BFKL Evolution Equation

G.G. Silveira

gustavo.silveira@ufrgs.br

High Energy Phenomenology Group

Instituto de Física

Universidade Federal do Rio Grande do Sul

Porto Alegre, Brazil

GFPAE - UFRGS

http://www.if.ufrgs.br/gfpae
Outline

• Physics at small-x;

• Quark-Quark Scattering in LLA;

• The BFKL Equation;

• Solution to Zero Momentum Transfer;

• Application: $qq \rightarrow qq$;

• BFKL Equation in NLLA;

• BFKL Equation in DIS;

• Application: Truncated BFKL Series;

• Application: LO versus NLO BFKL Equation;

• Conclusions.
Resummation in pQCD

- The experimental data is well described by DGLAP Equation when \( \ln Q^2 \gg \ln \frac{1}{x} \);

- When \( Q^2 \) is **large**, the leading terms need to be **resummed**:
  \[ \rightarrow \text{Resum over the leading terms to subtract the divergences.} \]

  - **LLA Limit**:
    At each perturbative order only the highest power in \( \ln Q^2 \) is retained
    \[
    \sum_n \alpha_s^n \ln^{(n)} Q^2 \left( \ln^{(n)} \frac{1}{x} + \ln^{(n-1)} \frac{1}{x} + \ldots \right) \tag{1}
    \]

  - **NLLA Limit**:
    It is retained subdominant powers in \( \ln Q^2 \)
    \[
    \sum_n \alpha_s^n \ln^{(n-1)} Q^2 \left( \ln^{(n)} \frac{1}{x} + \ln^{(n-1)} \frac{1}{x} + \ldots \right) \tag{2}
    \]
Resummation at small-$x$

- When the **Small-$x$ Limit** is reached, other resummations should be applied:
  - **DLLA Limit:**
    - In the LLA limit we retain only dominant terms in $\ln(1/x)$
    
    $$
    \sum_n \alpha_s^n \ln^n Q^2 \ln^n \frac{1}{x}
    $$
    
    - **DGLAP resums** the terms $\alpha_s^n \ln^n Q^2$ and $\alpha_s^n \ln^n Q^2 \ln^n \frac{1}{x}$.
    - $\Leftarrow$ But it does **not** resum the **leading** terms $\alpha_s^n \ln^n \frac{1}{x}$

- **LL$_x$A Limit:** $x \ll 1$, $Q^2$ not large $\Rightarrow \ln Q^2 \ll \ln \frac{1}{x}$
  
  - Resummation of $\sum_n \alpha_s^n \ln^n \frac{1}{x} (\ln^n Q^2 + \ln^{n-1} Q^2 + \ldots)$
  
  $\Leftarrow$ **In this limit the BFKL Equation operates!**
Structure Function $F_2$

- From the HERA data:
  - Steep rise of $F_2$ at low-$x$: ($F_2 \leftrightarrow \sigma \leftrightarrow g$)
    - Increase of the gluon density!
  - DGLAP Equation $\Rightarrow$ still OK!
    - In the kinematic range of HERA.
- If it is reached much lower values of $x$ . . .
  - Does DGLAP still describe the data?

NLO DGLAP fit for HERA data.

(Marage, hep-ph/9911426, 1999)
Information from HERA

- Parametrize $F_2$ for $x < 0.1$ in the form

$$F_2(x, Q^2) = A(Q^2) x^{-\lambda}$$

- For small $Q^2$ ($\lesssim 1$ GeV$^2$):
  $\lambda \approx 0.1$

- For large $Q^2$ ($\sim 10 - 100$ GeV$^2$):
  $\lambda \approx 0.25 - 0.35$

- For $x \to 0$:
  - In the perturbative regime ($Q^2 \gtrsim 1$ GeV$^2$) $\Rightarrow$ DGLAP Equation;
  - The region which $Q^2$ is small ($< 1$ GeV$^2$) $\Rightarrow$ Regge Theory.

☐ BFKL Equation resums the leading terms $\ell n \frac{1}{x}$ for $Q^2 < 1$ GeV$^2$
Regge Theory

• Low values of \( x \) correspond to large values of \( s \) → **Here the Regge Theory takes place!**

• In the hadronic process, particles are exchanged as
  - **Nuclear Physics:** mesons (\( \rho, \omega, \ldots \));
  - **High-Energy Phenomenology:** 'trajectories' or Reggeons \( R \).

• What says Regge Theory to us? What means 'trajectories'?
  - Extending the angular momentum to complex values one found **singularities**;
  - These singularities give rise to resonances that can be exchanged in the \( t \)-channel;
  - When a family of resonances is exchanged it is called **Regge trajectory exchange**;

• A Regge trajectory exchanged is said a exchange of a
  \[
  \begin{align*}
  &\text{Reggeized particle} \\
  &\text{Reggeon} \ R
  \end{align*}
  \]
Reggeized Gluon

- **Reggeized Particle:**
  - The amplitude for the exchange of a particle in the $t$-channel is written as $A \sim s^{\alpha(t)}$;  
  - The exponent $\alpha(t)$ is related to the particle trajectory;

- **BFKL Equation** in LO $\rightarrow$ resummation of $\sum_n \alpha_s^n \ln(n) \frac{s}{t}$ with $s \gg Q^2, t$;
  - In this order, the leading process is the exchange of gluons;
  - It will be studied the reggeized gluons exchange in all orders of perturbation theory;
The Pomeron

• In Regge Theory this exchange is the **Pomeron Exchange**: $A_P \sim s^{\alpha_P(t)}$

• Experimentally the cross section has the form

$$\sigma \sim s^\lambda \quad \rightarrow \quad \lambda \sim 0.08 - 0.10$$  \hspace{1cm} (5)

• The Regge Theory predicts a cross section of the form

$$\sigma \sim s^{\alpha_P(0)-1} \quad \rightarrow \quad \alpha_P(0) \simeq 1$$  \hspace{1cm} (6)

• **Interesting feature**: Pomeron has the vacuum quantum numbers:

$$P = +1, \quad C = +1, \quad I = 0$$  \hspace{1cm} (7)

and the Pomeron is the dominant trajectory in the **elastic** and **diffractive** processes!
The Pomeron in QCD

- To incorporate the Pomeron in QCD → consider an exchange of the vacuum quantum numbers!

- Using the QCD degrees of freedom (quarks and gluons): two-gluon exchange!

- In high-energy processes ($x \ll 1$) the Pomeron contribution is essential;
- In this sense, the DGLAP Equation does not take into account the Pomeron contribution!
- It is needed to sum the contributions of the leading terms in $\ell n s$!
One-Gluon Exchange

- **First contribution to the Pomeron:** 2-gluons exchange!
  
  \( \rightarrow \) We start calculating the one-gluon exchange amplitude and then work to higher orders!

- Quark-quark scattering in the Regge limit \((s \gg -t)\)

\[
\begin{aligned}
\mathbf{p}_1, j & \quad \rightarrow \quad \mathbf{p}_1', i \\
\mathbf{p}_2, l & \quad \rightarrow \quad \mathbf{p}_2', k
\end{aligned}
\]

- Computing the amplitude of the process with the Feynman Rules in the Feynman gauge:

\[
A_{ijlm}^{(0)} = \bar{u}(p_1 - q) \left(-ig_s \gamma^\mu t^{a}_{ij}\right) u(p_1) \left(-i\delta_{ab}g_{\mu\nu} \frac{1}{q^2}\right) \bar{u}(p_2 + q) \left(-ig_s \gamma^\nu t^{b}_{lm}\right) u(p_2)
\]  

\( (8) \)
Kinematic Regime

- In the center-of-mass reference frame one takes $p_1$ and $p_2$ along the $z$ axis

\[ p_1 = \frac{\sqrt{s}}{2} (1, 0, 1), \quad p_2 = \frac{\sqrt{s}}{2} (1, 0, -1) \tag{9} \]

- Using the Sudakov Parametrization:

\[ q^\mu = \alpha p_1^\mu + \beta p_2^\mu + q^\mu = \left( \frac{\sqrt{s}}{2} [\alpha + \beta], \quad q, \quad \frac{\sqrt{s}}{2} [\alpha - \beta] \right) \tag{10} \]

where the constants $\alpha$ and $\beta$ are the momentum fraction of the quarks carried by the gluon and

\[ p_1^2 = p_2^2 = 0 \quad 2 (p_1 \cdot p_2) = s \]

- The momentum transfer squared has the form

\[ t = q^2 = 2\alpha\beta (p_1 \cdot p_2) - q^2 = \alpha\beta s - q^2 \]
Final State Condition

- Taking the mass-shell conditions for the outgoing quarks

\[
(p_1 - q)^2 = -\beta s + \alpha \beta s - q^2 = t - \beta s = 0 \quad \left\{ \begin{array}{l}
\beta = t/s \\
(p_2 + q)^2 = \alpha s + \alpha \beta s - q^2 = t + \alpha s = 0 \\
\alpha = -t/s
\end{array} \right.
\]

so

\[q^\mu = -\frac{t}{s} (p_1^\mu - p_2^\mu) + q^\mu \simeq q^\mu\]  \hspace{1cm} (12)

- The momentum transfer squared now is

\[t \equiv q^2 \sim -q^2\]  \hspace{1cm} (13)

- In the large-\(s\) limit one can state that:
  - All components of the exchanged momentum \(q\) are much smaller than \(p_1\) and \(p_2\)!
Scattering Amplitude

- Writing the scattering amplitude

\[ iA_{ijlm}^{(0)}(s, t) = ig_s^2 \left(t^a_{ij} t^a_{lm}\right) \bar{u}(p_1 - q) \gamma^\mu u(p_1) \left(\frac{1}{q^2}\right) \bar{u}(p_2' + q) \gamma_\mu u(p_2) \]  

(14)

- The amplitude squared, averaged and summed over colors is

\[ |A^{(0)}|^2 = 2g_s^4 \left(\frac{N_c^2 - 1}{4N_c^2}\right) \left(\frac{s^2 + u^2}{t^2}\right) \xrightarrow{s \to \infty} \left(\frac{8}{9}\right) g_s^4 \left(\frac{s^2}{t^2}\right) \]  

(15)

where the color factor for \( N_c = 3 \) is

\[ \frac{1}{N_c^2} (t^a_{ij} t^a_{lm}) (t^b_{ij} t^b_{lm})^* = \frac{1}{N_c^2} t^a_{ij} t^a_{lm} t^b_{ji} t^b_{ml} = \frac{1}{N_c^2} \text{Tr}(t^a t^b) \text{Tr}(t^a t^b) = \frac{N_c^2 - 1}{4N_c^2} = \frac{2}{9}. \]
Eikonal Approximation

- The general form of the $qqg$ vertex is

$$V^\mu = -ig_s \bar{u}(p_1 + q) \gamma^\mu u(p_1)$$

- Due to the smallness of $q$ one can approximate

$$V^\mu \simeq -ig_s \bar{u}(p_1) \gamma^\mu u(p_1) = -2ig_sp_1^\mu$$

which is the called *quark-gluon eikonal vertex* that represents a *soft* particle exchange!

- From this one rewrites the amplitude as

$$A_{ijlm}^{(0)} = 2g_s^2 \left( t_i^a t_l^a \right) \left( \frac{1}{q^2} \right) (2p_1 \cdot p_2) = 8\pi\alpha_s \left( t_i^a t_l^a \right) \left( \frac{s}{t} \right)$$

- This approximation does not change the squared amplitude, having the same form as before

$$|A^{(0)}|^2 = \frac{8g_s^4}{9} \left( \frac{s^2}{t^2} \right)$$
Two-Gluon Exchange

- Corrections of the order $\mathcal{O}(\alpha_s^2)$: ONE-LOOP DIAGRAM

\[ \text{Im} A^{(1)}(s, t) = \frac{1}{2} \int d\Pi_2 A^{(0)}(s, k^2) A^{(0)}(s, [k - q]^2) \]

which amplitudes are the one-gluon exchange amplitudes computed previously.

- It will be computed the scattering amplitude using the Cutkosky Rules: $t \equiv q^2 \simeq -q^2$
Subleading Diagrams

• It has been taken the **leading terms** of the type $\ell \ln s$;

• Some diagrams will yield **subleading terms**, like

  Vertex Correction diagrams;  
  
  ![Vertex Correction Diagram](image)

  Self-energy diagrams.  
  
  ![Self-energy Diagram](image)
Phase Space

• One takes the two-body phase space

\[ \int d\Pi_2 = \int \frac{d^4\kappa_1}{(2\pi)^3} \frac{d^4\kappa_2}{(2\pi)^3} \delta(\kappa_1^2) \delta(\kappa_2^2) (2\pi)^4 \delta^{(4)}(p_1 + p_2 - \kappa_1 - \kappa_2) \]

\[ = \int \frac{d^4k}{(2\pi)^2} \delta([p_1 - k]^2) \delta([p_2 + k]^2) \]

• As before, one introduces the Sudakov variables

\[ k = \alpha p_1 + \beta p_2 + k_\perp \]

(20)

\[ d^4k = \left( \frac{s}{2} \right) d\alpha d\beta d^2k \]

(21)

• The Two-body phase space with the Sudakov variables is written as

\[ \int d\Pi_2 = \frac{s}{8\pi^2} \int d\alpha d\beta d^2k \delta(-\beta[1 - \alpha]s + k^2) \delta(\alpha[1 + \beta]s - k^2) \]

(22)
High Energy Limit

- When one works in the large-$s$ limit, the Sudakov variables can be approximate to

\[ \alpha = |\beta| \simeq \frac{k^2}{s} \ll 1 \]  

(23)

\[ k^2 \simeq -k^2, \quad (k - q)^2 \simeq -(k - q)^2 \]  

(24)

\[ k^2 \simeq (k - q)^2 \simeq q^2 \]  

(25)

where one rewrites the two-body phase space like

\[ \int d\Pi_2 = \frac{1}{8\pi^2 s} \int d\alpha \, d\beta \, d^2 k \, \delta \left( \beta + \frac{k^2}{s} \right) \delta \left( \alpha - \frac{k^2}{s} \right) = \frac{1}{8\pi^2 s} \int d^2 k \]

that is

\[ k^\mu = - \left( \frac{k^2}{s} \right) p_1^\mu + \left( \frac{k^2}{s} \right) p_2^\mu + k^\mu \]
Amplitudes from one-gluon exchange:

\[
A^{(0)}(s, k^2) = -8\pi\alpha_s \left( t_m^a t_n^a \right) \frac{s}{k^2},
\]

\[
A^{(0)\dagger}(s, [k - q]^2) = -8\pi\alpha_s \left( t_m^b t_n^b \right)^* \frac{s}{(k - q)^2},
\]

that is

\[
\text{Im} A^{(1)}_a (s, t) = 4\alpha_s^2 s \left( t^a t^b \right)_{ij} \left( t^a t^b \right)_{kl} \int d^2 k \left[ \frac{1}{k^2(k - q)^2} \right].
\]
Dispersion Relations

- In the leading $\ln \frac{1}{x}$ approximation one can express the amplitude as

\[
A = \text{Re} A + i \text{Im} A = C \ln \left( \frac{s}{t} \right) + \ldots = C \ln \left| \frac{s}{t} \right| - i\pi C
\]  

which yields

\[
\text{Re} A = C \ln \left| \frac{s}{t} \right| \quad \text{Im} A = -\pi C
\]  

- The $C$ coefficient expresses the relation between the real and imaginary parts of the amplitude

\[
\text{Re} A = -\frac{1}{\pi} \text{Im} A \ln \left| \frac{s}{t} \right|
\]

which, for the full scattering amplitude, all these can be expressed as

\[
A = -\frac{1}{\pi} \text{Im} A \left( \ln \left| \frac{s}{t} \right| - i\pi \right) = -\frac{1}{\pi} \ln \left( \frac{s}{t} \right) \text{Im} A
\]
Using the dispersion relations one can find the full amplitude for the square diagram

\[ A^{(1)}_{\square}(s, t) = -\frac{4}{\pi} \alpha_s^2 s (t^a t^b)_{ij} (t^a t^b)_{kl} \ln \left( \frac{s}{t} \right) \int d^2 k \left[ \frac{1}{k^2 (k - q)^2} \right] \]

\[ = -16 \left( \frac{\pi \alpha_s}{N_c} \right) (t^a t^b)_{ij} (t^a t^b)_{kl} \ln \left( \frac{s}{t} \right) \epsilon(t) \]

where the dimensionless function \( \epsilon(t) \) incorporates the transverse-momentum integration

\[ \epsilon(t) = \frac{N_c \alpha_s}{4\pi^2} \int d^2 k \left[ \frac{-q^2}{k^2 (k - q)^2} \right] \]  \( \text{(30)} \)

This function is very important to express the Pomeron exchange in perturbative QCD.

It will result from here the trajectory of the pQCD Pomeron!
One can compute this amplitude using the fact that in the Regge Limit

\[ \text{Im} A_{\times}^{(1)} = \text{Im} A_{\Box}^{(1)}(s \to u, t) \]  

(31)

Thus the imaginary part of the amplitude can be expressed as

\[ \text{Im} A_{\times}^{(1)}(s, t) = -16 \left( \frac{\pi \alpha_s}{N_c} \right) (t^a t^b)_{ij} (t^b t^a)_{kl} \left( \frac{u}{t} \right) \ln \left( \frac{u}{t} \right) \epsilon(t) \]  

(32)
Full Amplitude in the High Energy Limit

- In the high energy limit the channels are related through \( s \simeq -u \);

\[
\text{Im} A^{(1)}_x(s, t) = 16 \left( \frac{\pi \alpha_s}{N_c} \right) (t^a t^b)_{ij} (t^b t^a)_{kl} \left( \frac{s}{t} \right) \ln \left( \frac{s}{|t|} \right) \epsilon(t)
\]  

(33)

- One can compute the full amplitude through dispersion relations getting

\[
A^{(1)}_{ijkl}(s, t) = A^{(1)}(s, t) + A^{(1)}_x(s, t) =
\]

\[
= -16 \left( \frac{\pi \alpha_s}{N_c} \right) (t^a t^b)_{ij} \left( \frac{s}{t} \right)
\]

\[
\times \left\{ [t^a, t^b]_{kl} \ln \left( \frac{s}{|t|} \right) - i\pi (t^a t^b)_{kl} \right\} \epsilon(t)
\]

- It is clear that there is a different contribution from the imaginary part;

  - This term is important because it will receive contribution only from the color-singlet term.

    - The color-singlet term is crucial due to its contribution to the Pomeron exchange!
Color Projectors

- The quark-quark scattering amplitude can be decomposed in the SU(3) representation:

\[ A_{ijkl}(s,t) = \sum_R \mathcal{P}_{ijkl}(R)A_R(s,t) \quad (34) \]

- The color-singlet \((\mathbf{1})\) and color-octet \((\mathbf{8})\) amplitudes are expressed as

\[
\begin{align*}
\mathcal{P}_{ijkl}(\mathbf{1}) &= \left( \frac{1}{N_c} \right) \delta_{ij} \delta_{kl} \\
\mathcal{P}_{ijkl}(\mathbf{8}) &= 2 t^a_{ij} t^a_{kl} \\
A_{ijkl}^{(\mathbf{1})}(s,t) &= \mathcal{P}_{ijkl}(\mathbf{1})A_{\mathbf{1}}(s,t) \\
A_{ijkl}^{(\mathbf{8})}(s,t) &= \mathcal{P}_{ijkl}(\mathbf{8})A_{\mathbf{8}}(s,t) \quad (35)
\end{align*}
\]

- For these projectors there is the normalization:

\[
\mathcal{P}_{ijkl}(R)\mathcal{P}^{lkmn}(R') = \mathcal{P}_{ij}^{mn}(R)\delta_{RR'} \quad (37)
\]

- From this one gets

\[
A_{\mathbf{1}}(s,t) = \mathcal{P}_{kl}^{ij}(\mathbf{1})A_{kl}^{ij}(s,t) \quad A_{\mathbf{8}}(s,t) = \left( \frac{1}{N_c^2 - 1} \right) \mathcal{P}_{lk}^{ij}(\mathbf{8})A_{kl}^{ij}(s,t) \quad (37)
\]
Color-Octet Exchange

- Applying the color-octet projector one can extract the amplitude

\[
A_{8}^{(1)}(s, t) = -16 \left( \frac{\pi \alpha_s}{N_c} \right) C_{8}^{(1)} \left( \frac{s}{t} \right) \ln \left( \frac{s}{|t|} \right) \epsilon(t) \tag{38}
\]

where

\[
C_{8}^{(1)} = \left( \frac{1}{N_c^2 - 1} \right) \mathcal{P}_{kl}(g)(t^a t^b)_{ij} [t^a, t^b]_{kl} = -\frac{N_c}{4} \tag{39}
\]

- From the decomposition one can obtain the quark-quark amplitude via color-octet exchange

\[
A_{8}^{(1)}(s, t) = 8\pi \alpha_s (t^a_{ij} t^a_{kl}) \left( \frac{s}{t} \right) \ln \left( \frac{s}{|t|} \right) \epsilon(t) \tag{40}
\]

- Note that color-octet amplitude is real and \( \mathcal{O}(\ln s) \) at one-loop level.
Color-Singlet Exchange

• Proceeding in the same way one can extract the amplitude in the color-singlet case

\[
A_1^{(1)}(s, t) = 16 \left( i \frac{\pi^2 \alpha_s}{N_c} \right) C_1^{(1)} \left( \frac{s}{t} \right) \epsilon(t)
\]

(41)

where

\[
C_1^{(1)} = P_{ij}^{jk}(1)(t^a t^b)_{ij}(t^a t^b)_{kl} = \frac{N_c^2 - 1}{4 N_c}
\]

• As before one can obtain the quark-quark amplitude via color-singlet exchange

\[
A_1^{(1)}(s, t) = 4i\pi^2 \alpha_s (\delta_{ij} \delta_{kl}) \left( \frac{N_c^2 - 1}{4 N_c} \right) \frac{s}{t} \epsilon(t)
\]

(42)

• One can see that the contribution \( \ln \left( \frac{s}{|t|} \right) \) from the two diagrams cancel each other;

• This amplitude starts at order \( \mathcal{O}(\alpha_s^2) \) and is suppressed by a factor \( \ln s \) with respect to the color-octet case.

○ Color-singlet and color-octet amplitudes have opposite signatures

\[
\begin{aligned}
\xi_1 &= +1 \\
\xi_2 &= -1
\end{aligned}
\]
Two-Loop Diagrams

+ ...
In the same way one can introduce the Sudakov parametrization:

\[ k_1 = \alpha_1 p_1 + \beta_1 p_2 + k_{1\perp} \]

\[ k_2 = \alpha_2 p_1 + \beta_2 p_2 + k_{2\perp} \]
Kinematic Regime

- The leading $\ln s$ contribution comes from the Kinematic regime of strong ordering of the longitudinal momenta

\[
1 \gg \alpha_1 \gg \alpha_2 \quad (43)
\]
\[
1 \gg |\beta_2| \gg |\beta_1| \quad (44)
\]

- Taking the gluons on mass-shell

\[
(k_1 - k_2)^2 = k_1^2 + k_2^2 - 2(k_1 \cdot k_2) = 0
\]
\[
= -k_1^2 - k_2^2 - \alpha_1 \beta_2 s - \alpha_2 \beta_1 s + k_1 \cdot k_2 = 0
\]
\[
\approx -(k_1 - k_2)^2 - \alpha_1 \beta_2 s = 0
\]

which results in a non-ordering in the transverse momenta

\[
\alpha_1 \beta_2 s = -(k_1 - k_2)^2 \quad (45)
\]
\[
k_1^2 \approx k_2^2 \approx q^2 \quad (46)
\]
Central Emission

- Computing the scattering amplitude one can find

\[
iA_{2\to3,a}^\rho = (-2ig_s p_1^\mu) t_{mj}^a \left( - \frac{i}{k_1^2} \right) \]

\[\times g_s f_{abc} [(k_1 + k_2)^\rho g^{\mu\nu} + (k_1 - 2k_2)^\mu g^{\nu\rho} + (k_2 - 2k_1)^\nu g^{\rho\mu}] \]

\[\times \left( - \frac{i}{k_2^2} \right) (-2ig_s p_2^\nu) t_{nl}^b \]

- Taking into account the kinematics expressed before one can obtain the amplitude

\[
A_{2\to3,a}^\rho = -2i g_s^3 f_{abc} (t_{mj}^a t_{nl}^b) \left( \frac{1}{k_1^2 k_2^2} \right) [\alpha_1 p_1^\rho + \beta_2 p_2^\rho - (k_1^\rho + k_2^\rho)]
\]
Gluon Emission from Upper Quarks

- In the same way one can write the amplitude for the first diagram of gluon emission

\[ iA_{2\rightarrow 3,b}^\rho = (-2ig_sp_1^{\rho}) t^c_{j'j} \left[ \frac{i}{(p_1 - k_1 + k_2)^2} \right] \]

\[ \times (2ig_s)(p_1^\mu - k_1^\mu + k_2^\mu) t^b_{m,j'} \]

\[ \times \left( -\frac{i}{k_2^2} \right) (-2ig_sp_2^\mu) t^b_{n,l} \]

- Using the information from the Kinematic regime, the amplitude takes the form

\[ A_{2\rightarrow 3,b}^\rho = -4g_s(t^b t^c)_{m,j} t^b_{n,l} \left[ \frac{1}{\beta_2 s k_2^2} \right] p_1^{\rho} \]
Gluon Emission from Upper Quarks

- Again taking the amplitude but for the second diagram one has

\[ iA_{2 \rightarrow 3,c}^\rho = (-2ig_sp_1\mu)t_{j',j}\rho (\frac{i}{k_2^2}) (-2ig_sp_2\mu)t_{n'l} \]

\[ \times \left[ \frac{i}{(p_1 - k_2)^2} \right] (-2ig_s)(p_1^\rho - k_2^\rho)t_{m,j'} \]

- From the Kinematic regime one can see that

\[ A_{2 \rightarrow 3,c}^\rho = 4g_s^3s f_{abc}t_{m,j}^a t_{n'l}^b \left( \frac{1}{\beta_2 s k_2^2 p_1^\rho} \right) \]  \hspace{1cm} (47)

- Finally, for the full scattering amplitude one can obtain, using \([t^b, t^c] = if_{abc}t^a\)

\[ A_{2 \rightarrow 3,b+c}^\rho = -4i g_s^3 s f_{abc} (t_{m,j}^a t_{n'l}^b) \left( \frac{1}{\beta_2 s k_2^2} \right) p_1^\rho \]  \hspace{1cm} (48)
Gluon Emission from Lower Quarks

Following the same procedure one finds the amplitude of gluon emission from the lower quarks

\[ A_\rho^{2\to3,d+e} = -4i g_s^3 f_{abc} (t^a_{mj} t^b_{nl}) \left( \frac{1}{\alpha_1 s k_1^2} \right) p_2^\rho \]  

(49)
Lipatov Vertex

- Summing the amplitudes obtained before one can find the full amplitude in $O(g_s^3)$

$$A_{2 \rightarrow 3}^\rho = -4i g_s^3 \left( \frac{p_1^\mu p_2^\nu}{k_1^2 k_2^2} \right) (t^a_{mj} t^b_{nl}) f_{abc} \Gamma_{\mu\nu}$$

(50)

where the quantity $\Gamma_{\mu\nu}$ is called the **Lipatov effective vertex** and has the form

$$\Gamma_{\mu\nu}^\rho(k_1, k_2) = \frac{2p_2^\mu p_1^\nu}{s} \left[ \left( \alpha_1 + \frac{2k_1^2}{\beta_2 s} \right) p_1^\rho + \left( \beta_2 + \frac{2k_2^2}{\alpha_1 s} \right) p_2^\rho - (k_1^\rho + k_2^\rho) \right]$$

(51)

- Physically this effective vertex incorporates the **propagators** of the emitted gluons.

- This vertex has the important property of being **gauge-invariant**, that is

$$(k_1^\rho - k_2^\rho) \Gamma_{\mu\nu}^\rho(k_1, k_2) = 0$$
All graphs with one gluon in the final state are summed up by the effective diagram

\[ iA_{2\rightarrow 3}^\rho = (-2i g_s p_1^{\mu}) t^a_{m j} \left( -\frac{i}{k_1^2} \right) f_{abc} g_s \Gamma_{\mu\nu}^\rho (k_1, k_2) \left( -\frac{i}{k_2^2} \right) (-2i g_s p_2^\nu) t^b_{n l} \]  

which, obviously, coincides with the amplitude obtained before.
It’s interesting to introduce the quantity below for convenience

\[
C^\rho(k_1, k_2) = \left( \alpha_1 + \frac{2k_1^2}{\beta_2 s} \right) p_1^\rho + \left( \beta_2 + \frac{2k_2^2}{\alpha_1 s} \right) p_2^\rho - (k_1^\rho + k_2^\rho) \tag{53}
\]

so that

\[
\Gamma^\rho_{\mu\nu} = \left( \frac{2}{s} \right) p_{2\mu} p_{1\nu} C^\rho \tag{54}
\]

\[
C^\rho = \left( \frac{2}{s} \right) p_1^\mu p_2^\nu \Gamma^\rho_{\mu\nu} \tag{55}
\]

Through this new quantity the full amplitude is rewritten as

\[
A_{2 \to 3}^\rho = 2i g_s t_m^a \left( \frac{i}{k_1^2} \right) f_{abc} g_s C^\rho(k_1, k_2) \left( \frac{i}{k_2^2} \right) g_s t_n^b
\]
Real Gluon Contribution

\[ \text{Im} A_{\text{real}}^{(2)}(s, t) = -\frac{g_{\rho\sigma}}{2} \int d\Pi_3 A_{2\rightarrow3}^\rho(k_1, k_2) A_{2\rightarrow3}^{\sigma\dagger}(k_1 - q, k_2 - q) \]  \hspace{1cm} (56)

where is needed the sum over gluon helicities: \( \sum_\lambda \epsilon_\lambda^\mu(p) \epsilon_\lambda^{\mu*}(p) = -g^{\mu\nu} \)

Following the procedure applied before one can use the Cutkosky Rules
Phase Space

- A little bit more difficulty and one can compute the three-body phase space

\[
\int d\Pi_3 = \int \frac{d^4 k_1}{(2\pi)^3} \frac{d^4 k_2}{(2\pi)^3} \frac{d^4 k_3}{(2\pi)^3} \delta(k^2_1) \delta(k^2_2) \delta(k^2_3) (2\pi)^4 \delta^4(p_1 + p_2 - k_1 - k_2 - k_3)
\]

\[
= \frac{1}{(2\pi)^5} \int d^4 k_1 d^4 k_2 \delta(k^2_1) \delta(k^2_2) \delta([p_1 + p_2 - k_1 - k_3]^2)
\]

\[
= \frac{1}{(2\pi)^5} \int d^4 k_1 d^4 k_2 \delta([p_1 - k_1]^2) \delta([p_2 + k_2]^2) \delta([k_1 - k_2]^2)
\]

- Again using the Sudakov parametrization one finds for the phase space

\[
\int d\Pi_3 = \frac{s^2}{4(2\pi)^5} \int d\alpha_1 d\beta_1 d^2 k_1 \int d\alpha_2 d\beta_2 d^2 k_2
\]

\[
\times \delta(-\beta_1 [1 - \alpha_1] s - k^2_1) \delta(\alpha_2 [1 + \beta_2] s - k^2_2)
\]

\[
\times \delta([\alpha_1 - \alpha_2] [\beta_1 - \beta_2] s - [k_1 - k_2]^2)
\]
Approximation

- As done previously the Kinematic regime implies that

\[ 1 \gg \alpha_1 \gg \alpha_2 \quad , \quad 1 \gg |\beta_2| \gg |\beta_1| \quad (57) \]

\[ 1 \gg |\beta_2| \gg |\beta_1| \quad (58) \]

\[ k^2_i \approx -k^2_i \]

and finally the phase space is

\[ \int d\Pi_3 = \frac{s^2}{4(2\pi)^5} \int d\alpha_1 \, d\beta_1 \, d^2k_1 \int d\alpha_2 \, d\beta_2 \, d^2k_2 \]

\[ \times \delta(-\beta_1 s - k^2_1) \delta(\alpha_2 s - k^2_2) \delta(-\alpha_1 \beta_2 s - [k_1 - k_2]^2) \]

\[ = \frac{1}{4(2\pi)^5} \int_{\alpha_2}^{1} \frac{d\alpha_1}{\alpha_1} \int_{0}^{1} d\alpha_2 \int d^2k_1 \int d^2k_2 \delta(\alpha_2 - k^2_2) \]

\[ = \frac{1}{4(2\pi)^5 s} \int_{q^2/s}^{1} \frac{d\alpha_1}{\alpha_1} \int d^2k_1 \int d^2k_2 \quad (59) \]
Full Amplitude

- In order to use the Cutkosky rules one needs to compute the amplitude of the right hand side diagram

\[ A_{2\rightarrow 3}^{\rho \dagger} = -2i g_s t_{im}^{a'} \left[ -\frac{i}{(k_1 - q)^2} \right] (-f_{a'bc'} g_s) C_{\rho}(-[k_1 - q],-[k_2 - q]) \left( -\frac{i}{k_2^2} \right) g_s t_{kn}^{b'} \]  \hspace{1cm} (60)

- Making the product of the both sides of the effective diagram one gets

\[ A_{tot} = A_{2\rightarrow 3}(k_1, k_2) A_{2\rightarrow 3,\rho}^{\dagger}(k_1 - q, k_2 - q) = \]

\[ = 4g_s^6 s^2 G_{\text{real}} \left[ \frac{C_{\rho}(k_1, k_2) C_{\rho}(-k_1 + q, -k_2 + q)}{k_1^2 k_2^2 (k_1 - q)^2 (k_1 - q)^2} \right] \]

where the color factor is

\[ G_{\text{real}} = -(t^{a'} t^a)_{ij} (t^{b'} t^b)_{kl} f_{abc} f_{a'b'c} \]  \hspace{1cm} (61)
Imaginary Part for the Real Radiative Correction

• Performing the product of the vectors

\[ C^\rho(k_1, k_2) C^\rho(-k_1 + q, -k_2 + q) = -2 \left[ \mathbf{q}^2 - \frac{k_1^2 (k_2^2 - \mathbf{q}^2)}{(k_1 - k_2)^2} - \frac{k_2^2 (k_1 - 1)^2}{(k_1 - k_2)^2} \right] \]  

(62)

• Join the phase space integral and the total amplitude one obtain the amplitude in \( \mathcal{O}(\alpha_s^2) \)

\[ \text{Im} A_{\text{real}}^{(2)}(s, t) = \left( \frac{2\alpha_s^2}{\pi^2} \right) \mathcal{G}_{\text{real}} s \ln \left( \frac{s}{|t|} \right) \int d^2k_1 \int d^2k_2 \]

\[ \times \left[ \frac{\mathbf{q}^2}{k_1^2 k_2^2 (k_1 - \mathbf{q})^2 (k_2 - \mathbf{q})^2} - \frac{1}{k_2^2 (k_1 - \mathbf{q})^2 (k_1 - k_2)^2} \right. \]

\[ \left. - \frac{1}{k_1^2 (k_2 - \mathbf{q})^2 (k_1 - k_2)^2} \right] \]
Virtual Contribution

Considering gluon exchanges in the $t$-channel one computes this amplitude by

$$\text{Im} A^{(2)}_{\text{virtual}}(s, t) = \frac{1}{2} \int d\Pi_2 A^{(1)}(s, k_2^2) A^{(0)*}(s, [k_2 - q]^2)$$

$$+ \frac{1}{2} \int d\Pi_2 A^{(0)}(s, k_1^2) A^{(1)*}(s, [k_1 - q]^2)$$
For the first case, the tree amplitude for both sides of the square diagram is

\[
A^{(1)}(s, k_2^2) = 8\pi \alpha_s \left( t^b_{m\bar{j}} t^b_{n\bar{l}} \right) \left( \frac{s}{k_2^2} \right) \ln \left( \frac{s}{k_2^2} \right) \epsilon(t)
\]

(63)

and

\[
A^{(0)\dagger}(s, [k_2 - q]^2) = 8\pi \alpha_s \left( t^a_{m\bar{i}} t^a_{n\bar{k}} \right)^* \left[ \frac{s}{(k_2 - q)^2} \right]
\]

(64)

which by using the relations

\[
\ln \left( \frac{s}{k_2^2} \right) \sim \ln \left( \frac{s}{|t|} \right)
\]

one gets

\[
\text{Im} A^{(2)}_{\text{virtual, } \Box}(s, t) = - \left( \frac{N_c \alpha_s^3}{\pi^2} \right) G_{\text{virtual}} s \ln \left( \frac{s}{|t|} \right)
\]

\[
\times \int d^2k_1 \int d^2k_2 \left[ \frac{1}{k_1^2(k_2 - q)^2(k_1 - k_2)^2} \right]
\]
Cross Diagram and Full Amplitude

- The same procedure can be done to obtain the contribution from the crossed diagram

\[ \text{Im} A_{\text{virtual, } \times}^{(2)} (s, t) = - \left( \frac{N_c \alpha_s^3}{\pi^2} \right) G_{\text{virtual}} s \ln \left( \frac{s}{|t|} \right) \times \int d^2 k_1 \int d^2 k_2 \left[ \frac{1}{k_1^2 (k_1 - q)^2 (k_1 - k_2)^2} \right] \]

- Finally, the full contribution from the virtual gluon exchange in the \( t \)-channel is

\[ \text{Im} A_{\text{virtual}}^{(2)} (s, t) = - \left( \frac{N_c \alpha_s^3}{\pi^2} \right) G_{\text{virtual}} s \ln \left( \frac{s}{|t|} \right) \int d^2 k_1 \int d^2 k_2 \left[ \frac{1}{k_1^2 (k_1 - q)^2 (k_1 - k_2)^2} + \frac{1}{k_2^2 (k_2 - q)^2 (k_1 - k_2)^2} \right] \]
Color-Octet Exchange

• For color-octet exchange, one can account the contribution from the $u$-channel by symmetry

  ○ Remembering, in high energy the relation $s \simeq -u$ is valid;

• The $u$-channel contribution can be obtained from that of $s$-channel by the interchange:

\[
\begin{align*}
  t^b & \leftrightarrow t^{b'} \\
  (t^a t^b)_{kl} & \leftrightarrow (t^{b'} t^a)_{kl}
\end{align*}
\]

(65)

• With this, the $u$-channel terms are accounted by the replacements

\[
\begin{align*}
  \mathcal{G}_{\text{real}} & \rightarrow \mathcal{G}'_{\text{real}} = -(t^{a'} t^a)_{ij} [t^{b'} t^b]_{kl} f_{abc} f_{a'b'c} \\
  \mathcal{G}_{\text{virtual}} & \rightarrow \mathcal{G}'_{\text{virtual}} = -(t^a t^b)_{ij} [t^a t^b]_{kl}
\end{align*}
\]

(66)

(67)
Real and Virtual Contributions

- Once made the replacement one accounts for the real-gluon contribution

\[ \text{Im} A_{8,\text{real}}^{(2)}(s, t) = \left( \frac{2\alpha_s^3}{\pi^2} \right) 2 C_{8,\text{real}}^{(2)} (t_{ij} t_{kl}^a) s \ln \left( \frac{s}{|t|} \right) \int d^2 k_1^2 \int d^2 k_2^2 \times \frac{q^2}{k_1^2 k_2^2 (k_1 - q)^2 (k_2 - q)^2} - \frac{1}{k_2^2 (k_1 - q)^2 (k_1 - k_2)^2} - \frac{1}{k_1^2 (k_2 - q)^2 (k_1 - k_2)^2} \]

and for the virtual gluon emission contribution one has

\[ \text{Im} A_{8,\text{virtual}}^{(2)}(s, t) = \left( \frac{N_c \alpha_s^3}{\pi^2} \right) 2 C_{8,\text{virtual}}^{(2)} (t_{ij} t_{kl}^a) s \ln \left( \frac{s}{|t|} \right) \int d^2 k_1 \int d^2 k_2 \times \frac{1}{k_1^2 (k_2 - q)^2 (k_1 - k_2)^2} + \frac{1}{k_2^2 (k_1 - q)^2 (k_1 - k_2)^2} \]

with \( C_{8,\text{real}}^{(2)} = \left( \frac{1}{N_c^2 - 1} \right) \mathcal{P}_{ij}^{lk} (8) \ G'_{\text{real}} = \frac{N_c^2}{8} \) and \( C_{8,\text{virtual}}^{(2)} = \left( \frac{1}{N_c^2 - 1} \right) \mathcal{P}_{ij}^{lk} (8) \ G'_{\text{virtual}} = -\frac{N_c}{4} \).
Full Contribution

- Computing the **full contribution** for Color-Octet exchange

\[
\text{Im} A_8^{(2)}(s, t) = \text{Im} A_8^{(2)}\text{real}(s, t) + \text{Im} A_8^{(2)}\text{virtual}(s, t)
\]

\[
= \left( \frac{N_c^2 \alpha_s^3}{2\pi^3} \right) (t_{ij}^a t_{kl}^a) s \ln \left( \frac{s}{|t|} \right) \int d^2k_1 d^2k_2 \left[ \frac{q^2}{k_1^2 k_2^2 (k_1 - q)^2 (k_2 - q)^2} \right]
\]

which can be rewritten as

\[
\text{Im} A_8^{(2)}(s, t) = 8\pi^2 \alpha_s (t_{ij}^a t_{kl}^a) s \ln \left( \frac{s}{|t|} \right) \epsilon^2(t)
\]

(69)

where \( \epsilon^2(t) = \left( \frac{N_c \alpha_s}{4\pi^2} \right)^2 \int d^2k \left[ \frac{-q^2}{k^2 (k - q)^2} \right] \).

- Via dispersion relations one gets the leading \( \ln s \) contribution

\[
A_8^{(2)}(s, t) = 4\pi \alpha_s (t_{ij}^a t_{kl}^a) s \ln^2 \left( \frac{s}{|t|} \right) \epsilon^2(t) \equiv \left( \frac{1}{2} \right) \epsilon^2(t) \ln^2 \left( \frac{s}{|t|} \right) A_8^{(0)}
\]

(70)

which is **real**.
Amplitude in order $\mathcal{O}(\alpha_s^3)$

- Joining the three contributions one finds the Full Amplitude in the LLA limit

$$A_8(s, t) = 8\pi \alpha_s \left( \frac{s}{t} \right) (t_{ij}^a t_{kl}^a) \left[ 1 + \epsilon(t) \ln \left( \frac{s}{|t|} \right) + \frac{1}{2} \epsilon^2(t) \ln^2 \left( \frac{s}{|t|} \right) + \ldots \right]$$ (71)

which corresponds to the first three terms in the expansion of

$$A_8(s, t) = 8\pi \alpha_s (t_{ij}^a t_{kl}^a) \frac{s}{t} \left( \frac{s}{|t|} \right)^{\epsilon(t)} \equiv 8\pi \alpha_s (t_{ij}^a t_{kl}^a) \left( \frac{s}{|t|} \right)^{\alpha_g(t)}$$ (72)

where the quantity

$$\alpha_g(t) = 1 + \epsilon(t)$$ (73)

is the constructed reggeized gluon trajectory in the $t$-channel. → Not the Pomeron yet!
Contribution of the Color-singlet Exchange

For completeness and following what was done in the color-octet case one accounts the leading $\ln s$ contribution in the color-singlet exchange

\[
A^{(2)}_{1,\text{real}}(s, t) = \left( \frac{2 \alpha_s^3}{\pi^2} \right) C^{(2)}_{1,\text{real}} \delta_{ij} \delta_{kl} \ln \left( \frac{s}{|t|} \right) \int d^2 k_1 \int d^2 k_2 \times \left[ \frac{q^2}{k_1^2 k_2^2 (k_1 - q)^2 (k_2 - q)^2} - \frac{1}{k_2^2 (k_1 - q)^2 (k_1 - k_2)^2} \right]
\]

\[
A^{(2)}_{1,\text{virtual}}(s, t) = -i \left( \frac{N_c \alpha_s^3}{\pi^2} \right) C^{(2)}_{1,\text{virtual}} \delta_{ij} \delta_{kl} \ln \left( \frac{s}{|t|} \right) \int d^2 k_1 \int d^2 k_2 \times \left[ \frac{1}{k_2^2 (k_1 - q)^2 (k_1 - k_2)^2} + \frac{1}{k_1^2 (k_2 - q)^2 (k_1 - k_2)^2} \right]
\]

where $C^{(2)}_{1,\text{real}} = \mathcal{P}^{ij}_{kl}(1)$, $G_{\text{real}} = -\frac{N_c^2 - 1}{4}$ and $C^{(2)}_{1,\text{virtual}} = \mathcal{P}^{ij}_{kl}(1)$, $G_{\text{virtual}} = -\frac{N_c^2 - 1}{4N_c}$.

BFKL Evolution Equation, G.G. Silveira, GFPAE – p. 50
BFKL Ladder

- Previously we introduced the **Eikonal Approximation** which has an important **property**:
  - Independence on the spin of the particle which emits the soft gluon!

- Extending the process for $n$ gluon exchanges in the $s$-channel:
  - Can be constructed a diagram like a **ladder** with $n$ ‘rungs’ or $n$ gluon exchanges;
  - It is considered an exchange of $n$ reggeized gluons in $t$-channel.

- The algebra will be done in the **Multi-Regge Kinematics**
  - It will yield the **leading** $\ell n s$ contributions.

- The procedure will be to account with the mathematical tools calculated previously.
  - Tree amplitudes, Sudakov variables, phase space, . . .
BFKL Ladder

\[ p_1, j \quad \rightarrow \quad p_1', i \]

\[ k_1 \quad a_1, \mu_1 \quad b_1, \rho_1 \]

\[ k_2 \quad a_2, \mu_2 \]

\[ \vdots \]

\[ k_{i-1} \quad a_{i-1}, \mu_{i-1} \quad b_{i-1}, \rho_{i-1} \]

\[ k_i \quad a_i, \mu_i \quad b_i, \rho_i \]

\[ k_{i+1} \quad a_{i+1}, \mu_{i+1} \]

\[ \vdots \]

\[ k_n \quad a_n, \mu_n \quad b_n, \rho_n \]

\[ k_{n+1} \quad a_{n+1}, \mu_{n+1} \]

\[ p_2, l \quad \rightarrow \quad p_2', k \]
’Multi-Regge’ Kinematics

- Like before it can be introduced the Sudakov parametrization

\[ k_i = \alpha_i p_1 + \beta_i p_2 + k_i \perp \quad (i = 1, 2, \ldots, n + 1) \] (74)

- The Multi-Regge regime corresponds to
  - All transverse momenta being of the same order
    \[ s \gg k_1^2 \simeq k_2^2 \simeq \ldots \simeq k_n^2 \simeq k_{n+1}^2 \simeq q^2 \] (75)
  - Strong ordering of the longitudinal momenta
    \[ 1 \gg \alpha_1 \gg \alpha_2 \gg \ldots \gg \alpha_{n+1} \gg \frac{q^2}{s} \]
    \[ 1 \gg |\beta_{n+1}| \gg |\beta_n| \gg \ldots \gg \beta_2 \gg |\beta_1| \gg \frac{q^2}{s} \]

- These two properties will be important in featuring the ladder further on.
Tree Amplitude

- From the gluon exchange one can obtain the amplitude for $n$ gluons emitted

$$i A_{\rho_1 \ldots \rho_n}^{2 \rightarrow n+2} = (-2i g_s) p_1^{\mu_1} t_1^{a_1} \left( -\frac{i}{k_1^2} \right)$$

$$\times g_s f_{a_1 a_2 b_1} \Gamma_{\mu_1 \mu_2}^{\rho_1} (k_1, k_2) \left( -\frac{i}{k_2^2} \right)$$

$$\times g_s f_{a_2 a_3 b_2} \Gamma_{\mu_3}^{\rho_2} (k_2, k_3) \left( -\frac{i}{k_3^2} \right)$$

$$\vdots$$

$$\times g_s f_{a_n a_{n+1} b_n} \Gamma_{\mu_n \mu_{n+1}}^{\rho_n} (k_n, k_{n+1}) \left( -\frac{i}{k_{n+1}^2} \right)$$

$$\times (-2i g_2) p_2^{\mu_{n+1}} t_{k_{n+1}}^{a_{n+1}}$$

- Note that we are treating the process through $n$ effective diagrams attached to each other.
Using the relation between the Lipatov vertices

\[ p_1^{\mu_1} \Gamma_{\mu_1 \mu_2}^{\rho_1} (k_1, k_2) \Gamma_{\mu_3}^{\rho_2} (k_2, k_3) \ldots \Gamma_{\mu_{n+1}}^{\rho_n} (k_n, k_{n+1}) p_2^{\mu_{n+1}} = \]

\[ = \left( \frac{s}{2} \right) C^{\rho_1} (k_1, k_2) C^{\rho_2} (k_2, k_3) \ldots C^{\rho_n} (k_n, k_{n+1}) \]

\[ = \left( \frac{s}{2} \right) \prod_{i=1}^{n} C^{\rho_i} (k_i, k_{i+1}) \]

where it was defined the \( C \) vector as

\[ C^{\rho} (k_i, k_{i+1}) = \left( \alpha_i + \frac{2k_i^2}{\beta_i s} \right) p_1^{\rho} + \left( \beta_{i+1} + \frac{2k_{i+1}^2}{\alpha_i s} \right) p_2^{\rho} - (k_i^{\rho} + k_{i+1}^{\rho}) \]  \hspace{1cm} (76)

and it is related to the Lipatov vertex through the relation

\[ \Gamma_{\mu \nu}^{\rho} = \left( \frac{2}{s} \right) p_{2 \mu} p_{1 \nu} C^{\rho} \]
Scattering Amplitude

- So that, the amplitude can be rewritten as

\[ A_{2 \rightarrow n+2}^{\rho_1 \cdots \rho_n} = 2 i s g_s t_{ij}^{a_1} \left( \frac{i}{k_1^2} \right) \]

\[ \times g_s f_{a_1 a_2 b_1} C^{\rho_1} (k_1, k_2) \left( \frac{i}{k_2^2} \right) \]

\[ \times g_s f_{a_2 a_3 b_2} C^{\rho_2} (k_2, k_3) \left( \frac{i}{k_3^2} \right) \]

\[ \vdots \]

\[ \times g_s f_{a_n a_{n+1} b_n} C^{\rho_n} (k_n, k_{n+1}) \left( \frac{i}{k_{n+1}^2} \right) \]

\[ \times g_s t_{kl}^{a_{n+1}} \]

- However this is only the tree amplitude!

- It does not take into account virtual radiative corrections in the \( t \)-channel.
Radiative Corrections

- It was proposed an ansatz by Lipatov, Kuraev and Fadin
  
  - A modification in the gluon propagator in the $t$-channel to account for these corrections in all orders in $\alpha_s$
  
  - Modification proposed inspired in Regge Theory

\[
- \frac{i}{k_i^2} \rightarrow - \frac{i}{k_i^2} \left( - \frac{s_i}{k_i^2} \right) \epsilon(k_i^2) \simeq - \frac{i}{k_i^2} \left( \frac{\alpha_i - 1}{\alpha_i} \right) \epsilon(k_i^2) \tag{77}
\]

where

\[
s_i = (k_{i-1} - k_{i+1})^2 \simeq \left( \frac{\alpha_i - 1}{\alpha_i} \right) (k_i - k_{i+1})^2
\]

is the center-of-mass energy in the $i$-th section of the ladder, and

\[
\epsilon(k_i^2) = \frac{N_c \alpha_s}{4 \pi^2} \int d^2 h \left[ \frac{-k_i^2}{h^2(h - k_i)^2} \right] \tag{78}
\]

is the dimensionless function already seen before with an auxiliary vector $h$. 
Then, in the Feynman Gauge, the modified gluon propagator is

$$D_{\mu \nu}(s, k^2_i) = -i g_{\mu \nu} \frac{g^2}{k_i^2} \left( \frac{s}{k^2} \right) \epsilon(t)$$  \hspace{1cm} (79)

⇒ Radiative Corrections are directly included in the propagator.

Exemplifying for the elastic $qq$ scattering one obtains

$$A(s, t) = 8\pi \alpha_s \left( t^a_{ij} t^a_{kl} \right) \left( \frac{s}{t} \right) \left( \frac{s}{|t|} \right) \epsilon(t)$$

which coincides with the result obtained in the LLA amplitude expansion.
'Reggeized' BFKL Ladder

- Rewriting the tree amplitude with the modified propagator one gets

\[ A_{2\rightarrow n+2}^{\rho_1 \cdots \rho_n} = 2i s g_s t_{ij}^{a_1} \left( \frac{i}{k_1^2} \right) \left( \frac{1}{\alpha_1} \right) e(k_1^2) \]

\[ \times g_s f_{a_1 a_2 b_1} C_{\rho_1}^{\alpha_1}(k_1, k_2) \left( \frac{i}{k_2^2} \right) \left( \frac{\alpha_1}{\alpha_2} \right) e(k_2^2) \]

\[ \vdots \]

\[ \times g_s f_{a_n a_{n+1} b_n} C_{\rho_n}^{\alpha_n}(k_n, k_{n+1}) \left( \frac{i}{k_{n+1}^2} \right) \left( \frac{\alpha_n}{\alpha_{n+1}} \right) e(k_{n+1}^2) \]

\[ \cdot g_s t_{kl}^{a_{n+1}} \]

\[ = 2i s g_s^2 \left( t_{ij}^{a_1} t_{kl}^{a_{n+1}} \right) \left( \frac{i}{k_1^2} \right) \left( \frac{1}{\alpha_1} \right) e(k_1^2) \]

\[ \times \prod_{i=1}^{n} \left\{ g_s f_{a_i a_{i+1} b_i} C_{\rho_i}^{\alpha_i}(k_i, k_{i+1}) \left( \frac{i}{k_{i+1}^2} \right) \left( \frac{\alpha_i}{\alpha_{i+1}} \right) e(k_{i+1}^2) \right\} \]
 BFKL Evolution Equation, G.G. Silveira, GFPAE – p. 60
Imaginary Ladder Amplitude

- Carrying out the contraction of the $C^i$'s vectors one gets the imaginary part of the scattering amplitude of the Gluon Ladder

\[
\text{Im } \mathcal{A}_R(s, t) &= \frac{1}{2} \sum_{n=0}^{\infty} 4s^2 g_s^4 G_R \int d\Pi_{n+2} \left[ \frac{1}{k_1^2 (k_1 - q)^2} \right] \left( \frac{1}{\alpha_1} \right)^{\epsilon(k_1^2) + \epsilon([k_1 - q]^2)} \\
&\times \prod_{i=1}^{n} \left\{ \left[ \frac{g_s^2}{k_{i+1}^2 (k_{i+1} - q)^2} \right] (-2\eta_R) K(k_i, k_{i+1}) \right\} \times \left( \frac{\alpha_i}{\alpha_{i+1}} \right)^{\epsilon(k_{i+1}^2) + \epsilon([k_i - q]^2)} \right\},
\]

(80)

where

\[
\begin{align*}
G_1 &= \frac{N_c^2 - 1}{4N_c} & G_8 &= -\frac{N_c}{8} \\
\eta_1 &= \frac{N_c}{2} & \eta_8 &= N_c
\end{align*}
\]

(81)
Mellin Transform

- The full Amplitude can be obtained using the **Dispersion Relation** as made before;

- We’ll adopt a **new proposal** that suggests work in the **complex angular momentum plane**!

- Instead of working directly with $A_R$, it will be convenient to calculate its Mellin transform

$$f_R = \int_1^\infty d\left(\frac{s}{|t|}\right) \left(\frac{s}{t}\right)^{-\omega - 1} \frac{\text{Im}A_R(s, t)}{s}$$  \hspace{1cm} (82)

in the **Froissart-Gribov** representation of the partial-wave amplitude.

- The inverse Mellin transform is

$$\frac{\text{Im}A_R(s, t)}{s} = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} d\omega \left(\frac{s}{|t|}\right)^\omega f_R(\omega, t)$$  \hspace{1cm} (83)
Watson-Sommerfeld Transform

- One can take the \( u \)-channel contribution using the property

\[
\text{Im} \mathcal{A}_R(s, t) = -\xi_R \text{Im} \mathcal{A}_R(u, t)
\]

(84)

- The quantities \( \xi_R \) are the signatures defined as

\[
\xi_1 = +1 \quad \xi_8 = -1
\]

(85)

- Since \( u \approx -s \), the \( u \)-channel term is taken into account by the replacement

\[
f_R(\omega, t) \rightarrow (1 + \xi_R e^{-i\pi \omega}) f_R(\omega, t)
\]

(86)

- The partial-wave amplitude \( f_R(\omega, t) \) is related to the amplitude \( \mathcal{A}_R \) by the relation

\[
\mathcal{A}_R(s, t) = -\frac{1}{4\pi i} \int_{c-i\infty}^{c+i\infty} d\omega \left( \frac{s}{|t|} \right)^{\omega+1} \left[ \frac{\xi_R - e^{-i\pi \omega}}{\sin \pi \omega} \right] f_R(\omega, t)
\]

(87)

which is called the \textbf{Watson-Sommerfeld Transform}.
Starting the calculation of the BFKL Equation, one takes the \((n + 2)\)–body phase space

\[
d\Pi_{n+2} = \frac{s^{n+1}}{2^{n+1}(2\pi)^{3n+2}} \int \prod_{i=1}^{n+1} d\alpha_i \, d\beta_i \, d^2k_i
\]

\[
\times \delta(-\beta_1[1 - \alpha_1]s - k_1^2) \delta(\alpha_{n+1}[1 + \beta_{n+1}]s - k_{n+1}^2)
\]

\[
\times \prod_{j=1}^{n} \delta([\alpha_j - \alpha_{j+1}][\beta_j - \beta_{j+1}]s - [k_j - k_{j+1}]^2)
\]

which in the Multi-Regge kinematics is simplified to

\[
d\Pi_{n+2} = \frac{1}{2^{n+1}(2\pi)^{3n+2}} \prod_{i=1}^{n} \int_{\alpha_{i+1}}^{1} \frac{d\alpha_i}{\alpha_i} \int_0^{1} d\alpha_{n+1}
\]

\[
\times \prod_{j=1}^{n+1} \int d^2k_j \, \delta(\alpha_{n+1}s - k^2)
\]
Partial-Wave Amplitude

- Computing the amplitude using the Mellin transform one can find

\[
f_R(\omega, q^2) = (4\pi \alpha_s)^2 g_R \sum_{n=0}^{\infty} \prod_{i=1}^{n+1} \frac{d^2 k_i}{(2\pi)^2} \times \frac{1}{k_1^2(k_1 - q)^2} \frac{1}{\omega - \epsilon(k_1^2) - \epsilon([k_1 - q]^2)} \times (-2\alpha_s \eta_R) K(k_1, k_2) \times \frac{1}{k_2^2(k_2 - q)^2} \frac{1}{\omega - \epsilon(k_2^2) - \epsilon([k_2 - q]^2)} \times (-2\alpha_s \eta_R) K(k_n, k_{n+1}) \times \frac{1}{k_{n+1}^2(k_{n+1} - q)^2} \frac{1}{\omega - \epsilon(k_{n+1}^2) - \epsilon([k_{n+1} - q]^2)}
\]

(90)
The BFKL Equation

- Writing the amplitude in the recursive form

\[
 f_R(\omega, q^2) = (4\pi \alpha_s)^2 g_R \int \frac{d^2 k}{(2\pi)^2} \frac{F_R(\omega, k, q)}{k^2 (k - q)^2}
\]  

(91)

that is

\[
 [\omega - \epsilon(-k^2) - \epsilon(-[k - q]^2)] F_R(\omega, k, q) = 1 - \frac{2\alpha_s N_c}{4\pi^2} \int d^2 h \left[ \frac{K(k, h)}{h^2(x - q)^2} \right] F_R(\omega, h, q)
\]  

(92)

- This is the general form of the BFKL Equation:

This equation describes the evolution of the gluon ladder in the LL\(x\)A limit.
Color-Octet from BFKL Equation

- Using explicitly the expressions for the reggeized gluon trajectories as seen before

\[
\epsilon(-k^2) = -\frac{N_c \alpha_s}{4\pi^2} \int d^2h \left[ \frac{-k^2}{h^2(h-k)^2} \right] 
\]

\[
\epsilon(-[k-q]^2) = -\frac{N_c \alpha_s}{4\pi^2} \int d^2h \left[ \frac{(k-q)^2}{(h-q)^2(k-h)^2} \right] 
\]

(93a)

(93b)

this terms will cancel with those of the expression \( K(k, h) \) related to the virtual corrections (\( \epsilon \)'s terms) and yielding

\[
\omega \mathcal{F}_8(\omega, k, q) = 1 - \frac{N_c \alpha_s}{4\pi^2} \int d^2h \left[ \frac{q^2}{h^2(h-q)^2} \right] \mathcal{F}_8(\omega, h, q) 
\]

(94)

which admits the \( k \)-independent solution

\[
\mathcal{F}_8 = \frac{1}{\omega - \epsilon(-q^2)} 
\]

(95)
Octet Partial-Wave Amplitude

- From the recursive relation we get

\[
 f_8(\omega, q^2) = 2\pi^2 \alpha_s \left[ \frac{\epsilon(-q^2)}{q^2} \right] \frac{1}{\omega - \epsilon(-q^2)}
\]  \hfill (96)

- In terms of the complex angular momentum \( \ell \equiv \omega + 1 \), the octet partial-wave amplitude behaves as

\[
 f_8(\ell, t) \sim \frac{1}{\ell - \alpha_g(t)}
\]  \hfill (97)

where the \( \alpha_g(t) = 1 + \epsilon(t) \).

- We can see that \( f_8(\ell, t) \) has a **pole singularity** as \( \ell = \alpha_g(t) \).

- Computing the inverse Mellin transform one gets the imaginary part of the amplitude

\[
 \text{Im} \mathcal{A}_8(s, t) = 2\pi^2 \alpha_s \epsilon(t) \left( \frac{s}{|t|} \right)^{1+\epsilon(t)}
\]  \hfill (98)
Color-Octet Amplitude

• Taking the total amplitude from dispersion relations and adding the $u$-channel contribution we obtain the full amplitude for the color-octet exchange

$$A_\text{8}(s, t) = -4\pi \alpha_s \left( t_{ij}^a t_{kl}^a \right) \left[ 1 - e^{-i\pi\alpha_g(t)} \right] \left( \frac{s}{|t|} \right)^{\alpha_g(t)}$$

which is the **Regge-type amplitude** for the $qq$ elastic scattering.

• In the Multi-Regge regime one can approximate $\alpha_g(t) \approx 1$

$$A_\text{8}(s, t) \approx -8\pi \alpha_s \left( t_{ij}^a t_{kl}^a \right) \left( \frac{s}{|t|} \right)^{\alpha_g(t)}$$

which coincides with the result obtained from one-loop exchange and justifies the ansatz proposed previously.
Color-Singlet from BFKL Equation

- The Gluon ladder in color-singlet configuration contributes directly to the QCD Pomeron!

- For this configuration we can rewrite the BFKL equation as

\[
[\omega - \epsilon(-k^2) - \epsilon(-[k - q]^2)] F(\omega, k, k', q) = \\
= \delta^2(k - k') - \frac{\alpha_s N_c}{2\pi^2} \int d^2h \left[ \frac{K(k, h)}{h^2(h - q)^2} \right] F(\omega, h, k', q)
\]

- We can introduce the function \( F(\omega, k, k', q) \) related to \( F_1(\omega, k, q) \) by

\[
F_1(\omega, k, q) = \int \frac{d^2k'}{k'^2} k'^2 F(\omega, k, k', q)
\]
The Color-Singlet BFKL Equation

- With some algebra we get

\[
\omega F'(\omega, k, k', q) = \delta^2(k - k') + \frac{\alpha_s N_c}{2\pi^2} \int d^2h \left\{ \left( \frac{-q^2}{(h - q)^2 k^2} \right) F(\omega, h, k', q) \right. \\
+ \frac{1}{(h - k)^2} \left[ F(\omega, h, k', q) - \frac{k^2 F(\omega, h, k', q)}{h^2 + (k - h)^2} \right] \\
+ \left. \frac{1}{(h - k)^2} \left[ \frac{(k - q)^2 h^2 F(\omega, h, k', q)}{(h - q)^2 k^2} - \frac{(k - q)^2 F(\omega, h, k', q)}{(h - q)^2 + (k - h)^2} \right] \right\}
\]

- This is the standard form of the color-singlet BFKL equation.

- From this solution one can find the inverse Mellin transform as

\[
F(s, k, k', q) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} d\omega \left( \frac{s}{|t|} \right)^\omega F(\omega, k, k', q)
\]

\[ (102) \]
Some Properties

- Analyzing the BFKL equation for the color-singlet case we see:
  - **Ultraviolet** finite in the limits $h^2 \to \infty$ and $k^2 \to \infty$;
  - **Infrared** divergences:
    - Regular *infrared* behavior for $h \to 0$ and $k = h$;
      - The singularities that arise from $1/(h - k)^2$ are cancelled by the other terms!
    - Problem in the infrared case:
      - Singularities from the virtual-gluon terms when $k^2 \to 0$;
  - **Answer** to this problem (thanks to Lipatov)
  - A Colorless particle has quarks and gluons confined and it regulates the divergences!
    - The confinement limits the quarks and gluons to be **on-mass shell**!
The Integro-Differential Equation

- We can write the BFKL Equation for zero momentum transfer, so

\[ \omega F(\omega, k, k', q) = \delta^2(k - k') + \int d^2h \mathcal{K}(k, h) F(\omega, h, k') \]  

(103)

where the function \( \mathcal{K} \) is called "BFKL kernel" and has the form

\[ \mathcal{K}(k, h) = \mathcal{K}_{\text{virtual}}(k, h) + \mathcal{K}_{\text{real}}(k, h) \]

\[ = 2\epsilon(-k^2) \delta^2(k - h) + \left( \frac{N_c\alpha_s}{\pi^2} \right) \frac{1}{(k - h)^2} \]  

(104)

- Expressing the BFKL Equation with the inverse Mellin transform we get

\[ \frac{\partial F(s, k, k')}{\partial \ln(s/k^2)} = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} d\omega \left( \frac{s}{k^2} \right) \omega F(\omega, k, k') \]

\[ = \frac{N_c\alpha_s}{\pi^2} \int \frac{d^2h}{(k - h)^2} \left[ F(s, h, k') - \left( \frac{k^2}{h^2 + (k - h)^2} \right) F(s, k, k') \right] \]  

(105)

which describes the evolution of the BFKL amplitude \( F(s, k, k') \).
**Eigenvalues of $\mathcal{K}$**

- In order to solve the BFKL equation for zero momentum transfer, it can be done rewriting

\[
\omega F = 1 + \mathcal{K} \otimes F
\]  

(106)

- Solving this equation, one founds the eigenfunctions $\phi_\alpha$ of $\mathcal{K}$

\[
\mathcal{K} \otimes \phi_\alpha = \omega_\alpha \phi_\alpha.
\]  

(107)

- Some algebra leads to an expression for the eigenfunctions of $\mathcal{K}$

\[
\phi_{n\nu}(|k|, \vartheta) = \frac{1}{\pi \sqrt{2}} (k^2)^{-\frac{1}{2} + i\nu} e^{-n\vartheta}
\]  

(108)

and the eigenvalues can be obtained from this expression as

\[
\omega_n(\nu) = \frac{2 \alpha_s N_c}{\pi} \text{Re} \int_0^1 dx \left[ x^{\frac{|n|+1}{2}} - x^{\frac{-i\nu}{2}} - 1 \right] = -\frac{2 \alpha_s N_c}{\pi} \text{Re} \left[ \psi \left( \frac{|n|+1}{2} + i\nu \right) - \psi(1) \right]
\]  

(109)

where the function $\psi$ is the Digamma function such that $\psi(1) = -\gamma_E = -0.577215\ldots$. 
Solution for $t = 0$

The solution of the BFKL equation for zero momentum transfer reads

$$F(\omega, k, k') = \frac{1}{2\pi^2 (k^2 k'^2)^{1/2}} \sum_{n=0}^{\infty} e^{i n (\vartheta - \vartheta')} \int_{-\infty}^{+\infty} d\nu \left[ \frac{\nu \ell n \left( \frac{k^2}{k'^2} \right)}{\nu - \omega_n(\nu)} \right]$$

The leading $\ell n s$ behavior of $F(s, k, k', q)$ retain only the contribution from $n = 0$

$$\omega_0(\nu) \simeq \lambda - \frac{1}{2} \lambda' \nu^2$$

This result lead us to the LLA pomeron solution of the BFKL Equation

$$F(s, k, k') = \frac{1}{\sqrt{2\pi^3 \lambda' k^2 k'^2}} \left( \frac{1}{\sqrt{\ln(s/k^2)}} \right) \left( \frac{s}{k^2} \right)^\lambda \exp \left[ \frac{\ln^2(k^2/k'^2)}{2\lambda' \ln(s/k^2)} \right]$$
Applications: $qq \rightarrow qq$

- Applying the result to the quark-quark scattering it gives us

$$A_{1}^{qq}(s, t) = (8\pi^{2}\alpha_{s})^{2} \left[ \frac{N_{c}^{2} - 1}{4N_{c}} \right] \delta_{ij} \delta_{kl} i s \int \frac{d^{2}k}{(2\pi)^{2}} \int \frac{d^{2}k'}{(2\pi)^{2}} \frac{F(s, k, k', q)}{k^{2}k'^{2}}$$

(113)

- The total cross section is obtained as

$$\sigma_{qq}^{\text{total}} = \frac{1}{s} \text{Im} A_{1}(s, t = 0) = 4\alpha_{s}^{2} \left( \frac{N_{c}^{2} - 1}{4N_{c}^{2}} \right) \int d^{2}k \int d^{2}k' \frac{F(s, k, k')}{k^{2}k'^{2}}$$

(114)

with rapidity defined as $y = \ln(s/k_{\text{min}}^{2})$ it results

$$\sigma_{qq}^{\text{total}} = \frac{\pi(N_{c}^{2} - 1)}{N_{c}^{2}} \left( \frac{\alpha_{s}^{2}}{k_{\text{min}}^{2}} \right) \frac{e^{\lambda y}}{\sqrt{\pi \lambda' y/8}}$$

(115)
Unitarity Violation

- The unitarity of the $S$-matrix

$$SS^\dagger = S^\dagger S = 1 \quad (116)$$

which implies that

$$|f\rangle = S|i\rangle = SS^\dagger |f\rangle \quad (117)$$

- From this feature arises the **Froissart-Martin Theorem**:
  - When $s \rightarrow \infty$

$$\sigma_{\text{total}} \leq C \ln^2 s \quad (118)$$

- In the case of quark-quark scattering we have

$$\sigma_{\text{total}}^{qq} \sim \frac{s^\lambda}{\sqrt{\ln s}} \quad (119)$$

that violates asymptotically the Froissart-Martin bound, since $\lambda = N_c \alpha_s 4\ln 2/\pi > 1$. 

BFKL Evolution Equation, G.G. Silveira, GFPAE – p. 77
Diffusion

- Features of **BFKL Equation** in the case of **LLA Pomeron**:
  - BFKL amplitude $F(s, k, k')$:
    - Gaussian distribution in $\ell n (k^2/k'^2)$;
    - Width growing with $y \equiv \ell n (s/k^2)$.
  - As the energy increases, the *non-perturbative region* can be probed;

- Setting the LLA BFKL solution as $(N \rightarrow$ iteration step)
  \[
  F^{(N)}(\omega, k_i) \sim (k_i^2)^{-\frac{1}{2}} \psi_N \left( \ell n \left[ \frac{k_i^2}{k_0^2} \right] \right) \equiv (k_i^2)^{-\frac{1}{2}} \psi_N (\xi_i) 
  \]  
  (120)

- Some algebra leads to a typical **diffusion equation**
  \[
  \lambda \frac{\partial \psi(N, \xi)}{\partial N} = \frac{\lambda'}{2} \frac{\partial^2 \psi(N, \xi)}{\partial \xi^2} 
  \]  
  (121)
Perturbative Region

- Taking "time" as $N = 0$ the wave function as a solution of the Diffusion Equation is

$$\psi(0, \xi) = \frac{1}{(\pi\sigma^2)^{\frac{1}{4}}} \exp \left( -\frac{\xi^2}{2\sigma^2} \right)$$

(122)

and neglecting the initial width we obtain

$$\psi(N, \xi) \sim \left( \frac{\lambda}{2\lambda'N} \right)^{\frac{1}{2}} \exp \left( -\frac{\lambda\xi^2}{2\lambda'N} \right)$$

(123)

- With the correspondence $N/\lambda \rightarrow y = \ln(s/k^2)$ we see that

  - A diffusion spreading equivalent to the behavior of LLA Pomeron solution.

- Important:

  - As the energy grows the infrared region of transverse momenta becomes more relevant:

    □ The perturbative treatment fails!
Running Coupling

- The Diffusion Phenomenon suggests the use of **running coupling**
  - From LLA the self-energy and vertex correction diagrams were **neglected**!
  - Which implies that the coupling constant $\alpha_s$ had been taken as a **constant**!
  - **Strategy**: Solution for the LLA BFKL equation with $\alpha_s \rightarrow \alpha_s(k^2)$.
    - One finds a discrete spectrum to the BFKL kernel;
    - A pole series being the Pomeron amplitude the leading one.
  - Another important feature:
    - Upper and lower limits to the intersection of the Pomeron’s trajectory:
      \[
      1 + 1.2 \left[ \frac{N_c \alpha_s(k_0^2)}{\pi} \right] \leq \alpha_P(0) \leq 1 + 4\ln \left[ \frac{2N_c \alpha_s(k_0^2)}{\pi} \right].
      \] (124)
Going to NLLA limit, the structure of the BFKL kernel has the form

\[ K(k, h) = 2\epsilon(-k^2)\delta^2(k - h) + K_{\text{real}}(k, h) \]  \hspace{1cm} (125)

- Reggeized gluon calculated in two-loop precision;
- Real part receives contribution from the tree level and production of \( gg \) and \( q\bar{q} \).

The eigenvalues of the BFKL kernel \( K \) in NLO are

\[ \omega(\gamma) = \frac{N_c\alpha_s(k^2)}{\pi} \left[ \chi^{(0)}(\gamma) + \left( \frac{N_c\alpha_s(k^2)}{\pi} \right) \chi^{(1)}(\gamma) \right] \]  \hspace{1cm} (126)

where

- \( \chi^{(0)}(\gamma) = 2\psi(1) - \psi(\gamma) - \psi(1 - \gamma) \) is the LLA contribution;
- \( \chi^{(1)} \) represents the NLO correction.
The correction from $\chi^{(1)}(\gamma)$ has the form

$$
\chi^{(1)}(\gamma) = -\frac{1}{4} \left\{ \frac{1}{2} \left( \frac{11}{3} - \frac{2n_f}{3N_c} \right) \left[ \left( \chi^{(0)}(\gamma) \right)^2 - \psi'(\gamma) + \psi'(1 - \gamma) \right] 
- 6\zeta(3) + \frac{\pi^2 \cos \pi \gamma}{(\sin \pi \gamma)(1 - 2\gamma)} \left[ 3 + \left( 1 + \frac{n_f}{N_c^3} \right) \frac{2 + 3\gamma(1 - \gamma)}{(3 - 2\gamma)(1 + 2\gamma)} \right] 
- \left( \frac{67}{9} - \frac{\pi^2}{3} - \frac{10}{9} \frac{n_f}{N_c} \right) \chi^{(0)}(\gamma) - \psi''(\gamma) - \psi''(1 - \gamma) 
- \frac{\pi^3}{\sin \pi \gamma} + 4\phi(\gamma) \right\}.
$$

The running coupling constant has a correction of the form

$$
\alpha_s(k^2) \simeq \alpha_s(\mu^2) \left[ 1 - \frac{\alpha_s(\mu^2)}{4\pi} \left( \frac{11N_c}{3} - \frac{2n_f}{3} \right) \ln \left( \frac{k^2}{\mu^2} \right) \right].
$$
In this approach the eigenvalues have two types of corrections

- From the derivative of the strong running coupling;
- Energy-scale independence of the due to $\chi^{(1)}(\gamma)$.

Thus, the eigenvalues can be expressed under these corrections as

$$\omega(\gamma) = \left[ \bar{\alpha}_s(\mu^2)\chi_0(\gamma) + \bar{\alpha}_s^2(\mu^2)\chi_1(\gamma) \right] + \left[ \bar{\alpha}_s(\mu^2) \left( \frac{11}{12} - \frac{n_f}{6N_c} \right) \ell n \left( \frac{k^2}{\mu^2} \right) \chi_0(\gamma) \right].$$  \hspace{1cm} (129)

Leading eigenvalue is that with $\gamma = 1/2$:

$$\omega_0 = \bar{\alpha}_s \chi_0 \left( \frac{1}{2} \right) = 2.77\bar{\alpha}_s.$$

At NLO this eigenvalue is

$$\bar{\alpha}_s \chi(\gamma) \big|_{\gamma=\frac{1}{2}} = \bar{\alpha}_s \chi_0(\gamma) + \bar{\alpha}_s^2 \chi_1(\gamma) \big|_{\gamma=\frac{1}{2}} = \omega_0 \left( 1 - 6.61\bar{\alpha}_s \right) = 2.77\bar{\alpha}_s - 18.34\bar{\alpha}_s^2,$$  \hspace{1cm} (130)

- **HERA regime**: Correction $\chi^{(1)}(\gamma)$ so large that dominates over $\chi^{(0)}(\gamma)$!
BFKL Equation in DIS

• Using the integro-differential equation we obtain for \( f(x, k^2) \)

\[
\frac{\partial f(x, k^2)}{\partial \ln(1/x)} = \frac{3 \alpha_s k^2}{\pi} \int_0^\infty \frac{d^2 h}{h^2} \left[ \frac{f(x, h^2) - f(x, k^2)}{|h^2 - k^2|} \right] + \frac{f(x, k^2)}{(4h^4 + k^4)^{1/2}}
\]

(131)

we obtain the **BFKL equation for DIS** in the leading \( \ln(1/x) \) approximation with a fixed coupling constant.

• The solution for this equation gives an unintegrated gluon distribution

\[
f(x, k^2) \sim \left( \frac{x}{x_0} \right)^{-\lambda} \left[ \frac{k^2 / k_0^2}{\ln(x/x_0)} \right]^{1/2} \exp \left[ -\frac{\ln^2(k^2 / \tilde{k}_0^2)}{2 \lambda' \ln(x_0/x)} \right]
\]

(132)

with the leading behavior at low-\( x \)

\[
f(x, k^2) \sim x^{-\lambda}
\]

(133)
Unintegrated Gluon Distribution

- It is clearly visible:
  - The diffusion in $k^2$; and
  - Growth of the type $x^{-\lambda}$.

Predictions for $F_2$

(Bojak and Ernst, Phys. Rev. D53, 80, 1996)

BFKL prescription for $F_2$ compared with HERA data.
Applications: Truncated BFKL Series

- Considering only the first two orders in LO perturbative theory we have for the partial-wave amplitudes

\[
f_1(\omega, k_1, k_2, q) = \frac{1}{\omega} \delta^2(k_1 - k_2)
\]

\[
f_2(\omega, k_1, k_2, q) = -\frac{N_c \alpha_s}{2\pi^2} \frac{1}{\omega^2} \left[ \frac{q^2}{k_1^2(k_2 - q)^2} - \frac{1}{2} \frac{1}{(k_1 - k_2)^2} \left\{ 1 + \frac{k_2^2(k_1 - q)^2}{k_1^2(k_2 - q)^2} \right\} \right]
\]

which corresponds to taking the two-gluon exchange and the one-rung ladder into account only.

- **Truncating** the BFKL series at two orders, a parametrization is proposed to *proton-proton* and *proton-anti-proton* the total cross section goes like

\[
\sigma_{\text{total}}^{pp(p\bar{p})} = C_R (s/s_0)^{\alpha_R(0)-1} + C_{\text{Born}} + C_{\text{NO}} \ln(s/s_0) \tag{134}
\]

where \( k^2 = s_0 = 1 \text{ GeV}^2 \).
Fits for $pp$ (lower line) and $p\bar{p}$ (upper line) total cross section from PDG data.
Application: LO versus NLO BFKL Equation

- Using an effective kernel and a saddle point approximation to compute $F_2^P$ in NLO-BFKL
  → **problem**: deviations at $Q^2 < 10 \text{ GeV}^2$

(Schoeffel, hep-ph/0505114, 2005)

Fits at LO (solid line) and NLO (dashed line) BFKL for $F_2^P$ from H1 data.
Application: Meson Production (I)

- It can be studied the meson production via pomeron exchange in $e^+e^-$ colliders;
- A possible process is illustrated below

\[ e^\pm \rightarrow e^\pm + \gamma \]

\[ s_{ee} \]

\[ W^2 \]

\[ \theta_1 \]

\[ K_{\text{BFKL}} \]

\[ \gamma \]

\[ \theta_2 \]

\[ t \]

\[ I_{\gamma V_1} \]

\[ I_{\gamma V_2} \]

\[ V_1 \]

\[ V_2 \]
The cross section is expressed in the form

$$
\sigma_{e^+e^- \rightarrow e^+e^- V_1 V_2 (\sqrt{s}ee)} = \int dx_a \, dx_b \, f_{\gamma/e}(x_a) \, f_{\gamma/e}(x_b) \frac{d\sigma_{\gamma\gamma \rightarrow V_1 V_2}}{dt}(\hat{s})
$$

(135)

The cross section of the subprocess depends on the BFKL amplitude $F$

$$
\frac{d\sigma(\gamma\gamma \rightarrow V_1 V_2)}{dt} = \frac{16\pi}{81t^4} |F_{BFKL}(z, \tau)|^2
$$

(136)

These functions represent the incoming photons and are related to the BFKL Amplitude

$$
F_{BFKL}(z, \tau) = \frac{t^2}{(2\pi)^3} \int d\nu \frac{\nu^2}{(\nu^2 + 1/4)^2} e^{\chi(\nu) z} I_{\gamma V_1}^\nu (Q_\perp) I_{\gamma V_2}^\nu (Q_\perp)^* 
$$

(137)

where the quantities $I_{\nu V_i}^\gamma$ are called impact factors and the quantity $\chi(\nu)$ depends of the BFKL Kernel eigenvalues

$$
\chi(\nu) = 4 \text{Re} \left( \psi(1) - \psi \left( \frac{1}{2} + i\nu \right) \right)
$$

(138)
Finally, the results for the production of several mesons are described in the next table:

<table>
<thead>
<tr>
<th>Meson</th>
<th>$\sqrt{s_{ee}} = 200$ GeV</th>
<th>$\sqrt{s_{ee}} = 500$ GeV</th>
<th>$\sqrt{s_{ee}} = 1000$ GeV</th>
<th>$\sqrt{s_{ee}} = 3000$ GeV</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\rho J/\Psi$</td>
<td>0.90 (0.015)</td>
<td>5.80 (0.049)</td>
<td>21.87 (0.097)</td>
<td>178.19 (0.22)</td>
</tr>
<tr>
<td>$\phi J/\Psi$</td>
<td>0.11 (0.0023)</td>
<td>0.69 (0.0073)</td>
<td>2.58 (0.014)</td>
<td>20.77 (0.033)</td>
</tr>
<tr>
<td>$\omega J/\Psi$</td>
<td>0.075 (0.0013)</td>
<td>0.48 (0.0041)</td>
<td>1.85 (0.0081)</td>
<td>15.03 (0.019)</td>
</tr>
<tr>
<td>$J/\Psi J/\Psi$</td>
<td>0.045 (0.0021)</td>
<td>0.27 (0.0066)</td>
<td>0.98 (0.012)</td>
<td>7.56 (0.031)</td>
</tr>
<tr>
<td>$\rho \Upsilon$</td>
<td>0.0013 (0.000055)</td>
<td>0.0093 (0.00017)</td>
<td>0.036 (0.00034)</td>
<td>0.31 (0.00080)</td>
</tr>
<tr>
<td>$\omega \Upsilon$</td>
<td>0.00011 (0.0000055)</td>
<td>0.00078 (0.000017)</td>
<td>0.0030 (0.000034)</td>
<td>0.026 (0.000080)</td>
</tr>
<tr>
<td>$\phi \Upsilon$</td>
<td>0.0002 (0.000011)</td>
<td>0.0013 (0.000034)</td>
<td>0.0050 (0.000068)</td>
<td>0.040 (0.00016)</td>
</tr>
<tr>
<td>$J/\Psi \Upsilon$</td>
<td>0.00025 (0.000027)</td>
<td>0.0015 (0.000086)</td>
<td>0.0052 (0.00017)</td>
<td>0.038 (0.00040)</td>
</tr>
<tr>
<td>$\Upsilon \Upsilon$</td>
<td>0.0000072 (0.0000014)</td>
<td>0.000038 (0.0000045)</td>
<td>0.00012 (0.0000088)</td>
<td>0.0008 (0.000020)</td>
</tr>
</tbody>
</table>

The double vector meson production cross sections in $e^+e^-$ processes at different energies, $|t|_{min} = 0$ and $\theta_{max} = 30$ mrad, assuming the BFKL Pomeron (Two-gluon) exchange. Cross sections are given in pb.
Application: Higgs Boson Production (I)

- A way to study the production of the Higgs boson is the double-pomeron exchange;
- In order to find the Higgs boson, two processes are accounted (Royon, C. hep-ph/0601226)
- Exclusive Process:

\[
d\sigma_{h_{\text{exc}}}^e(s) = C_h \left( \frac{s}{M_h^2} \right)^{2\epsilon} \delta \left( \xi_1 \xi_2 - \frac{M_h^2}{s} \right) \times \prod_{i=1,2} \left\{ \frac{d^2 v_i}{1 - \xi_i} \frac{d\xi_i}{\xi_i} 2^{\alpha' v_i^2} \exp \left( -2\lambda_H v_i^2 \right) \right\} \sigma(gg \rightarrow h) \tag{139}\]

- Inclusive Process:

\[
d\sigma_{H_{\text{incl}}}^i = C_H \left( \frac{x_1^g x_2^g s}{M_H^2} \right)^{2\epsilon} \delta \left( \xi_1 \xi_2 - \frac{M_H^2}{x_1^g x_2^g s} \right) \times \prod_{i=1,2} \left\{ G_P(x_i^g, \mu) \ dx_i^g \frac{d^2 v_i}{1 - \xi_i} \frac{d\xi_i}{\xi_i} 2^{\alpha' v_i^2} \exp \left( -2v_i^2 \lambda_H \right) \right\} ; \tag{140}\]
Higgs boson signal-to-background ratio as a function of the resolution on the missing-mass, in GeV.

\( m_H = 120 \text{ GeV} \)
The QCD Evolution Landscape

The BFKL evolution and its limitations.
Conclusions

- Clarifies the knowledge about the High-Energy particle phenomenology;
  - Goal for Regge Theory.
- Good agreement with Low-$x$ data beyond the DGLAP Equation;
- Require some analysis in non-perturbative region.
References


