}) = z f^{(-)}(+\infty, \vec{p})$$

- There are boundary conditions both at $x_0 = -\infty$ and $x_0 = +\infty$ \quad not an initial value problem \quad hard...
So far, we have considered only inclusive quantities – i.e. the $P_n$ are defined as probabilities of producing particles anywhere in phase-space.

What about events where a part of the phase-space remains unoccupied? e.g. rapidity gaps.
Main issues

1. How do we calculate the probabilities $P_{excl}^n$ with an excluded region in the phase-space? Can one calculate the total gap probability $P_{gap} = \sum_n P_{excl}^n$?

2. What is the appropriate distribution of sources $W_{Y}^{excl}[\rho]$ to describe a projectile that has not broken up?


4. Are there some factorization results, and for which quantities do they hold?
Exclusive probabilities

- The probabilities $P_n^{\text{excl}}[\Omega]$, for producing $n$ particles – only in the region $\Omega$ – can also be constructed from the vacuum diagrams, as follows:

$$P_n^{\text{excl}}[\Omega] = \frac{1}{n!} \mathcal{D}_\Omega^n e^{iV} e^{-iV^*}$$

where $\mathcal{D}_\Omega$ is an operator that removes two sources and links the corresponding points by a cut (on-shell) line, for which the integration is performed only in the region $\Omega$

- One can define a generating function,

$$F_\Omega(z) \equiv \sum_n P_n^{\text{excl}}[\Omega] z^n,$$

whose derivative is given by the same diagram topologies as the derivative of the generating function for inclusive probabilities
Exclusive probabilities

- Differences with the inclusive case:
  - In the diagrams that contribute to \( \frac{F'_{\Omega}(z)}{F_{\Omega}(z)} \), the cut propagators are restricted to the region \( \Omega \) of the phase-space.
    - at leading order, this only affects the boundary conditions for the classical fields in terms of which one can write \( \frac{F'_{\Omega}(z)}{F_{\Omega}(z)} \)
    - very similar to the inclusive case
  - Contrary to the inclusive case – where we know that \( F'(1) = 1 \) – the integration constant needed to go from \( \frac{F'_{\Omega}(z)}{F_{\Omega}(z)} \) to \( F_{\Omega}(z) \) is non-trivial. This is due to the fact that the sum of all the exclusive probabilities is smaller than unity.
    - \( F_{\Omega}(1) \) is in fact the probability of not having particles in the complement of \( \Omega \) – i.e. the gap probability.
Survival probability

We can write:

$$F_{\Omega}(z) = F_{\Omega}(1) \exp \left\{ \int_{1}^{z} d\tau \frac{F'_{\Omega}(\tau)}{F_{\Omega}(\tau)} \right\}$$

- the prefactor $F_{\Omega}(1)$ will appear in all the exclusive probabilities

This prefactor is nothing but the famous “survival probability” for a rapidity gap

- One can in principle calculate it by the general techniques developed for calculating inclusive probabilities:

$$F_{\Omega}(1) = F_{1-\Omega}^{\text{incl}}(0)$$

- Note: it is incorrect to say that a certain process with a gap can be calculated by multiplying the probability of this process without the gap by the survival probability
Factorization ?

- In order to discuss factorization for exclusive quantities, one must calculate their 1-loop corrections, and study the structure of the divergences... Not done yet.

- Except for the case of Deep Inelastic Scattering, nothing is known regarding factorization for exclusive processes in a high density environment.

- For the overall framework to be consistent, one should have factorization between the gap probability, $F_\Omega(1)$, and the source density studied in Hentschinski, Weigert, Schafer (2005) (and the ordinary $W_\gamma[\rho]$ on the other side).

- The total gap probability is the “most inclusive” among the exclusive quantities one may think of. For what quantities – if any – does factorization work?
Loop corrections
1-loop corrections to $N$
1-loop corrections to $N$

- 1-loop diagrams for $\overline{N}$

- This can be seen as a perturbation of the initial value problem encountered at LO, e.g.:
1-loop corrections to $N$

- 1-loop diagrams for $\overline{N}$

- This can be seen as a perturbation of the initial value problem encountered at LO, e.g.:
The 1-loop correction to $\overline{N}$ can be written as a perturbation of the initial value problem encountered at LO:
The 1-loop correction to $\overline{N}$ can be written as a perturbation of the initial value problem encountered at LO:

$$\delta \overline{N} = \left[ \int_{\vec{u} \in \text{light cone}} \delta \mathcal{A}_{\text{in}}(\vec{u}) \, T_{\vec{u}} \right] \overline{N}_{LO}$$

- $\overline{N}_{LO}$ is a functional of the initial fields $\mathcal{A}_{\text{in}}(\vec{u})$ on the light-cone
- $T_{\vec{u}}$ is the generator of shifts of the initial condition at the point $\vec{u}$ on the light-cone, i.e.: $T_{\vec{u}} \sim \delta / \delta \mathcal{A}_{\text{in}}(\vec{u})$
The 1-loop correction to $\overline{N}$ can be written as a perturbation of the initial value problem encountered at LO:

$$\delta \overline{N} = \left[ \int_{\vec{u} \in \text{light cone}} \delta A_{\text{in}}(\vec{u}) \ T_{\vec{u}} + \int_{\vec{u}, \vec{v} \in \text{light cone}} \frac{1}{2} \Sigma(\vec{u}, \vec{v}) \ T_{\vec{u}} \ T_{\vec{v}} \right] \overline{N}_{LO}$$

- $\overline{N}_{LO}$ is a functional of the initial fields $A_{\text{in}}(\vec{u})$ on the light-cone
- $T_{\vec{u}}$ is the generator of shifts of the initial condition at the point $\vec{u}$ on the light-cone, i.e.: $T_{\vec{u}} \sim \delta / \delta A_{\text{in}}(\vec{u})$
- $\delta A_{\text{in}}(\vec{u})$ and $\Sigma(\vec{u}, \vec{v})$ are in principle calculable analytically
The first two terms involve:

$$\delta A(x) \equiv \frac{g}{2} \int d^4 z \sum_{\epsilon = \pm} \epsilon G_{++}(x, z) G_{\epsilon\epsilon}(z, z)$$

The third term involves $G_{+-}(x, y)$

The propagators $G_{\pm\pm}$ are propagators in the background $A$, in the Schwinger-Keldysh formalism. They obey:

$$\left\{ \begin{array}{l}
G_{+-} = G_R G_R^0 -1 G_A^0 -1 G_A \\
G_{\pm\pm} = \frac{1}{2} [G_R G_R^0 -1 (G_{+-} + G_{-+}) G_A^0 -1 G_A \pm (G_R + G_A)]
\end{array} \right.$$
Sketch of a proof – II

- $G_{++}$ and $G_{--}$ are only needed with equal endpoints
  - they are both equal to

\[
G_{++}(z, z) = G_{--}(z, z) = \frac{1}{2} \left[ G_R G_R^0 G_A^0 G_A^0 -1 (G_{++}^0 + G_{--}^0) G_A^0 G_A^0 -1 \right] (z, z)
\]

- thus, $\delta A$ can be simplified into :

\[
\delta A(x) = \frac{g}{2} \int d^4 z \left[ G_{++}(x, z) - G_{+-}(x, z) \right] G_{++}(z, z)
\]

\[
= \frac{g}{2} \int d^4 z \ G_R(x, z) G_{++}(z, z)
\]

- $G_R G_R^0 G_A^0 -1 G_A^0 -1$ can be written as :

\[
\left[ G_R G_R^0 G_A^0 -1 G_A^0 -1 \right] (x, y) = \int \frac{d^3 \vec{p}}{(2\pi)^3 2E_{\vec{p}}} \ \zeta_{\vec{p}}(x) \zeta_{\vec{p}}^*(y),
\]

with

\[
[\Box x + m^2 + g A(x)] \ \zeta_{\vec{p}}(x) = 0 \quad \text{and} \quad \lim_{x_0 \to -\infty} \zeta_{\vec{p}}(x) = e^{i p \cdot x}
\]
Green’s formulas:

\[ A(x) = \int_{\Omega} d^4 z \ G_R^0(x, z) \left[ j(z) - \frac{g}{2} A^2(z) \right] \]

\[ + \int_{\text{LC}} d^3 \vec{u} \ G_R^0(x, u) \left[ n \cdot \vec{\partial}_u - n \cdot \vec{\partial}_u \right] A_{\text{in}}(\vec{u}) \]

\[ \delta A(x) = \int_{\Omega} d^4 z \ G_R(x, z) \frac{g}{2} G_{++}(z, z) \]

\[ + \int_{\text{LC}} d^3 \vec{u} \ G_R(x, u) \left[ n \cdot \vec{\partial}_u - n \cdot \vec{\partial}_u \right] \delta A_{\text{in}}(\vec{u}) \]

\[ \zeta_{\vec{p}}(x) = \int_{\text{LC}} d^3 \vec{u} \ G_R(x, u) \left[ n \cdot \vec{\partial}_u - n \cdot \vec{\partial}_u \right] \zeta_{\vec{p}}_{\text{in}}(\vec{u}) \]

\[ G_R(x, y) = G^0_R(x, y) + g \int_{\Omega} d^4 z \ G_R^0(x, z) A(z) G_R(z, y) \]
Sketch of a proof – IV

Thanks to the operator

\[ a_{\text{in}}(\vec{u}) \cdot T \vec{u} \equiv a_{\text{in}}(\vec{u}) \frac{\delta}{\delta A_{\text{in}}(\vec{u})} + \left( n \cdot \partial_u \right) a_{\text{in}}(\vec{u}) \frac{\delta}{\delta (n \cdot \partial_u) A_{\text{in}}(\vec{u})} \]

we can write

\[ \zeta_{\vec{p}}(x) = \int_{\vec{u} \in \text{LC}} \left[ \zeta_{\vec{p} \text{in}}(\vec{u}) \cdot T \vec{u} \right] A(x) \]

\[ \delta A(x) = \int_{\Omega} d^4 z \ G_R(x, z) \frac{g}{2} G_{++}(\vec{z}, \vec{z}) + \int_{\vec{u} \in \text{LC}} \left[ \delta A_{\text{in}}(\vec{u}) \cdot T \vec{u} \right] A(x) \]

\[ \therefore \text{from the classical field } A(x), \text{ the operator } a_{\text{in}}(\vec{u}) \cdot T \vec{u} \text{ builds the fluctuation } a(x) \text{ whose initial condition on the light-cone is } a_{\text{in}}(\vec{u}) \]

The 3rd diagram can directly be written as:

\[ \int \frac{d^3 \vec{p}}{(2\pi)^3 2E_p} \int_{\vec{u}, \vec{v} \in \text{LC}} \left[ [\zeta_{\vec{p} \text{in}}(\vec{u}) \cdot T \vec{u}] A(x) \right] \left[ [\zeta_{\vec{p} \text{in}}^*(\vec{v}) \cdot T \vec{v}] A(y) \right] \]
One can finally prove that

$$\int_\Omega d^4 z \, G_R(x, z) \frac{g}{2} G_{++}(z, z) =$$

$$= \frac{1}{2} \int \frac{d^3 \vec{p}}{(2\pi)^3 2E_p} \int_{\vec{u}, \vec{v} \in \text{LC}} \left[ \zeta_{\vec{p} \text{ in}}(\vec{u}) \cdot T \vec{u} \right] \left[ \zeta^*_{\vec{p} \text{ in}}(\vec{v}) \cdot T \vec{v} \right] A(x)$$

$$\delta A(x) = \left[ \int_{\vec{u} \in \text{LC}} \left[ \delta A_{\text{in}}(\vec{u}) \cdot T \vec{u} \right] \right]$$

$$+ \frac{1}{2} \int \frac{d^3 \vec{p}}{(2\pi)^3 2E_p} \int_{\vec{u}, \vec{v} \in \text{LC}} \left[ \zeta_{\vec{p} \text{ in}}(\vec{u}) \cdot T \vec{u} \right] \left[ \zeta^*_{\vec{p} \text{ in}}(\vec{v}) \cdot T \vec{v} \right] A(x)$$

This leads to the announced formula for $\delta N$, with

$$\Sigma(\vec{u}, \vec{v}) \equiv \int \frac{d^3 \vec{p}}{(2\pi)^3 2E_p} \zeta_{\vec{p} \text{ in}}(\vec{u}) \zeta^*_{\vec{p} \text{ in}}(\vec{v})$$
[Conjecture: this result can be generalized to any observable that can be written in terms of the gauge field with retarded boundary conditions, \( \mathcal{O} \equiv \mathcal{O}[A] \):

\[
\delta \mathcal{O} = \left[ \int_{\vec{u} \in \text{light cone}} \delta A_{\text{in}}(\vec{u}) \ T_{\vec{u}} + \int_{\vec{u}, \vec{v} \in \text{light cone}} \frac{1}{2} \Sigma(\vec{u}, \vec{v}) \ T_{\vec{u}} T_{\vec{v}} \right] \mathcal{O}_{LO}
\]

▷ whatever we conclude for the multiplicity from this formula holds true for any such observable.
Divergences

- If taken at face value, this 1-loop correction is plagued by several divergences:
  - The two coefficients $\delta A_{\text{in}}(\vec{x})$ and $\Sigma(\vec{x}, \vec{y})$ are infinite, because of an unbounded integration over a rapidity variable.
  - At late times, $T^{\vec{x}}A(\tau, \vec{y})$ diverges exponentially,
    $$T^{\vec{x}}A(\tau, \vec{y}) \sim \frac{\delta A(\tau, \vec{y})}{\delta A_{\text{in}}(\vec{x})} \sim e^{\sqrt{\mu\tau}}$$
    because of an instability of the classical solution of Yang-Mills equations under rapidity dependent perturbations (Romatschke, Venugopalan (2005))
Initial state factorization

Anatomy of the full calculation:

\[
\begin{align*}
&\left\{ W_{Y_{\text{beam}}-Y}[\rho_1] \right. \\
&\left. N[ A_{\text{in}}(\rho_1, \rho_2) ] \right. \\
&\left. W_{Y_{\text{beam}}+Y}[\rho_2] \right. 
\end{align*}
\]
Initial state factorization

Anatomy of the full calculation:

\[ \begin{align*}
W_{Y_{\text{beam}}^+}[\rho_1] \\
N[ A_{\text{in}}(\rho_1, \rho_2) ] + \delta N \\
W_{Y_{\text{beam}}^-}[\rho_2]
\end{align*} \]

When the observable \( \overline{N}[A_{\text{in}}(\rho_1, \rho_2)] \) is corrected by an extra gluon, one gets **divergences** of the form \( \alpha_s \int dY \delta \overline{N} \) in \( \delta N \), one would like to be able to absorb these divergences into the \( Y \) dependence of the source densities \( W_Y[\rho_{1,2}] \).
Initial state factorization

- Anatomy of the full calculation:

\[
\begin{align*}
W_{Y_{\text{beam}} - Y_0} \rho_1] + \delta N \\
W_{Y_{\text{beam}} + Y'_{0}} \rho_2]
\end{align*}
\]

- When the observable \( \overline{N}[A_{\text{in}}(\rho_1, \rho_2)] \) is corrected by an extra gluon, one gets divergences of the form \( \alpha_s \int dY \in \delta \overline{N} \) one would like to be able to absorb these divergences into the \( Y \) dependence of the source densities \( W_Y [\rho_{1,2}] \)

- Equivalently, if one puts some arbitrary frontier \( Y_0 \) between the “observable” and the “source distributions”, the dependence on \( Y_0 \) should cancel between the various factors
Initial state factorization

The two kind of divergences don’t mix, because the divergent part of the coefficients is boost invariant. Given their structure, the divergent coefficients seem related to the evolution of the sources in the initial state.

In order to prove the factorization of these divergences in the initial state distributions of sources, one needs to establish:

$$\left[ \delta N \right]_{\text{divergent coefficients}} = \left[ (Y_0 - Y) H^\dagger[\rho_1] + (Y - Y_0') H^\dagger[\rho_2] \right] N_{LO}$$

where $H[\rho]$ is the Hamiltonian that governs the rapidity dependence of the source distribution $W_Y[\rho]$:

$$\frac{\partial W_Y[\rho]}{\partial Y} = H[\rho] W_Y[\rho]$$

FG, Lappi, Venugopalan (work in progress)
Initial state factorization

- Why is it plausible?
  - Reminder:
    \[
    \left[ \delta \bar{N} \right]_{\text{divergent coefficients}} = \left\{ \int_{\bar{x}} \left[ \delta A_{\text{in}}(\bar{x}) \right] \text{div} T_{\bar{x}} + \frac{1}{2} \int_{\bar{x}, \bar{y}} \left[ \Sigma(\bar{x}, \bar{y}) \right] \text{div} T_{\bar{x}}T_{\bar{y}} \right\} N_{LO}
    \]
  - Compare with the evolution Hamiltonian:
    \[
    \mathcal{H}[\rho] = \int_{\bar{x}_\perp} \sigma(\bar{x}_\perp) \frac{\delta}{\delta \rho(\bar{x}_\perp)} + \frac{1}{2} \int_{\bar{x}_\perp, \bar{y}_\perp} \chi(\bar{x}_\perp, \bar{y}_\perp) \frac{\delta^2}{\delta \rho(\bar{x}_\perp) \delta \rho(\bar{y}_\perp)}
    \]
  - The coefficients \( \sigma \) and \( \chi \) in the Hamiltonian are well known. There is a well defined calculation that will tell us if it works...
Unstable modes

Romatschke, Venugopalan (2005)

- Rapidity dependent perturbations to the classical fields grow like $\exp\left(\#\sqrt{\tau}\right)$ until the non-linearities become important:

$$
\max \frac{\tau^2 T^m}{g^4 \mu L^2} = c_0 + c_1 \exp(0.427 \sqrt{g^2 \mu \tau})
$$

$$
\max \frac{\tau^2 T^m}{g^4 \mu L^2} = c_0 + c_1 \exp(0.00544 g^2 \mu \tau)
$$
Unstable modes

- The coefficient $\delta A_{\text{in}}(\vec{x})$ is boost invariant.

  Hence, the divergences due to the unstable modes all come from the quadratic term in $\delta \mathcal{N}$:

  $$
  \left[ \delta \mathcal{N} \right]_{\text{unstable modes}} = \left\{ \frac{1}{2} \int \Sigma(\vec{x}, \vec{y}) \, T_{\vec{x}} T_{\vec{y}} \right\} \mathcal{N}_{\text{LO}} \left[ A_{\text{in}}(\rho_1, \rho_2) \right]
  $$

- When summed to all orders, this becomes a certain functional $Z[T_{\vec{x}}]$:

  $$
  \left[ \delta \mathcal{N} \right]_{\text{unstable modes}} = Z[T_{\vec{x}}] \, \mathcal{N}_{\text{LO}} \left[ A_{\text{in}}(\rho_1, \rho_2) \right]
  $$
Unstable modes

This can be arranged in a more intuitive way:

\[
\left[ \delta \bar{N} \right]_{\text{unstable modes}} = \int [Da] \ \tilde{Z}[a(\vec{x})] \ e^{i \int_{\vec{x}} a(\vec{x}) T_{\vec{x}}} \ \bar{N}_{LO} [A_{\text{in}}(\rho_1, \rho_2)] \\
= \int [Da] \ \tilde{Z}[a(\vec{x})] \ \bar{N}_{LO} [A_{\text{in}}(\rho_1, \rho_2) + a]
\]

- Summing these divergences simply requires to add fluctuations to the initial condition for the classical problem.
- The fact that \(\delta A_{\text{in}}(\vec{x})\) does not contribute implies that the distribution of fluctuations is real.

**Interpretation:**

Despite the fact that the fields are coupled to strong sources, the classical approximation alone is not good enough, because the classical solution has unstable modes that can be triggered by the quantum fluctuations.
Unstable modes

Fukushima, FG, McLerran (2006)

By a different method, one obtains Gaussian fluctuations characterized by:

\[
\langle a_i(\eta, \vec{x}_\perp) \, a_j(\eta', \vec{x}'_\perp) \rangle = \frac{1}{\tau \sqrt{-\left(\frac{\partial \eta}{\tau}\right)^2 - \vec{\partial}_\perp^2}} \left[ \delta_{ij} + \frac{\partial_i \partial_j}{\left(\frac{\partial \eta}{\tau}\right)^2} \right] \delta(\eta - \eta') \, \delta(\vec{x}_\perp - \vec{x}'_\perp)
\]

\[
\langle e^i(\eta, \vec{x}_\perp) \, e^j(\eta', \vec{x}'_\perp) \rangle = \tau \sqrt{-\left(\frac{\partial \eta}{\tau}\right)^2 - \vec{\partial}_\perp^2} \left[ \delta_{ij} - \frac{\partial_i \partial_j}{\left(\frac{\partial \eta}{\tau}\right)^2 + \vec{\partial}_\perp^2} \right] \delta(\eta - \eta') \, \delta(\vec{x}_\perp - \vec{x}'_\perp)
\]
Unstable modes

Classical solution in 2+1 dimensions
Unstable modes
Unstable modes
Combining everything, one should write

\[
\frac{dN}{dY \, d^2 \vec{p}_\perp} = \int [D\rho_1][D\rho_2] \, W_{Y_{\text{beam}}-Y}[\rho_1] \, W_{Y_{\text{beam}}+Y}[\rho_2] \\
\times \int [Da] \, \tilde{Z}[a] \, \frac{dN[A_{\text{in}}(\rho_1, \rho_2)+a]}{dY \, d^2 \vec{p}_\perp}
\]

This formula resums (all?) the divergences that occur at one loop.
Unstable modes – Interpretation

- Tree level:

\[ p \]
Unstable modes – Interpretation

- **Tree level:**

- **One loop** ▶ gluon pairs (includes Schwinger pairs): 

  ▶ The momentum $\vec{q}$ is integrated out
  ▶ If $\alpha_s^{-1} \lesssim |y_p - y_q|$, the correction is absorbed in $W[\rho_{1,2}]$
  ▶ If $|y_p - y_q| \lesssim \alpha_s^{-1}$: gluon splitting in the final state
Unstable modes – Interpretation

- After summing the fluctuations, things may look like this:

> these splittings may help to fight against the expansion?

Note: the timescale for this process is $\tau \sim Q_s^{-1} \ln^2(1/\alpha_s)$
Summary
Summary

- When the parton densities in the projectiles are large, the study of particle production becomes rather involved
  - non-perturbative techniques that resum all-twist contributions are needed

- At Leading Order, the inclusive gluon spectrum can be calculated from the classical solution with retarded boundary conditions on the light-cone

- At Next-to-Leading Order, the gluonic corrections can be seen as a perturbation of the initial value problem encountered at LO

- Resummation of the leading divergences to all orders:
  - Evolution with $Y$ of the distribution of sources
  - Quantum fluctuations on top of initial condition for the classical solution in the forward light-cone
Summary

- **Retarded classical fields**:  
  - Single inclusive gluon spectrum

- **Retarded classical fields + fluctuations**:  
  - Single inclusive quark spectrum  
  - Double inclusive gluon spectrum  
  - Factorization of the gluon spectrum  
  - Instabilities in the gluon spectrum

- **Non retarded classical fields**:  
  - Inclusive generating function $F(z)$  
  - Exclusive reactions, diffraction

- **Non retarded classical fields + fluctuations**:  
  - Factorization for exclusive quantities
Extra bits

- Quark production
- Longitudinal expansion
Quark production


\[
E_p \frac{d\langle n_{\text{quarks}} \rangle}{d^3 \vec{p}} = \frac{1}{16\pi^3} \int_{x,y} e^{i\vec{p} \cdot (x-y)} \phi_x \phi_y \langle \bar{\psi}(x)\psi(y) \rangle
\]

- Dirac equation in the classical color field:
Quark production


\[ E_P \frac{d\langle n_{\text{quarks}} \rangle}{d^3 \vec{p}} = \frac{1}{16\pi^3} \int_{x,y} e^{i\vec{p} \cdot (x-y)} \phi_x \phi_y \langle \bar{\psi}(x)\psi(y) \rangle \]

- **Dirac equation** in the classical color field:
Spectra for various quark masses

- Spectra for various quark masses

\[ \frac{dN}{dyd^2q_T} \text{ [arbitrary units]} \]

- CERN
- François Gelis – 2007
- UFRJ, Rio de Janeiro, April 2007 - p. 81
Longitudinal expansion

- For a system finite in the $\eta$ direction, the gluons will have a longitudinal velocity tied to their space-time rapidity.
For a system finite in the $\eta$ direction, the gluons will have a longitudinal velocity tied to their space-time rapidity.

- Longitudinal expansion

\[\text{Longitudinal expansion}\]
Longitudinal expansion

- For a system finite in the $\eta$ direction, the gluons will have a longitudinal velocity tied to their space-time rapidity

\[ v_z = \tanh(\eta) \]

- at late times: if particles fly freely, only one longitudinal velocity can exist at a given $\eta$.
- the expansion of the system is the main obstacle to local isotropy