BFKL Evolution Equation

G.G. Silveira
gustavo.silveira@ufrgs.br

High Energy Phenomenology Group

Instituto de Física

Universidade Federal do Rio Grande do Sul

Porto Alegre, Brazil

GFPAE - UFRGS

http://www.if.ufrgs.br/gfpae

Outline

- Physics at small-x;
- Quark-Quark Scattering in LLA;
- The BFKL Equation;
- Solution to Zero Momentum Transfer;
- Application: $qq \to qq$;
- BFKL Equation in NLLA;
- BFKL Equation in DIS;
- Application: Truncated BFKL Series;
- Application: LO *versus* NLO BFKL Equation;
- Conclusions.

Resummation in pQCD

- The experimental data is well described by DGLAP Equation when $\ell n\, Q^2 \gg \ell n\, \frac{1}{x}$;
- When Q^2 is **large**, the leading terms need to be resummed:

  $\hookrightarrow$ Resum over the leading terms to subtract the divergences.

  - **LLA Limit**:
    At each perturbative order only the highest power in $\ell n\, Q^2$ is retained

$$\sum_n \alpha_s^n \, \ell n^{(n)} \, Q^2 \left(\ell n^{(n)} \frac{1}{x} + \ell n^{(n-1)} \frac{1}{x} + \ldots \right) \tag{1}$$

  - **NLLA Limit**:
    It is retained subdominant powers in $\ell n\, Q^2$

$$\sum_n \alpha_s^n \, \ell n^{(n-1)} \, Q^2 \left(\ell n^{(n)} \frac{1}{x} + \ell n^{(n-1)} \frac{1}{x} + \ldots \right) \tag{2}$$

Resummation at small-x

- When the **Small-x Limit** is reached, other resummations should be applied:
  - **DLLA Limit:**
    In the LLA limit we retain only dominant terms in $\ell n\,(1/x)$

$$\sum_n \alpha_s^n\,\ell n^{(n)}\,Q^2\,\ell n^{(n)}\,\frac{1}{x} \tag{3}$$

    - DGLAP **resums** the terms $\alpha_s^n\,\ell n^{(n)}\,Q^2$ and $\alpha_s^n\,\ell n^{(n)}\,Q^2\,\ell n^{(n)}\,\frac{1}{x}$.

      $\hookrightarrow$ But it does **not** resum the leading terms $\alpha_s^n\,\ell n^{(n)}\,\frac{1}{x}$

  - **LL$_x$A Limit:** $x \ll 1$, Q^2 not large $\Rightarrow$ $\ell n\,Q^2 \ll \ell n\,\frac{1}{x}$
    - Resummation of $\sum_n \alpha_s^n\,\ell n^{(n)}\frac{1}{x}\,\left(\ell n^{(n)}\,Q^2 + \ell n^{(n-1)}\,Q^2 + \ldots\right)$

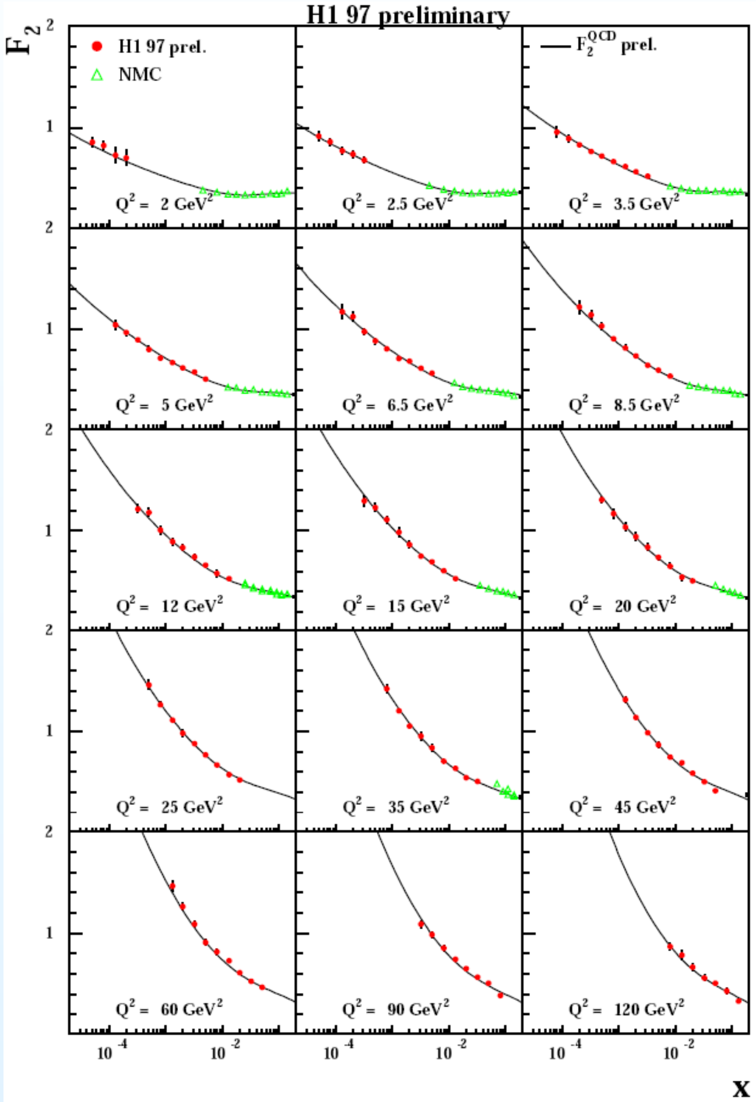
$\hookrightarrow$ **In this limit the BFKL Equation operates!**

Structure Function F_2

- From the HERA data:
  - Steep rise of F_2 at low-x: ($F_2 \leftrightarrow \sigma \leftrightarrow g$)
    - Increase of the gluon density!
  - **DGLAP Equation** $\Rightarrow$ still OK!

    $\hookrightarrow$ In the kinematic range of HERA.
- If it is reached much lower values of x ...
  - Does DGLAP still describe the data?

(Marage, hep-ph/9911426, 1999)



NLO DGLAP fit for HERA data.

Information from HERA

- Parametrize F_2 for $x < 0.1$ in the form

$$F_2(x, Q^2) = A(Q^2)\, x^{-\lambda} \tag{4}$$

  - For small Q^2 ($\lesssim 1\,\text{GeV}^2$): $\lambda \approx 0.1$
  - For large Q^2 ($\sim 10 - 100\,\text{GeV}^2$): $\lambda \approx 0.25 - 0.35$

- For $x \to 0$:
  - In the perturbative regime ($Q^2 \gtrsim 1\,\text{GeV}^2$) $\Rightarrow$ **DGLAP Equation**;
  - The region which Q^2 is small ($< 1\,\text{GeV}^2$) $\Rightarrow$ **Regge Theory**.

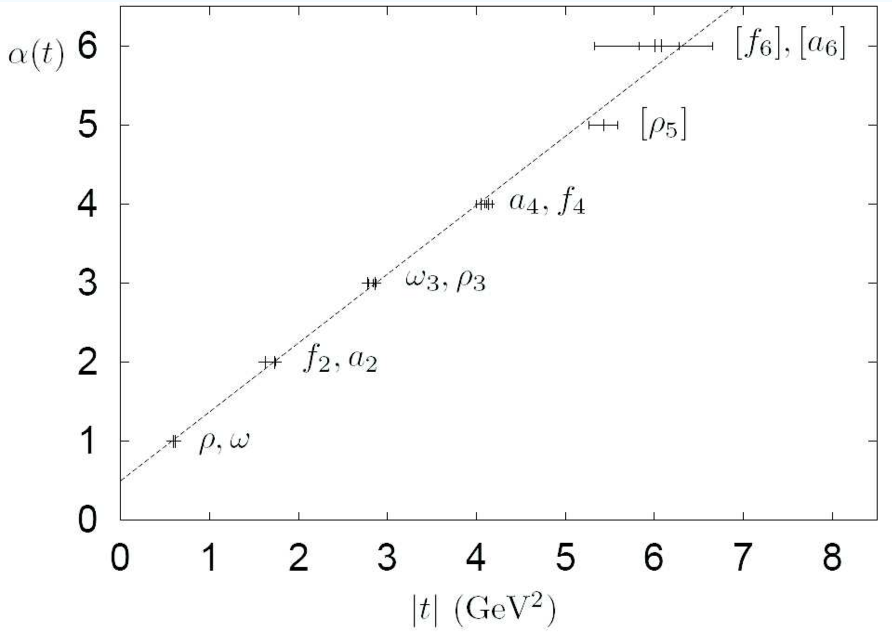
- BFKL Equation resums the leading terms $\ell n\, \frac{1}{x}$ for $Q^2 < 1\,\text{GeV}^2$

Regge Theory

- Low values of x correspond to large values of $s \longrightarrow$ **Here the Regge Theory takes place!**
- In the hadronic process, particles are exchanged as
  - Nuclear Physics: mesons (ρ, ω, $\ldots$);
  - High-Energy Phenomenology: '*trajectories*' or *Reggeons* $\mathbb{R}$.
- What says Regge Theory to us? What means '*trajectories*'?
  - Extending the angular momentum to complex values one found **singularities**;
  - These singularities give rise to resonances that can be exchanged in the t-channel;
  - When a family of resonances is exchanged it is called *Regge trajectory exchange*;
- A Regge trajectory exchanged is said a exchange of a $\begin{cases} \text{Reggeized particle} \\ \text{Reggeon } \mathbb{R} \end{cases}$

Reggeized Gluon

- **Reggeized Particle:**
  - The amplitude for the exchange of a particle in the t-channel is written as $A \sim s^{\alpha(t)}$;
    - The exponent $\alpha(t)$ is related to the particle trajectory;



- **BFKL Equation** in **LO** $\rightarrow$ resummation of $\sum_n \alpha_s^n \ln^{(n)} \frac{s}{t}$ with $s \gg Q^2,\, t$
  - In this order, the leading process is the exchange of gluons;
  - It will be studied the reggeized gluons exchange in all orders of perturbation theory;

The Pomeron

- In Regge Theory this exchange is the **Pomeron Exchange**: $A_{\mathbb{P}} \sim s^{\alpha_{\mathbb{P}}(t)}$
- Experimentally the cross section has the form

$$\sigma \sim s^{\lambda} \quad \longrightarrow \quad \lambda \sim 0.08 - 0.10 \tag{5}$$

- The Regge Theory predicts a cross section of the form

$$\sigma \sim s^{\alpha_{\mathbb{P}}(0)-1} \quad \longrightarrow \quad \alpha_{\mathbb{P}}(0) \simeq 1 \tag{6}$$

- Interesting feature: **Pomeron** has the *vaccum quantum numbers*:

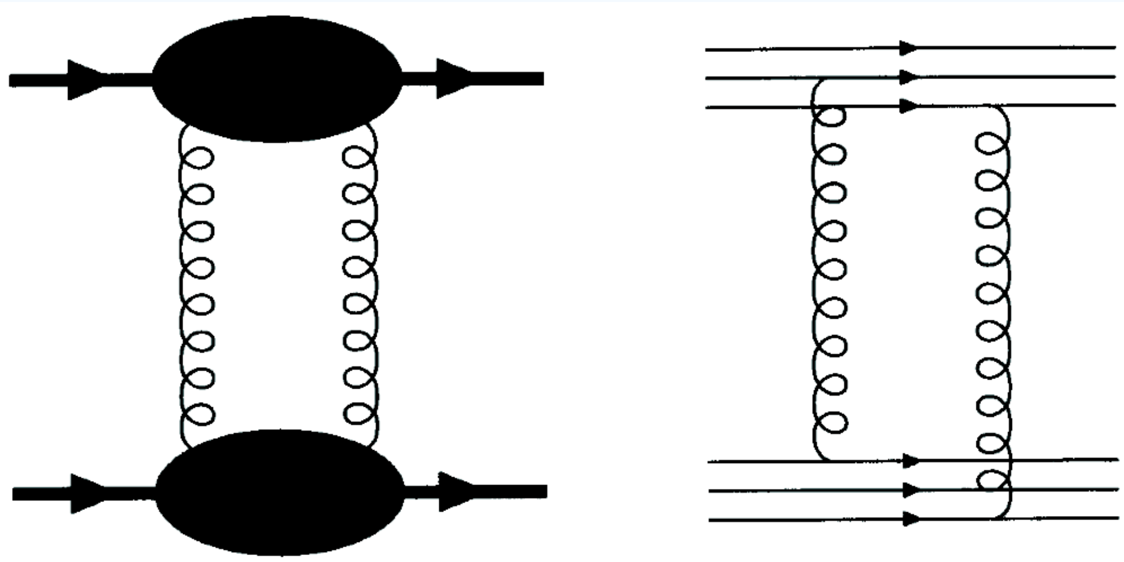
$$P = +1, \quad C = +1, \quad I = 0 \tag{7}$$

and the Pomeron is the dominant trajectory in the elastic and diffractive processes!

The Pomeron in QCD

- To incorporate the Pomeron in QCD $\longrightarrow$ consider an exchange of the vaccum quantum numbers!

- Using the QCD degrees of freedom (quarks and gluons): **two-gluon exchange**!



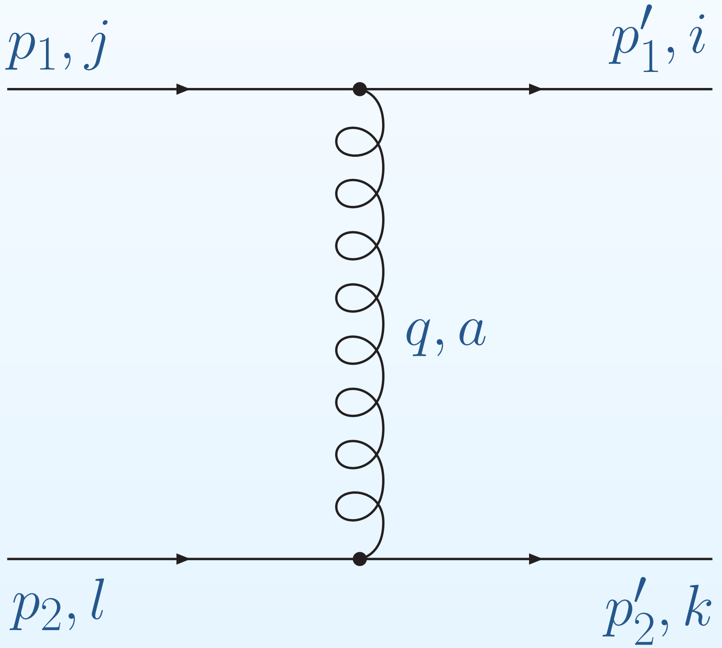
- In high-energy processes ($x \ll 1$) the Pomeron contribution is **essential**;
- In this sense, the DGLAP Equation does not take into account the Pomeron contribution!
- It is needed to sum the contributions of the leading terms in $\ell n\, s$!

One-Gluon Exchange

- First contribution to the Pomeron: **2-gluons exchange**!

  $\hookrightarrow$ We start calculating the one-gluon exchange amplitude and then work to higher orders!

- Quark-quark scattering in the Regge limit ($s \gg -t$)



- Computing the amplitude of the process with the Feynman Rules in the Feynman gauge:

$$A^{(0)}_{ijlm} = \bar{u}(p_1 - q)\left(-ig_s\gamma^\mu t^a_{ij}\right)u(p_1)\left(-\frac{i\delta_{ab}g_{\mu\nu}}{q^2}\right)\bar{u}(p_2 + q)\left(-ig_s\gamma^\nu t^b_{lm}\right)u(p_2) \qquad (8)$$

Kinematic Regime

- In the center-of-mass reference frame one takes p_1 and p_2 along the z axis

$$p_1 = \frac{\sqrt{s}}{2}\,(1,\,\mathbf{0},\,1) \quad , \quad p_2 = \frac{\sqrt{s}}{2}\,(1,\,\mathbf{0},\,-1) \tag{9}$$

- Using the Sudakov Parametrization:

$$q^\mu = \alpha p_1^\mu + \beta p_2^\mu + \mathbf{q}^\mu = \left(\frac{\sqrt{s}}{2}[\alpha+\beta],\,\mathbf{q},\,\frac{\sqrt{s}}{2}[\alpha-\beta]\right) \tag{10}$$

where the constants α and β are the momentum fraction of the quarks carried by the gluon and

$$p_1^2 = p_2^2 = 0 \qquad\qquad 2\,(p_1 \cdot p_2) = s$$

- The momentum transfer squared has the form

$$t = q^2 = 2\alpha\beta\,(p_1 \cdot p_2) - \mathbf{q}^2 = \alpha\beta s - \mathbf{q}^2$$

Final State Condition

- Taking the mass-shell conditions for the outgoing quarks

$$\left.\begin{aligned} (p_1 - q)^2 &= -\beta s + \alpha\beta s - \mathbf{q}^2 = t - \beta s = 0 \\ (p_2 + q)^2 &= \alpha s + \alpha\beta s - \mathbf{q}^2 = t + \alpha s = 0 \end{aligned}\right\} \quad \begin{aligned} \beta &= t/s \\ \alpha &= -t/s \end{aligned} \tag{11}$$

so

$$q^\mu = -\frac{t}{s}(p_1^\mu - p_2^\mu) + \mathbf{q}^\mu \simeq \mathbf{q}^\mu \tag{12}$$

- The momentum transfer squared now is

$$t \equiv q^2 \simeq -\mathbf{q}^2 \tag{13}$$

- In the large-s limit one can state that:
  - All components of the exchanged momentum q are **much smaller** than p_1 and p_2!

Scattering Amplitude

- Writing the scattering amplitude

$$iA^{(0)}_{ijlm}(s,t) = ig_s^2\,(t^a_{ij}t^a_{lm})\,\bar{u}(p_1-q)\gamma^\mu u(p_1)\left(\frac{1}{q^2}\right)\bar{u}(p'_2+q)\gamma_\mu u(p_2) \qquad (14)$$

- The amplitude **squared**, **averaged** and **summed over colors** is

$$|\overline{A^{(0)}}|^2 = 2g_s^4\left(\frac{N_c^2-1}{4N_c^2}\right)\left(\frac{s^2+u^2}{t^2}\right) \underset{s\simeq -u}{\overset{s\to\infty}{\equiv}} \left(\frac{8}{9}\right) g_s^4\left(\frac{s^2}{t^2}\right) \qquad (15)$$

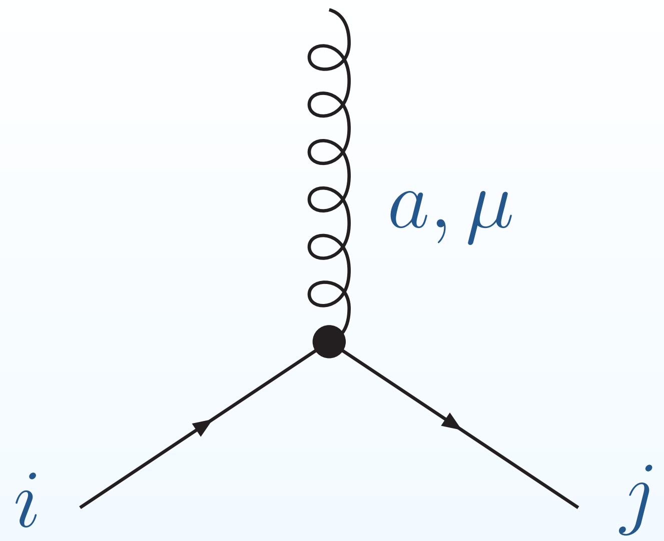
where the color factor for $N_c = 3$ is

$$\begin{aligned}\frac{1}{N_c^2}(t^a_{ij}t^a_{lm})(t^b_{ij}t^b_{lm})^* &= \frac{1}{N_c^2}\,t^a_{ij}t^a_{lm}t^b_{ji}t^b_{ml}\\ &= \frac{1}{N_c^2}\mathrm{Tr}(t^at^b)\mathrm{Tr}(t^at^b) = \frac{N_c^2-1}{4N_c^2} = \frac{2}{9}.\end{aligned}$$

Eikonal Approximation

- The general form of the qqg vertex is

$$V^\mu = -ig_s \bar{u}(p_1 + q)\gamma^\mu u(p_1)$$



- Due to the smallness of q one can approximate

$$V^\mu \simeq -ig_s \bar{u}(p_1)\gamma^\mu u(p_1) = -2ig_s p_1^\mu \quad (16)$$

which is the called *quark-gluon eikonal vertex* that represents a **soft** particle exchange!

- From this one rewrites the amplitude as

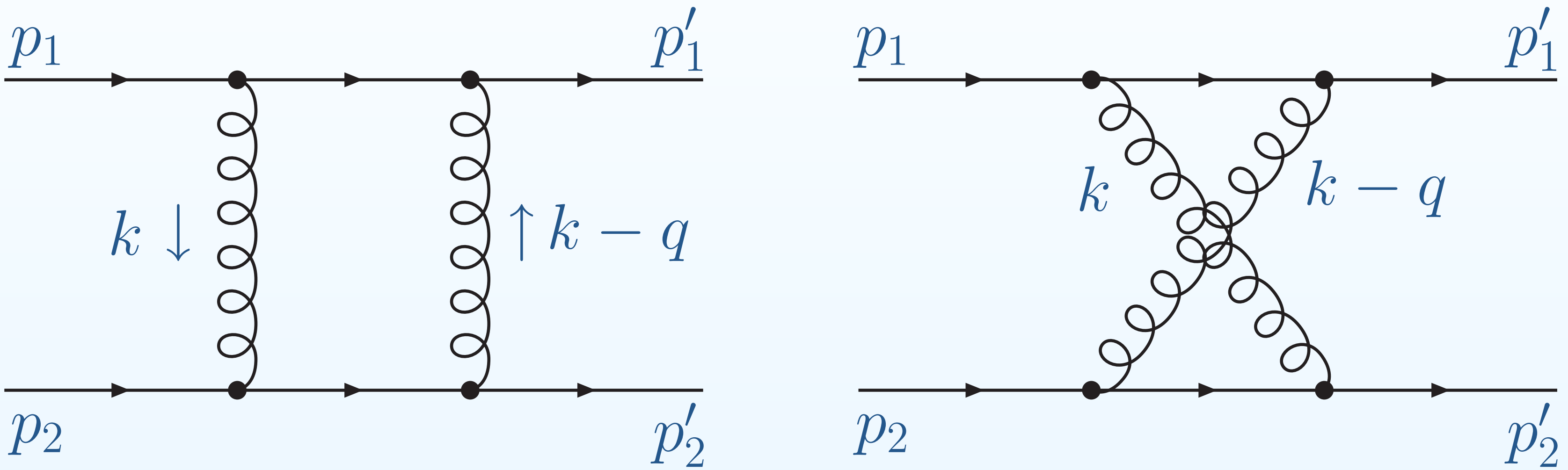
$$A_{ijlm}^{(0)} = 2g_s^2 \left(t_{ij}^a t_{lm}^a\right) \left(\frac{1}{q^2}\right) (2\, p_1 \cdot p_2) = 8\pi\alpha_s \left(t_{ij}^a t_{lm}^a\right) \left(\frac{s}{t}\right) \quad (17)$$

- This approximation **does not change** the squared amplitude, having the same form as before

$$|\overline{A^{(0)}}|^2 = \frac{8g_s^4}{9} \left(\frac{s^2}{t^2}\right) \quad (18)$$

Two-Gluon Exchange

- Corrections of the order $\mathcal{O}(\alpha_s^2)$: **ONE-LOOP DIAGRAM**



- It will be computed the scattering amplitude using the **Cutkosky Rules**: $t \equiv q^2 \simeq -\mathbf{q}^2$

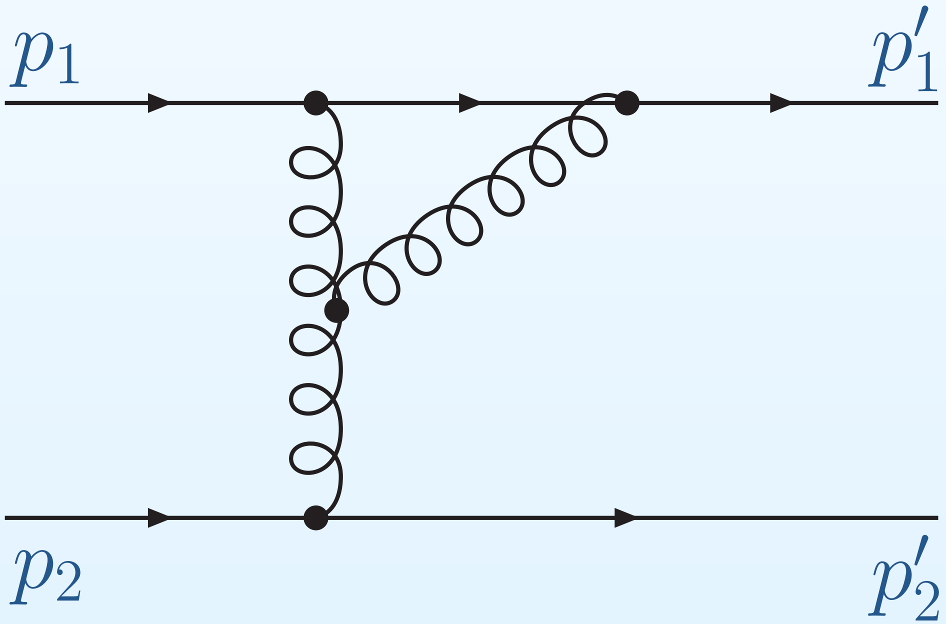
$$\text{Im}\, A^{(1)}(s,t) = \frac{1}{2} \int d\Pi_2\, A^{(0)}(s,k^2)\, A^{(0)\dagger}(s,[k-q]^2) \tag{19}$$

which amplitudes are the one-gluon exchange amplitudes computed previously.

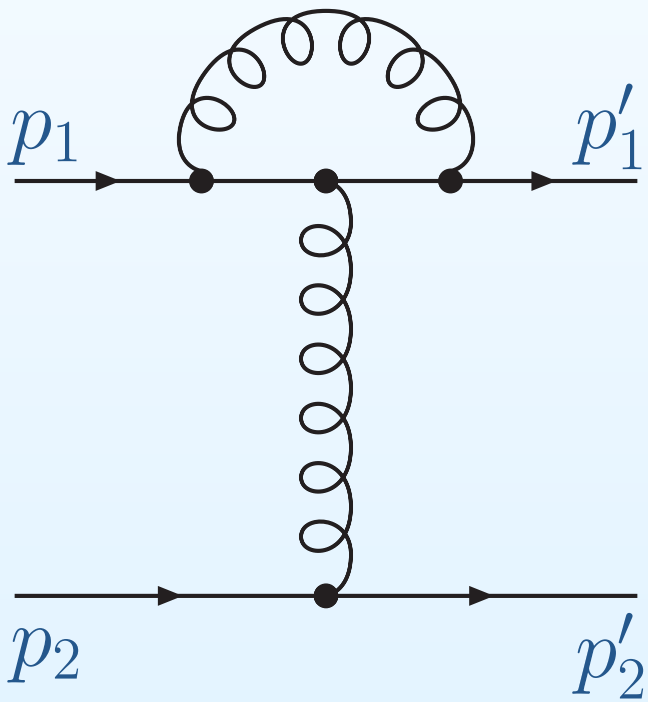
Subleading Diagrams

- It has been taken the **leading terms** of the type $\ell n\, s$;
- Some diagrams will yield subleading terms, like

Vertex Correction diagrams;



Self-energy diagrams.



Phase Space

- One takes the two-body phase space

$$\int d\Pi_2 = \int \frac{d^4\kappa_1}{(2\pi)^3}\frac{d^4\kappa_2}{(2\pi)^3}\,\delta(\kappa_1^2)\,\delta(\kappa_2^2)\,(2\pi)^4\,\delta^{(4)}(p_1+p_2-\kappa_1-\kappa_2)$$
$$= \int \frac{d^4k}{(2\pi)^2}\,\delta([p_1-k]^2)\,\delta([p_2+k]^2)$$

- As before, one introduces the Sudakov variables

$$k = \alpha p_1 + \beta p_2 + k_\perp \tag{20}$$

$$d^4k = \left(\frac{s}{2}\right)\,d\alpha\,d\beta\,d^2\mathbf{k} \tag{21}$$

- The Two-body phase space with the Sudakov variables is written as

$$\int d\Pi_2 = \frac{s}{8\pi^2}\int d\alpha\,d\beta\,d^2\mathbf{k}\,\delta(-\beta[1-\alpha]s+\mathbf{k}^2)\,\delta(\alpha[1+\beta]s-\mathbf{k}^2) \tag{22}$$

High Energy Limit

- When one works in the large-s limit, the Sudakov variables can be approximate to

$$\alpha = |\beta| \simeq \frac{\mathbf{k}^2}{s} \ll 1 \tag{23}$$

$$k^2 \simeq -\mathbf{k}^2, \quad (k-q)^2 \simeq -(\mathbf{k}-\mathbf{q})^2 \tag{24}$$

$$\mathbf{k}^2 \simeq (\mathbf{k}-\mathbf{q})^2 \simeq \mathbf{q}^2 \tag{25}$$

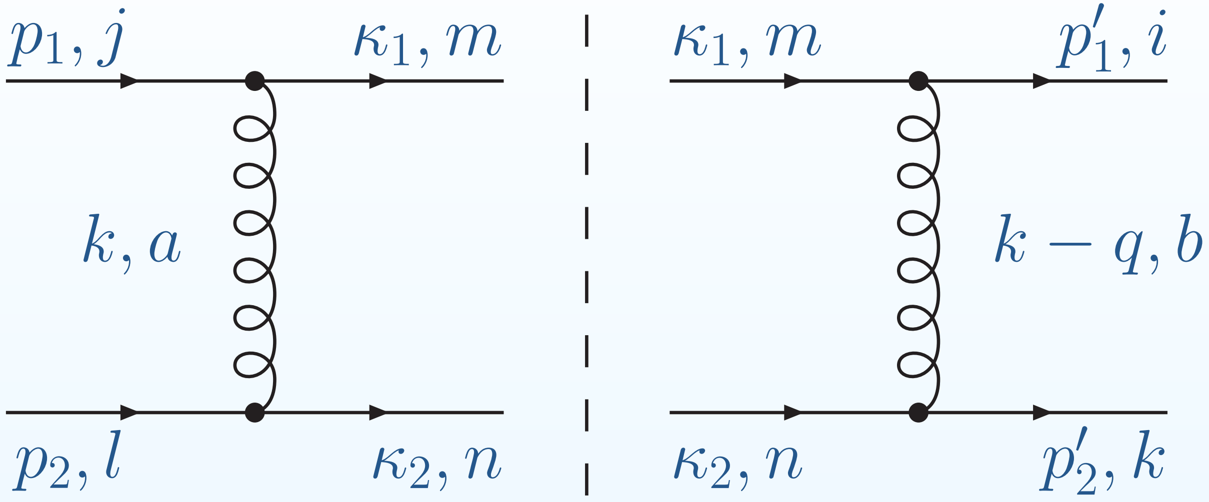
where one rewrites the two-body phase space like

$$\int d\Pi_2 = \frac{1}{8\pi^2 s}\int d\alpha\, d\beta\, d^2\mathbf{k}\, \delta\left(\beta + \frac{\mathbf{k}^2}{s}\right)\delta\left(\alpha - \frac{\mathbf{k}^2}{s}\right) = \frac{1}{8\pi^2 s}\int d^2\mathbf{k}$$

that is

$$k^\mu = -\left(\frac{\mathbf{k}^2}{s}\right)p_1^\mu + \left(\frac{\mathbf{k}^2}{s}\right)p_2^\mu + \mathbf{k}^\mu$$

Square Diagram



- Amplitudes from one-gluon exchange:

$$A^{(0)}(s, k^2) = -8\pi\alpha_s(t^a_{mj}t^a_{nl})\left[\frac{s}{\mathbf{k}^2}\right]$$

$$A^{(0)\dagger}(s, [k-q]^2) = -8\pi\alpha_s(t^b_{mi}t^b_{nk})^*\left[\frac{s}{(\mathbf{k}-\mathbf{q})^2}\right]$$

that is

$$\text{Im}A^{(1)}_{\text{a}}(s,t) = 4\alpha_s^2\, s\,(t^a t^b)_{ij}(t^a t^b)_{kl}\int d^2\mathbf{k}\left[\frac{1}{\mathbf{k}^2(\mathbf{k}-\mathbf{q})^2}\right]$$

Dispersion Relations

- In the leading $\ell n \, \frac{1}{x}$ approximation one can express the amplitude as

$$A = \text{Re}\, A + i\, \text{Im} A = \mathcal{C}\, \ell n \left(\frac{s}{t} \right) + \ldots = \mathcal{C}\, \ell n \left| \frac{s}{t} \right| - i \pi \mathcal{C} \tag{26}$$

which yields

$$\text{Re} A = \mathcal{C}\, \ell n \left| \frac{s}{t} \right| \qquad \text{Im} A = -\pi \mathcal{C} \tag{27}$$

- The $\mathcal{C}$ coefficient expresses the relation between the real and imaginary parts of the amplitude

$$\text{Re} A = -\frac{1}{\pi}\, \text{Im} A \ell n \left| \frac{s}{t} \right| \tag{28}$$

which, for the full scattering amplitude, all these can be expressed as

$$A = -\frac{1}{\pi}\, \text{Im} A \left(\ell n \left| \frac{s}{t} \right| - i\pi \right) = -\frac{1}{\pi}\, \ell n \left(\frac{s}{t} \right) \text{Im} A \tag{29}$$

Full Amplitude

- Using the dispersion relations one can find the full amplitude for the square diagram

$$A_{\square}^{(1)}(s,t) = -\frac{4}{\pi}\alpha_s^2 s(t^a t^b)_{ij}(t^a t^b)_{kl}\, \ell n\left(\frac{s}{t}\right)\int d^2\mathbf{k}\left[\frac{1}{\mathbf{k}^2(\mathbf{k}-\mathbf{q})^2}\right]$$

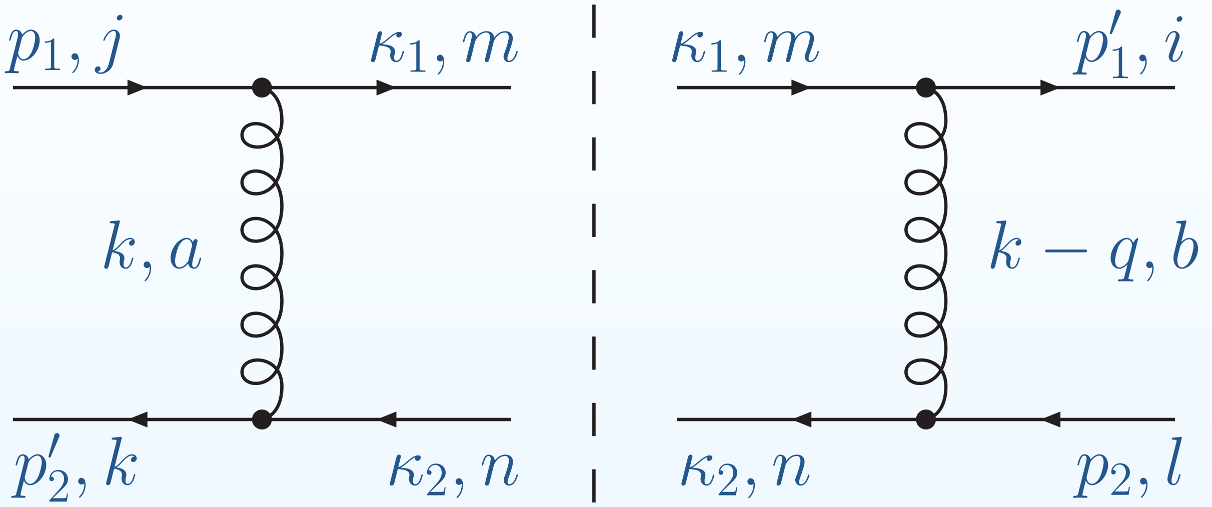
$$= -16\left(\frac{\pi\alpha_s}{N_c}\right)(t^a t^b)_{ij}(t^a t^b)_{kl}\left(\frac{s}{t}\right)\ell n\left(\frac{s}{t}\right)\epsilon(t)$$

where the dimensionless function $\epsilon(t)$ incorporates the transverse-momentum integration

$$\epsilon(t) = \frac{N_c\alpha_s}{4\pi^2}\int d^2\mathbf{k}\left[\frac{-\mathbf{q}^2}{\mathbf{k}^2(\mathbf{k}-\mathbf{q})^2}\right] \tag{30}$$

- This function is **very important** to express the Pomeron exchange in perturbative QCD.
  - It will result from here the **trajectory** of the **pQCD Pomeron**!

Cross Diagram



- One can compute this amplitude using the fact that in the Regge Limit

$$\text{Im}A_{\times}^{(1)} = \text{Im}A_{\square}^{(1)}(s \to u, t) \tag{31}$$

- Thus the imaginary part of the amplitude can be expressed as

$$\text{Im}A_{\times}^{(1)}(s,t) = -16\left(\frac{\pi\alpha_s}{N_c}\right)(t^a t^b)_{ij}(t^b t^a)_{kl}\left(\frac{u}{t}\right)\,\ell n\left(\frac{u}{t}\right)\,\epsilon(t) \tag{32}$$

Full Amplitude in the High Energy Limit

- In the high energy limit the channels are related through $s \simeq -u$;

$$\text{Im}A^{(1)}_{\times}(s,t) = 16\left(\frac{\pi\alpha_s}{N_c}\right)(t^a t^b)_{ij}(t^b t^a)_{kl}\left(\frac{s}{t}\right)\,\ell n\left(\frac{s}{|t|}\right)\,\epsilon(t) \qquad (33)$$

- One can compute the full amplitude through dispersion relations getting

$$\begin{aligned} A^{(1)}_{ijkl}(s,t) &= A^{(1)}_{\square}(s,t) + A^{(1)}_{\times}(s,t) = \\ &= -16\left(\frac{\pi\alpha_s}{N_c}\right)(t^a t^b)_{ij}\left(\frac{s}{t}\right) \\ &\times \left\{[t^a,t^b]_{kl}\,\ell n\left(\frac{s}{|t|}\right) - i\pi(t^a t^b)_{kl}\right\}\epsilon(t) \end{aligned}$$

- It is clear that there is a **different contribution** from the imaginary part;
  - This term is important because it will receive contribution only from the **color-singlet** term.
    - The color-singlet term is crucial due to its contribution to the **Pomeron exchange**!

Color Projectors

- The quark-quark scattering amplitude can be decomposed in the SU(3) representation:

$$A_{ijkl}(s,t) = \sum_R \mathcal{P}_{ijkl}(R)\mathcal{A}_R(s,t) \qquad (34)$$

- The color-singlet ($\underline{1}$) and color-octet ($\underline{8}$) amplitudes are expressed as

$$\left.\begin{array}{c} \mathcal{P}_{ijkl}(\underline{1}) = \left(\frac{1}{N_c}\right)\delta_{ij}\,\delta_{kl} \\ \mathcal{P}_{ijkl}(\underline{8}) = 2\,t^a_{ij}t^a_{kl} \end{array}\right\}$$

$$A^{(\underline{1})}_{ijkl}(s,t) = \mathcal{P}_{ijkl}(\underline{1})\mathcal{A}_{\underline{1}}(s,t) \qquad (35)$$

$$A^{(\underline{8})}_{ijkl}(s,t) = \mathcal{P}_{ijkl}(\underline{8})\mathcal{A}_{\underline{8}}(s,t) \qquad (36)$$

- For these projectors there is the normalization: $\mathcal{P}_{ijkl}(R)\mathcal{P}^{lkmn}(R') = \mathcal{P}^{mn}_{ij}(R)\delta_{RR'}$

- From this one gets

$$\mathcal{A}_{\underline{1}}(s,t) = \mathcal{P}^{ij}_{kl}(\underline{1})A^{ij}_{kl}(s,t) \qquad \mathcal{A}_{\underline{8}}(s,t) = \left(\frac{1}{N_c^2-1}\right)\mathcal{P}^{ij}_{lk}(\underline{8})A^{ij}_{kl}(s,t) \qquad (37)$$

Color-Octet Exchange

- Applying the color-octet projector one can extract the amplitude

$$\mathcal{A}_{\underline{8}}^{(1)}(s,t) = -16 \left(\frac{\pi\alpha_s}{N_c}\right) \mathcal{C}_{\underline{8}}^{(1)} \left(\frac{s}{t}\right) \ell n \left(\frac{s}{|t|}\right) \epsilon(t) \tag{38}$$

where

$$\mathcal{C}_{\underline{8}}^{(1)} = \left(\frac{1}{N_c^2 - 1}\right) \mathcal{P}_{lk}^{ji}(\underline{8})(t^a t^b)_{ij}[t^a, t^b]_{kl} = -\frac{N_c}{4} \tag{39}$$

- From the decomposition one can obtain the quark-quark amplitude via color-octet exchange

$$A_{\underline{8}}^{(1)}(s,t) = 8\pi\alpha_s(t_{ij}^a t_{kl}^a) \left(\frac{s}{t}\right) \ell n \left(\frac{s}{|t|}\right) \epsilon(t) \tag{40}$$

- Note that color-octet amplitude is **real** and $\mathcal{O}(\ell n\, s)$ at one-loop level.

Color-Singlet Exchange

- Proceeding in the same way one can extract the amplitude in the color-singlet case

$$\mathcal{A}^{(1)}_{\underline{1}}(s,t) = 16\left(\frac{i\,\pi^2\alpha_s}{N_c}\right)\,\mathcal{C}^{(1)}_{\underline{1}}\,\left(\frac{s}{t}\right)\,\epsilon(t) \tag{41}$$

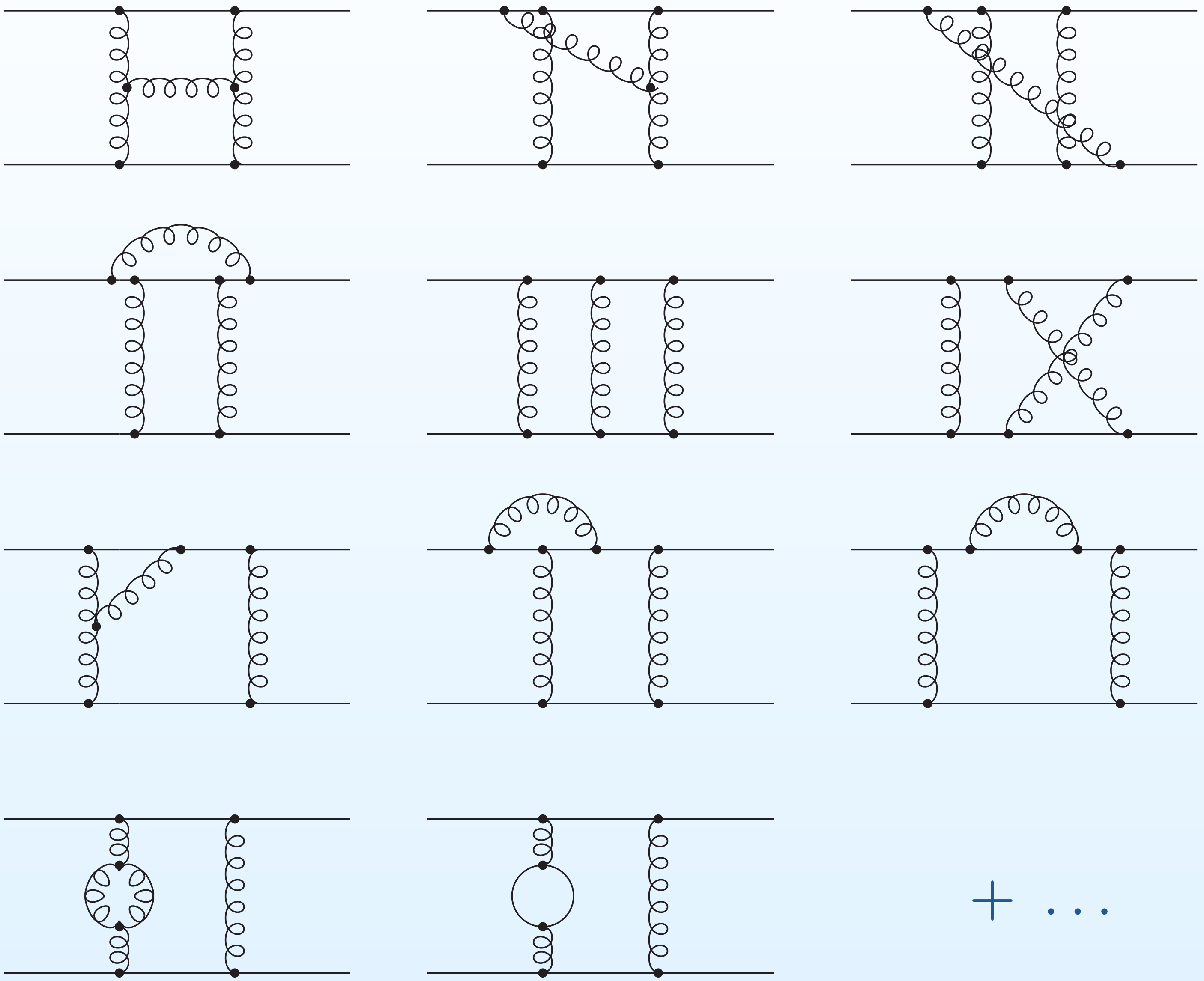
where $\mathcal{C}^{(1)}_{\underline{1}} = \mathcal{P}^{ji}_{lk}(\underline{1})(t^a t^b)_{ij}(t^a t^b)_{kl} = \frac{N_c^2-1}{4N_c}$

- As before one can obtain the quark-quark amplitude via color-singlet exchange

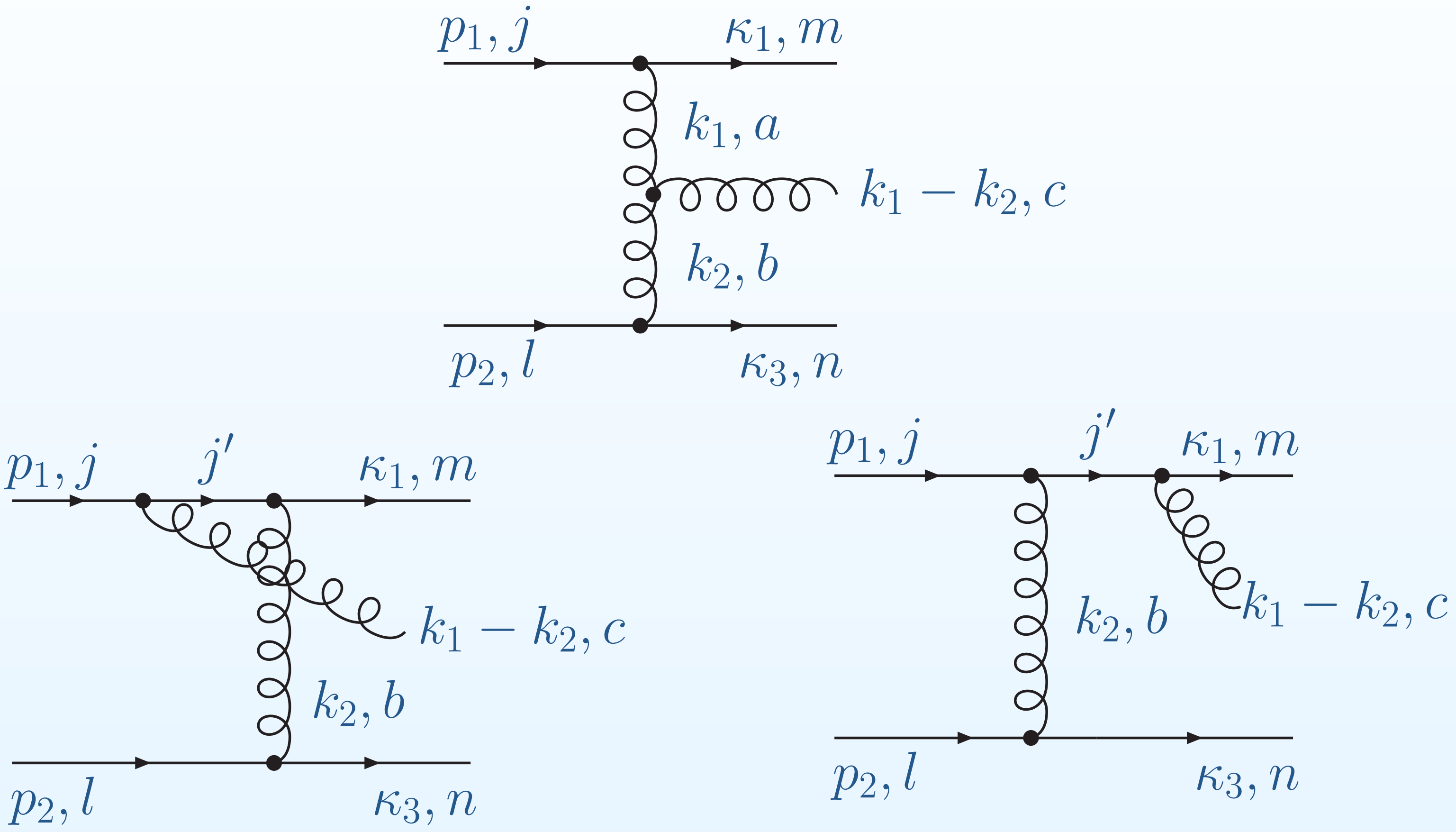
$$A^{(1)}_{\underline{1}}(s,t) = 4i\pi^2\alpha_s(\delta_{ij}\delta_{kl})\left(\frac{N_c^2-1}{4N_c}\right)\frac{s}{t}\,\epsilon(t) \tag{42}$$

- One can see that the contribution $\ell n\,(s/|t|)$ from the **two diagrams** cancel each other;
- This amplitude starts at order $\mathcal{O}(\alpha_s^2)$ and is **suppressed** by a factor $\ell n\; s$ with respect to the color-octet case.
  - Color-singlet and color-octet amplitudes have opposite signatures $\begin{cases} \xi_{\underline{1}} = +1 \\ \xi_{\underline{8}} = -1 \end{cases}$

Two-Loop Diagrams



Emission Diagrams



- In the same way one can introduce the Sudakov parametrization:

$$k_1 = \alpha_1 p_1 + \beta_1 p_2 + k_{1\perp}$$
$$k_2 = \alpha_2 p_1 + \beta_2 p_2 + k_{2\perp}$$

Kinematic Regime

- The leading $\ell n\, s$ contribution comes from the Kinematic regime of strong ordering of the longitudinal momenta

$$1 \gg \alpha_1 \gg \alpha_2 \tag{43}$$

$$1 \gg |\beta_2| \gg |\beta_1| \tag{44}$$

- Taking the gluons on mass-shell

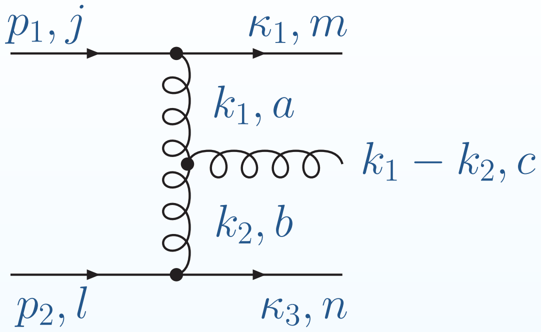
$$\begin{aligned}
(k_1 - k_2)^2 &= k_1^2 + k_2^2 - 2(k_1 \cdot k_2) = 0 \\
&= -\mathbf{k}_1^2 - \mathbf{k}_2^2 - \alpha_1\beta_2 s - \alpha_2\beta_1 s + \mathbf{k}_1 \cdot \mathbf{k}_2 = 0 \\
&\simeq -(\mathbf{k}_1 - \mathbf{k}_2)^2 - \alpha_1\beta_2 s = 0
\end{aligned}$$

which results in a non-ordering in the transverse momenta

$$\alpha_1\beta_2 s = -(\mathbf{k}_1 - \mathbf{k}_2)^2 \tag{45}$$

$$\mathbf{k}_1^2 \simeq \mathbf{k}_2^2 \simeq \mathbf{q}^2 \tag{46}$$

Central Emission



- Computing the scattering amplitude one can find

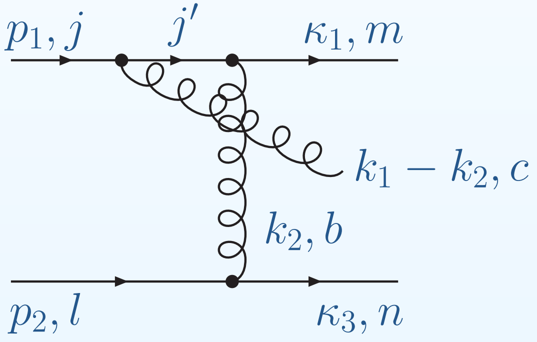
$$
\begin{aligned}
iA^{\rho}_{2\to 3,\mathrm{a}} &= (-2ig_s p_1^{\mu}) t^a_{mj} \left(-\frac{i}{k_1^2}\right) \\
&\times \quad g_s f_{abc} \left[(k_1 + k_2)^{\rho} g^{\mu\nu} + (k_1 - 2k_2)^{\mu} g^{\nu\rho} + (k_2 - 2k_1)^{\nu} g^{\rho\mu}\right] \\
&\times \quad \left(-\frac{i}{k_2^2}\right) (-2ig_s p_2^{\nu}) t^b_{nl}
\end{aligned}
$$

- Taking into account the kinematics expressed before one can obtain the amplitude

$$
A^{\rho}_{2\to 3,\mathrm{a}} = -2i\, g_s^3\, f_{abc} (t^a_{mj} t^b_{nl}) \left(\frac{1}{\mathbf{k}_1^2 \mathbf{k}_2^2}\right) \left[\alpha_1 p_1^{\rho} + \beta_2 p_2^{\rho} - (k_1^{\rho} + k_2^{\rho})\right]
$$

Gluon Emission from Upper Quarks

- In the same way one can write the amplitude for the first diagram of gluon emission



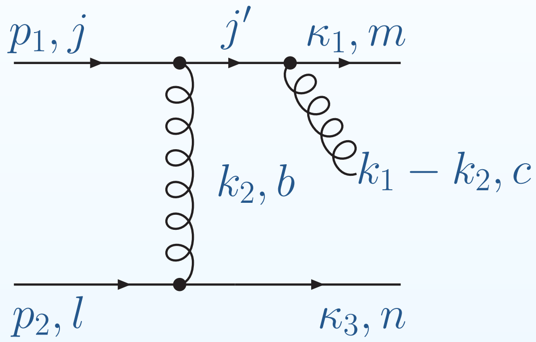
$$
\begin{aligned}
iA^{\rho}_{2\to 3,\mathtt{b}} &= (-2ig_s p_1^{\rho})\, t^{c}_{j'j} \left[\frac{i}{(p_1 - k_1 + k_2)^2}\right] \\
&\times (-2ig_s)(p_1^{\mu} - k_1^{\mu} + k_2^{\mu})\, t^{b}_{mj'} \\
&\times \left(-\frac{i}{k_2^2}\right) (-2ig_s p_{2\mu})\, t^{b}_{nl}
\end{aligned}
$$

- Using the information from the Kinematic regime, the amplitude takes the form

$$
A^{\rho}_{2\to 3,\mathtt{b}} = -4g_s (t^b t^c)_{mj}\, t^{b}_{nl} \left[\frac{1}{\beta_2 s \mathbf{k}_2^2}\right] p_1^{\rho}
$$

Gluon Emission from Upper Quarks

- Again taking the amplitude but for the second diagram one has



$$iA^{\rho}_{2\to 3,c} = (-2ig_s p_{1\mu}) t^b_{j'j} \left(-\frac{i}{k_2^2}\right) (-2ig_s p_2^{\mu}) t^b_{nl} \times \left[\frac{i}{(p_1 - k_2)^2}\right] (-2ig_s)(p_1^{\rho} - k_2^{\rho}) t^c_{mj'}$$

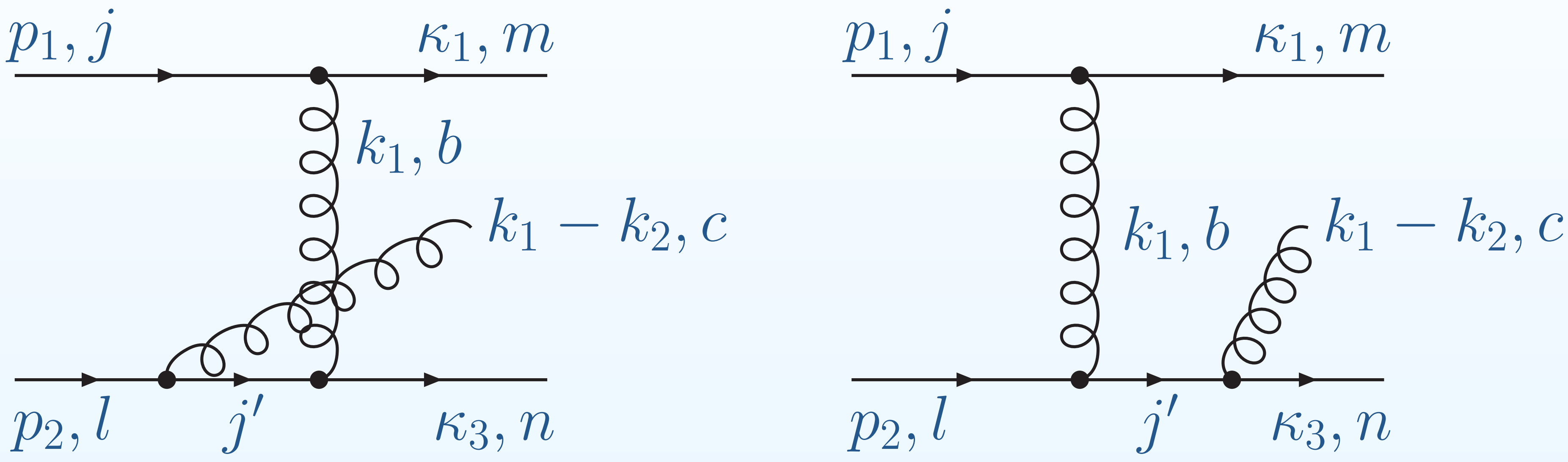
- From the Kinematic regime one can see that

$$A^{\rho}_{2\to 3,c} = 4 g_s^3 s f_{abc} t^a_{mj} t^b_{nl} \left(\frac{1}{\beta_2 s \mathbf{k}_2^2 p_1^{\rho}}\right) \quad (47)$$

- Finally, for the full scattering amplitude one can obtain, using $[t^b, t^c] = i f_{abc} t^a$

$$\boxed{A^{\rho}_{2\to 3,b+c} = -4i\, g_s^3\, s\, f_{abc}\, (t^a_{mj} t^b_{nl}) \left(\frac{1}{\beta_2 s \mathbf{k}_2^2}\right) p_1^{\rho}} \quad (48)$$

Gluon Emission from Lower Quarks



- Following the same procedure one finds the amplitude of gluon emission from the lower quarks

$$A^{\rho}_{2\to 3,d+e} = -4i\, g_s^3\, f_{abc}\, (t^a_{mj} t^b_{nl}) \left(\frac{1}{\alpha_1 s \mathbf{k}_1^2} \right) p_2^{\rho} \tag{49}$$

Lipatov Vertex

- Summing the amplitudes obtained before one can find the full amplitude in $\mathcal{O}(g_s^3)$

$$A^{\rho}_{2\to 3} = -4ig_s^3 \left(\frac{p_1^{\mu} p_2^{\nu}}{\mathbf{k}_1^2 \mathbf{k}_2^2}\right) (t^a_{mj} t^b_{nl}) \, f_{abc} \, \Gamma^{\rho}_{\mu\nu} \tag{50}$$

where the quantity $\Gamma^{\rho}_{\mu\nu}$ is called the **Lipatov effective vertex** and has the form

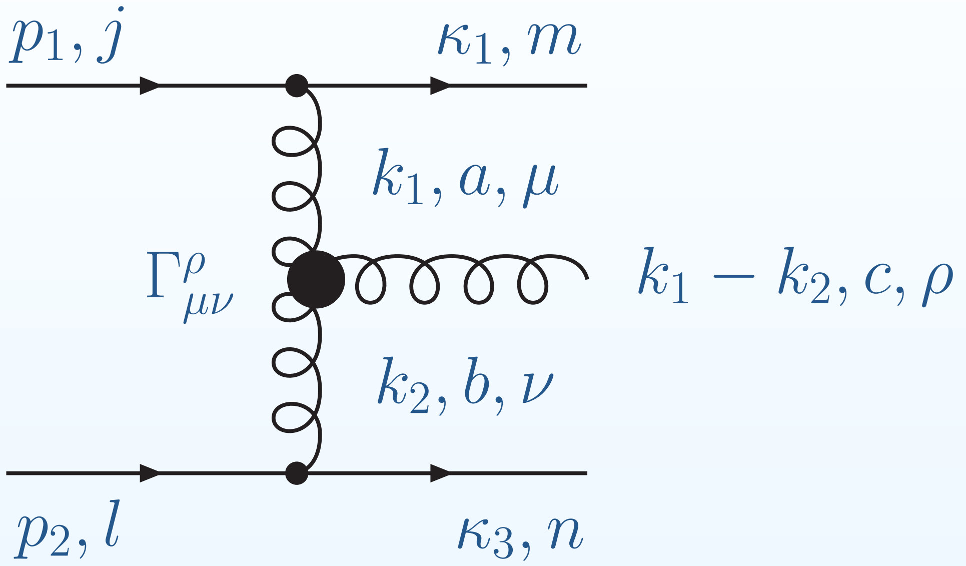
$$\Gamma^{\rho}_{\mu\nu}(k_1, k_2) \quad = \quad \frac{2p_{2\mu}p_{1\nu}}{s}\left[\left(\alpha_1 + \frac{2\mathbf{k}_1^2}{\beta_2 s}\right) p_1^{\rho} + \left(\beta_2 + \frac{2\mathbf{k}_2^2}{\alpha_1 s}\right) p_2^{\rho} - (\mathbf{k}_1^{\rho} + \mathbf{k}_2^{\rho})\right]$$

  - Physically this effective vertex incorporates the propagators of the emitted gluons.

- This vertex has the important property of being **gauge-invariant**, that is

$$(k_{1\rho} - k_{2\rho}) \; \Gamma^{\rho}_{\mu\nu} \, (k_1, k_2) = 0 \tag{51}$$

Effective Diagram



- All graphs with one gluon in the final state are summed up by the effective diagram

$$iA^{\rho}_{2\to 3} = (-2i\, g_s p_1^{\mu})\, t^a_{mj} \left(-\frac{i}{k_1^2} \right) f_{abc}\, g_s\, \Gamma^{\rho}_{\mu\nu}(k_1, k_2) \left(-\frac{i}{k_2^2} \right) (-2ig_s p_2^{\nu})\, t^b_{nl} \quad (52)$$

which, obviously, coincides with the amplitude obtained before.

Effective Diagram

- It's interesting to introduce the quantity below for convenience

$$C^\rho(k_1,k_2) = \left(\alpha_1 + \frac{2\mathbf{k}_1^2}{\beta_2 s}\right) p_1^\rho + \left(\beta_2 + \frac{2\mathbf{k}_2^2}{\alpha_1 s}\right) p_2^\rho - (\mathbf{k}_1^\rho + \mathbf{k}_2^\rho) \tag{53}$$

so that

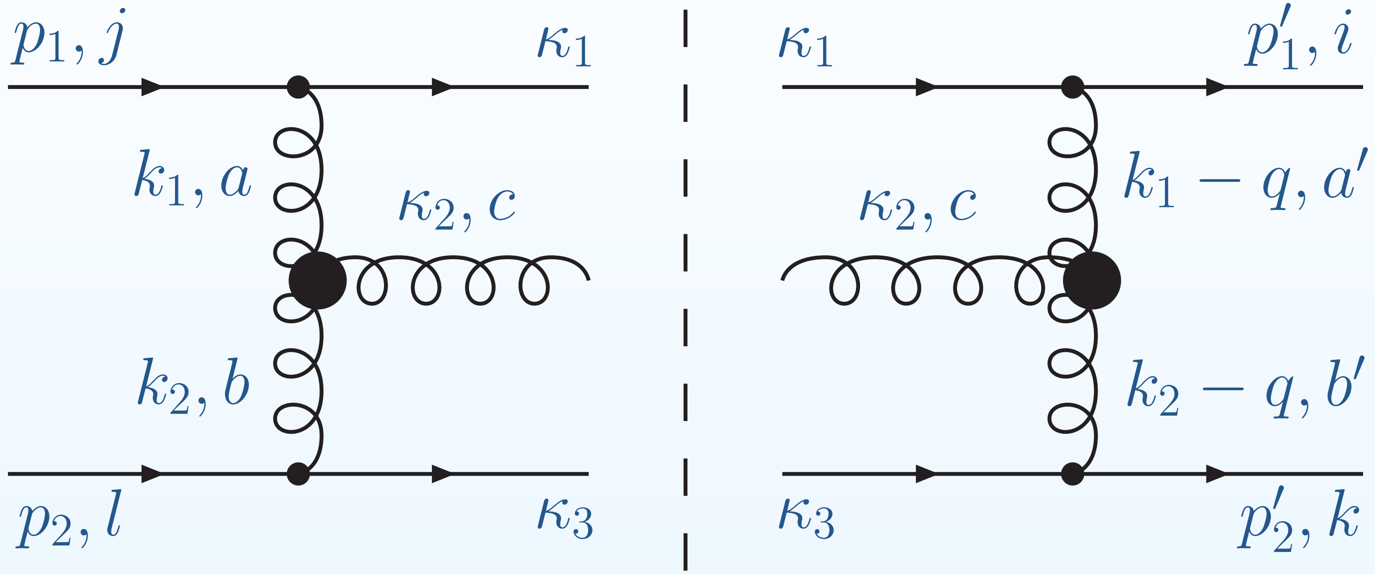
$$\Gamma^\rho_{\mu\nu} = \left(\frac{2}{s}\right) p_{2\mu}\, p_{1\nu}\, C^\rho \tag{54}$$

$$C^\rho = \left(\frac{2}{s}\right) p_1^\mu\, p_2^\nu\, \Gamma^\rho_{\mu\nu} \tag{55}$$

- Through this new quantity the full amplitude is rewritten as

$$A^\rho_{2\to3} = 2i\, g_s\, t^a_{mj} \left(\frac{i}{\mathbf{k}_1^2}\right) f_{abc}\, g_s\, C^\rho(k_1,k_2) \left(\frac{i}{\mathbf{k}_2^2}\right) g_s\, t^b_{nl}$$

Real Gluon Contribution



- Following the procedure applied before one can use the Cutkosky Rules

$$\text{Im}A^{(2)}_{\text{real}}(s,t) = -\frac{g_{\rho\sigma}}{2}\int d\Pi_3\, A^{\rho}_{2\to 3}(k_1,k_2)\, A^{\sigma\,\dagger}_{2\to 3}(k_1-q,k_2-q) \tag{56}$$

where is needed the sum over gluon helicities: $\sum_\lambda \varepsilon^{\mu}_{\lambda}(p)\,\varepsilon^{\nu *}_{\lambda}(p) = -g^{\mu\nu}$

Phase Space

- A little bit more difficulty and one can compute the <u>three-body</u> phase space

$$\int d\Pi_3 = \int \frac{d^4\kappa_1}{(2\pi)^3}\frac{d^4\kappa_2}{(2\pi)^3}\frac{d^4\kappa_3}{(2\pi)^3}\,\delta(\kappa_1^2)\,\delta(\kappa_2^2)\,\delta(\kappa_3^2)\,(2\pi)^4\,\delta^4(p_1+p_2-\kappa_1-\kappa_2-\kappa_3)$$

$$= \frac{1}{(2\pi)^5}\int d^4\kappa_1\,d^4\kappa_3\,\delta(\kappa_1^2)\,\delta(\kappa_3^2)\,\delta([p_1+p_2-\kappa_1-\kappa_3]^2)$$

$$= \frac{1}{(2\pi)^5}\int d^4k_1\,d^4k_2\,\delta([p_1-k_1]^2)\,\delta([p_2+k_2]^2)\,\delta([k_1-k_2]^2)$$

- Again using the Sudakov parametrization one finds for the phase space

$$\int d\Pi_3 = \frac{s^2}{4(2\pi)^5}\int d\alpha_1\,d\beta_1\,d^2\mathbf{k}_1\int d\alpha_2\,d\beta_2\,d^2\mathbf{k}_2$$

$$\times\ \delta(-\beta_1[1-\alpha_1]s-\mathbf{k}_1^2)\,\delta(\alpha_2[1+\beta_2]s-\mathbf{k}_2^2)$$

$$\times\ \delta([\alpha_1-\alpha_2][\beta_1-\beta_2]s-[\mathbf{k}_1-\mathbf{k}_2]^2)$$

Approximation

- As done previously the Kinematic regime implies that

$$1 \gg \alpha_1 \gg \alpha_2 \quad , \quad 1 \gg |\beta_2| \gg |\beta_1| \tag{57}$$

$$k_i^2 \simeq -\mathbf{k}_i^2 \tag{58}$$

and finally the phase space is

$$\begin{aligned} \int d\Pi_3 &= \frac{s^2}{4(2\pi)^5} \int d\alpha_1\, d\beta_1\, d^2\mathbf{k}_1 \int d\alpha_2\, d\beta_2\, d^2\mathbf{k}_2 \\ &\times \delta(-\beta_1 s - \mathbf{k}_1^2)\, \delta(\alpha_2 s - \mathbf{k}_2^2)\, \delta(-\alpha_1 \beta_2 s - [\mathbf{k}_1 - \mathbf{k}_2]^2) \\ &= \frac{1}{4(2\pi)^5} \int_{\alpha_2}^1 \frac{d\alpha_1}{\alpha_1} \int_0^1 d\alpha_2 \int d^2\mathbf{k}_1 \int d^2\mathbf{k}_2\, \delta(\alpha_2 - \mathbf{k}_2^2) \\ &= \frac{1}{4(2\pi)^5 s} \int_{\mathbf{q}^2/s}^1 \frac{d\alpha_1}{\alpha_1} \int d^2\mathbf{k}_1 \int d^2\mathbf{k}_2 \end{aligned} \tag{59}$$

Full Amplitude

- In order to use the Cutkosky rules one needs to compute the amplitude of the right hand side diagram

$$A^{\rho\,\dagger}_{2\to 3} = -2i\, g_s\, t^{a'}_{im} \left[-\frac{i}{(k_1 - q)^2} \right] (-f_{a'bc'}\, g_s)\, C^{\rho}(-[k_1 - q], -[k_2 - q]) \left(-\frac{i}{\mathbf{k}_2^2} \right) g_s\, t^{b'}_{kn} \tag{60}$$

- Making the product of the both sides of the effective diagram one gets

$$\begin{aligned} A_{tot} &= A^{\rho}_{2\to 3}(k_1, k_2)\, A^{\dagger}_{2\to 3,\rho}(k_1 - q, k_2 - q) = \\ &= 4 g_s^6 s^2\, \mathcal{G}_{\text{real}} \left[\frac{C^{\rho}(k_1, k_2)\, C_{\rho}(-k_1 + q, -k_2 + q)}{\mathbf{k}_1^2 \mathbf{k}_2^2 (\mathbf{k}_1 - \mathbf{q})^2 (\mathbf{k}_1 - \mathbf{q})^2} \right] \end{aligned}$$

where the color factor is

$$\mathcal{G}_{\text{real}} = -(t^{a'} t^{a})_{ij} (t^{b'} t^{b})_{kl}\, f_{abc}\, f_{a'b'c} \tag{61}$$

Imaginary Part for the Real Radiative Correction

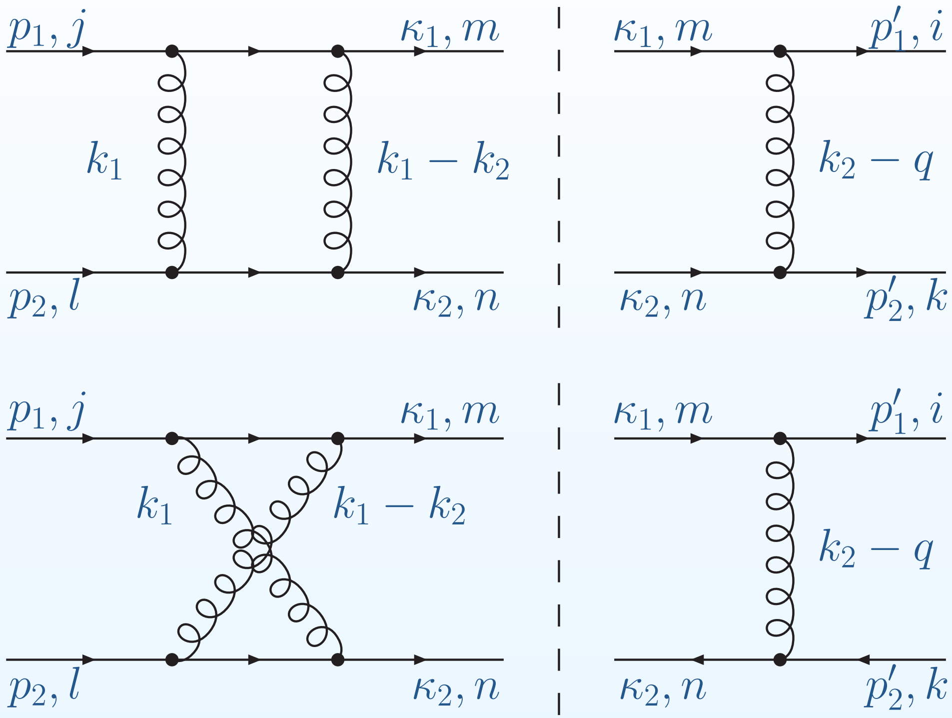
- Performing the product of the vectors

$$C^{\rho}(k_1,k_2)\,C_{\rho}(-k_1+q,-k_2+q) = -2\left[\mathbf{q}^2 - \frac{\mathbf{k}_1^2(k_2^2-\mathbf{q})^2}{(\mathbf{k}_1-\mathbf{k}_2)^2} - \frac{\mathbf{k}_2^2(\mathbf{k}_1-1)^2}{(\mathbf{k}_1-\mathbf{k}_2)^2}\right] \tag{62}$$

- Join the phase space integral and the total amplitude one obtain the amplitude in $\mathcal{O}(\alpha_s^2)$

$$\begin{aligned}
\mathrm{Im}A^{(2)}_{\text{real}}(s,t) &= \left(\frac{2\alpha_s^3}{\pi^2}\right)\mathcal{G}_{\text{real}}\, s\,\ell n\left(\frac{s}{|t|}\right)\int d^2\mathbf{k}_1^2\int d^2\mathbf{k}_2^2 \\
&\times \left[\frac{\mathbf{q}^2}{\mathbf{k}_1^2\mathbf{k}_2^2(\mathbf{k}_1-\mathbf{q})^2(\mathbf{k}_2-\mathbf{q})^2} - \frac{1}{\mathbf{k}_2^2(\mathbf{k}_1-\mathbf{q})^2(\mathbf{k}_1-\mathbf{k}_2)^2}\right. \\
&- \left.\frac{1}{\mathbf{k}_1^2(\mathbf{k}_2-\mathbf{q})^2(\mathbf{k}_1-\mathbf{k}_2)^2}\right]
\end{aligned}$$

Virtual Contribution



- Considering gluon exchanges in the t-channel one computes this amplitude by

$$\begin{aligned} \mathrm{Im}A^{(2)}_{\text{virtual}}(s,t) &= \frac{1}{2}\int d\Pi_2\, A^{(1)}(s,k_2^2)\, A^{(0)\dagger}(s,[k_2-q]^2) \\ &+ \frac{1}{2}\int d\Pi_2\, A^{(0)}(s,k_1^2)\, A^{(1)\dagger}(s,[k_1-q]^2) \end{aligned}$$

Square Diagram

- For the first case, the tree amplitude for both sides of the square diagram is

$$A^{(1)}(s,k_2^2) = 8\pi\,\alpha_s(t^b_{mj}t^b_{nl})\left(\frac{s}{k_2^2}\right)\ell n\left(\frac{s}{\mathbf{k}_2^2}\right)\epsilon(t) \tag{63}$$

$$A^{(0)\dagger}(s,[k_2-q]^2) = 8\pi\,\alpha_s(t^a_{mi}t^a_{nk})^*\left[\frac{s}{(k_2-q)^2}\right] \tag{64}$$

which by using the relations $\ell n(s/\mathbf{k}_2^2) \simeq \ell n(s/|t|)$ $\mathcal{G}_{\text{virtual}} = (t^a t^b)_{ij}(t^a t^b)_{kl}$

one gets

$$\begin{aligned}\text{Im}A^{(2)}_{\text{virtual},\Box}(s,t) &= -\left(\frac{N_c\alpha_s^3}{\pi^2}\right)\mathcal{G}_{\text{virtual}}\, s\,\ell n\left(\frac{s}{|t|}\right)\\ &\times \int d^2\mathbf{k}_1\int d^2\mathbf{k}_2\left[\frac{1}{\mathbf{k}_1^2(\mathbf{k}_2-\mathbf{q})^2(\mathbf{k}_1-\mathbf{k}_2)^2}\right]\end{aligned}$$

Cross Diagram and Full Amplitude

- The same procedure can be done to obtain the contribution from the crossed diagram

$$\text{Im}A^{(2)}_{\text{virtual},\times}(s,t) = -\left(\frac{N_c\alpha_s^3}{\pi^2}\right)\mathcal{G}_{\text{virtual}}\, s\,\ell n\left(\frac{s}{|t|}\right) \times \int d^2\mathbf{k}_1 \int d^2\mathbf{k}_2 \left[\frac{1}{\mathbf{k}_2^2(\mathbf{k}_1-\mathbf{q})^2(\mathbf{k}_1-\mathbf{k}_2)^2}\right]$$

- Finally, the full contribution from the virtual gluon exchange in the t-channel is

$$\text{Im}A^{(2)}_{\text{virtual}}(s,t) = -\left(\frac{N_c\alpha_s^3}{\pi^2}\right)\mathcal{G}_{\text{virtual}}\, s\,\ell n\left(\frac{s}{|t|}\right)\int d^2\mathbf{k}_1 \int d^2\mathbf{k}_2 \times \left[\frac{1}{\mathbf{k}_1^2(\mathbf{k}_2-\mathbf{q})^2(\mathbf{k}_1-\mathbf{k}_2)^2} + \frac{1}{\mathbf{k}_2^2(\mathbf{k}_1-\mathbf{q})^2(\mathbf{k}_1-\mathbf{k}_2)^2}\right]$$

Color-Octet Exchange

- For color-octet exchange, one can account the contribution from the u-channel by **symmetry**
  - Remembering, in high energy the relation $s \simeq -u$ is valid;
- The u-channel contribution can be obtained from that of s-channel by the interchange:

$$t^b \leftrightarrow t^{b\prime} \qquad (t^a t^b)_{kl} \leftrightarrow (t^b t^a)_{kl} \tag{65}$$

- With this, the u-channel terms are accounted by the replacements

$$\mathcal{G}_{\text{real}} \longrightarrow \mathcal{G}'_{\text{real}} = -(t^{a'} t^a)_{ij} [t^{b'}, t^b]_{kl} \, f_{abc} \, f_{a'b'c} \tag{66}$$

$$\mathcal{G}_{\text{virtual}} \longrightarrow \mathcal{G}'_{\text{virtual}} = -(t^a t^b)_{ij} [t^a, t^b]_{kl} \tag{67}$$

Real and Virtual Contributions

- Once made the replacement one accounts for the real-gluon contribution

$$\begin{aligned}\mathrm{Im}A^{(2)}_{\underline{8},\mathrm{real}}(s,t) &= \left(\frac{2\alpha_s^3}{\pi^2}\right) 2\,\mathcal{C}^{(2)}_{\underline{8},\mathrm{real}}\,(t^a_{ij}t^a_{kl})\, s\;\ell n\left(\frac{s}{|t|}\right)\int d^2\mathbf{k}_1^2\int d^2\mathbf{k}_2^2 \\ &\times \left[\frac{\mathbf{q}^2}{\mathbf{k}_1^2\mathbf{k}_2^2(\mathbf{k}_1-\mathbf{q})^2(\mathbf{k}_2-\mathbf{q})^2} - \frac{1}{\mathbf{k}_2^2(\mathbf{k}_1-\mathbf{q})^2(\mathbf{k}_1-\mathbf{k}_2)^2}\right. \\ &- \left.\frac{1}{\mathbf{k}_1^2(\mathbf{k}_2-\mathbf{q})^2(\mathbf{k}_1-\mathbf{k}_2)^2}\right]\end{aligned}$$

and for the virtual gluon emission contribution one has

$$\begin{aligned}\mathrm{Im}A^{(2)}_{\underline{8},\mathrm{virtual}}(s,t) &= -\left(\frac{N_c\alpha_s^3}{\pi^2}\right) 2\,\mathcal{C}^{(2)}_{\underline{8},\mathrm{virtual}}\,(t^a_{ij}t^a_{kl})\, s\;\ell n\left(\frac{s}{|t|}\right)\int d^2\mathbf{k}_1\int d^2\mathbf{k}_2 \\ &\times \left[\frac{1}{\mathbf{k}_1^2(\mathbf{k}_2-\mathbf{q})^2(\mathbf{k}_1-\mathbf{k}_2)^2} + \frac{1}{\mathbf{k}_2^2(\mathbf{k}_1-\mathbf{q})^2(\mathbf{k}_1-\mathbf{k}_2)^2}\right]\end{aligned}$$

with $\mathcal{C}^{(2)}_{\underline{8},\mathrm{real}} = \left(\frac{1}{N_c^2-1}\right)\mathcal{P}^{ij}_{lk}(\underline{8})\quad \mathcal{G}'_{\mathrm{real}} = \frac{N_c^2}{8}$ and $\mathcal{C}^{(2)}_{\underline{8},\mathrm{virtual}} = \left(\frac{1}{N_c^2-1}\right)\mathcal{P}^{ij}_{lk}(\underline{8})\quad \mathcal{G}'_{\mathrm{virtual}} = -\frac{N_c}{4}$.

Full Contribution

- Computing the full contribution for Color-Octet exchange

$$\text{Im}A_{\underline{8}}^{(2)}(s,t) = \text{Im}A_{\underline{8},\text{real}}^{(2)}(s,t) + \text{Im}A_{\underline{8},\text{virtual}}^{(2)}(s,t)$$
$$= \left(\frac{N_c^2\alpha_s^3}{2\pi^3}\right)(t_{ij}^a t_{kl}^a)\, s\, \ell n\left(\frac{s}{|t|}\right)\int d^2\mathbf{k}_1 d^2\mathbf{k}_2 \left[\frac{\mathbf{q}^2}{\mathbf{k}_1^2\mathbf{k}_2^2(\mathbf{k}_1-\mathbf{q})^2(\mathbf{k}_2-\mathbf{q})^2}\right] \quad (68)$$

which can be rewritten as

$$\text{Im}A_{\underline{8}}^{(2)}(s,t) = 8\pi^2\alpha_s(t_{ij}^a t_{kl}^a)\left(\frac{s}{|t|}\right)\ell n\left(\frac{s}{|t|}\right)\epsilon^2(t) \quad (69)$$

where $\epsilon^2(t) = \left(\frac{N_c\alpha_s}{4\pi^2}\right)^2 \int d^2\mathbf{k}\left[\frac{-\mathbf{q}^2}{\mathbf{k}^2(\mathbf{k}-\mathbf{q})^2}\right]$.

- Via dispersion relations one gets the leading $\ell n\, s$ contribution

$$A_{\underline{8}}^{(2)}(s,t) = 4\pi\,\alpha_s\,(t_{ij}^a t_{kl}^a)\left(\frac{s}{t}\right)\ell n^2\left(\frac{s}{|t|}\right)\epsilon^2(t) \equiv \left(\frac{1}{2}\right)\epsilon^2(t)\,\ell n^2\left(\frac{s}{|t|}\right)A_{\underline{8}}^{(0)} \quad (70)$$

which is **real**.

Amplitude in order $\mathcal{O}(\alpha_s^3)$

- Joining the three contributions one finds the Full Amplitude in the **LLA limit**

$$A_{\underline{8}}(s,t) = 8\pi\,\alpha_s\left(\frac{s}{t}\right)\,(t^a_{ij}t^a_{kl})\left[1+\epsilon(t)\ln\left(\frac{s}{|t|}\right)+\frac{1}{2}\,\epsilon^2(t)\ln^2\left(\frac{s}{|t|}\right)+\ldots\right] \tag{71}$$

which corresponds to the first three terms in the expansion of

$$A_{\underline{8}}(s,t) = 8\pi\alpha_s\,(t^a_{ij}t^a_{kl})\,\frac{s}{t}\left(\frac{s}{|t|}\right)^{\epsilon(t)} \equiv 8\pi\alpha_s\,(t^a_{ij}t^a_{kl})\left(\frac{s}{|t|}\right)^{\alpha_g(t)} \tag{72}$$

where the quantity

$$\boxed{\alpha_g(t) = 1+\epsilon(t)} \tag{73}$$

is the constructed **reggeized gluon trajectory** in the t-channel. $\rightarrow$ **Not the Pomeron yet!**

Contribution of the Color-singlet Exchange

- For completeness and following what was done in the color-octet case one accounts the leading $\ell n\, s$ contribution in the color-singlet exchange

$$
\begin{aligned}
A^{(2)}_{\underline{1},\text{real}}(s,t) &= i\left(\frac{2\,\alpha_s^3}{\pi^2}\right)\mathcal{C}^{(2)}_{\underline{1},\text{real}}\,\delta_{ij}\delta kl\, s\,\ell n\left(\frac{s}{|t|}\right)\int d^2\mathbf{k}_1\int d^2\mathbf{k}_2 \\
&\times \left[\frac{\mathbf{q}^2}{\mathbf{k}_1^2\mathbf{k}_2^2(\mathbf{k}_1-\mathbf{q})^2(\mathbf{k}_2-\mathbf{q})^2} - \frac{1}{\mathbf{k}_2^2(\mathbf{k}_1-\mathbf{q})^2(\mathbf{k}_1-\mathbf{k}_2)^2}\right. \\
&- \left.\frac{1}{\mathbf{k}_1^2(\mathbf{k}_2-\mathbf{q})^2(\mathbf{k}_1-\mathbf{k}_2)^2}\right]
\end{aligned}
$$

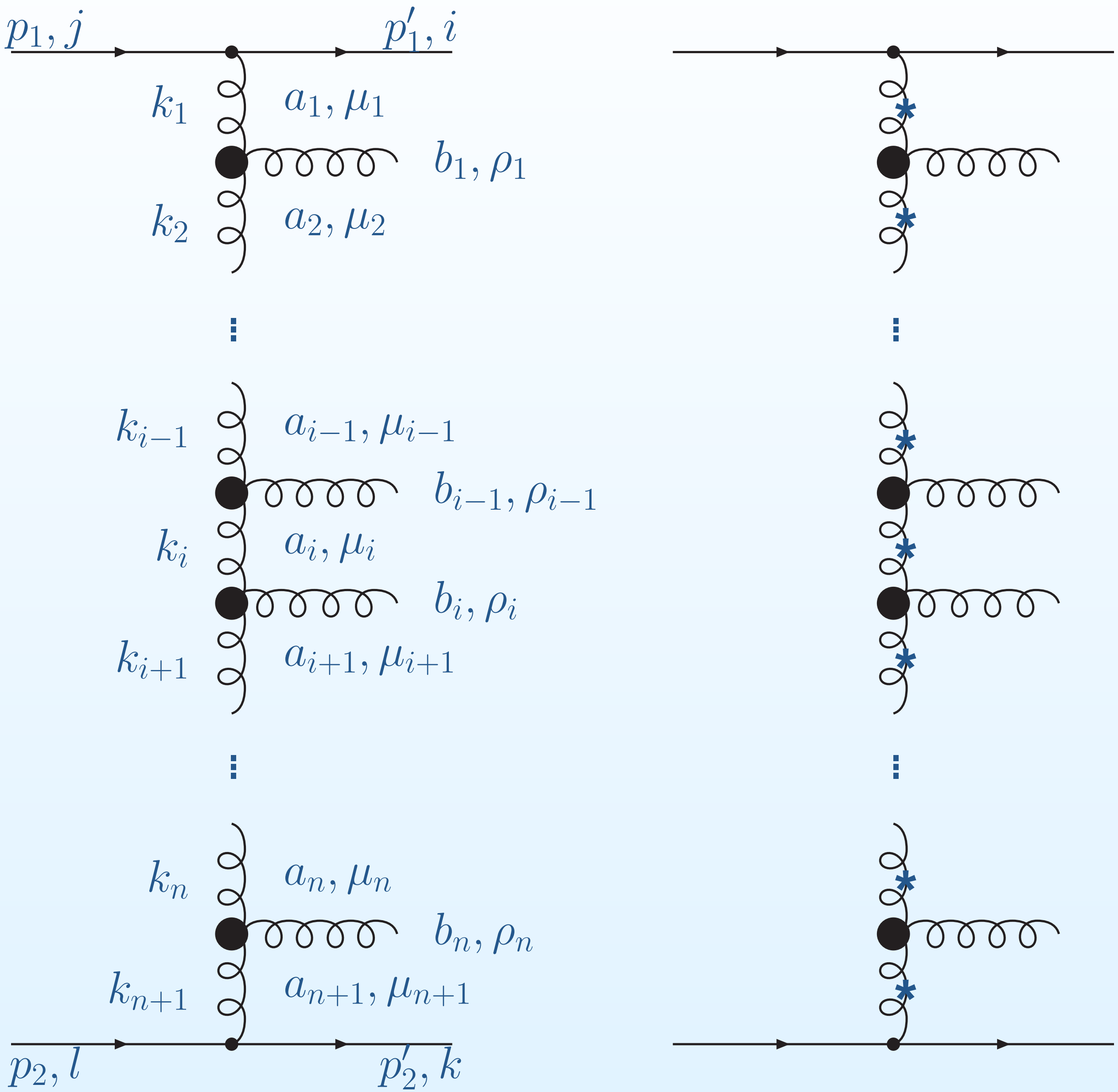
$$
\begin{aligned}
A^{(2)}_{\underline{1},\text{virtual}}(s,t) &= -i\left(\frac{N_c\,\alpha_s^3}{\pi^2}\right)\mathcal{C}^{(2)}_{\underline{1},\text{virtual}}\,\delta_{ij}\delta kl\, s\,\ell n\left(\frac{s}{|t|}\right)\int d^2\mathbf{k}_1\int d^2\mathbf{k}_2 \\
&\times \left[\frac{1}{\mathbf{k}_2^2(\mathbf{k}_1-\mathbf{q})^2(\mathbf{k}_1-\mathbf{k}_2)^2} + \frac{1}{\mathbf{k}_1^2(\mathbf{k}_2-\mathbf{q})^2(\mathbf{k}_1-\mathbf{k}_2)^2}\right]
\end{aligned}
$$

where $\mathcal{C}^{(2)}_{\underline{1},\text{real}} = \mathcal{P}^{ij}_{kl}(\underline{1})\quad \mathcal{G}_{\text{real}} = -\frac{N_c^2-1}{4}$ and $\mathcal{C}^{(2)}_{\underline{1},\text{virtual}} = \mathcal{P}^{ij}_{kl}(\underline{1})\quad \mathcal{G}_{\text{virtual}} = -\frac{N_c^2-1}{4N_c}$.

BFKL Ladder

- Previously we introduced the Eikonal Approximation which has an important **property**:
  - Independence on the spin of the particle which emits the soft gluon!
- Extending the process for n gluon exchanges in the s-channel:
  - Can be constructed a diagram like a **ladder** with n 'rungs' or n gluon exchanges;
  - It is considered an exchange of n reggeized gluons in t-channel.
- The algebra will be done in the **Multi-Regge Kinematics**
  - It will yield the **leading** $\ell n\, s$ contributions.
- The procedure will be to account with the mathematical tools calculated previously.
  - Tree amplitudes, Sudakov variables, phase space, . . .

BFKL Ladder



p_1, j
p'_1, i
k_1
a_1, μ_1
b_1, ρ_1
k_2
a_2, μ_2
k_{i-1}
a_{i-1}, μ_{i-1}
b_{i-1}, ρ_{i-1}
k_i
a_i, μ_i
b_i, ρ_i
k_{i+1}
a_{i+1}, μ_{i+1}
k_n
a_n, μ_n
b_n, ρ_n
k_{n+1}
a_{n+1}, μ_{n+1}
p_2, l
p'_2, k

'Multi-Regge' Kinematics

- Like before it can be introduced the Sudakov parametrization

$$k_i = \alpha_i p_1 + \beta_i p_2 + k_{i\perp} \qquad (i = 1,\, 2,\, \ldots,\, n+1) \tag{74}$$

- The Multi-Regge regime corresponds to
  - All transverse momenta being of the **same order**

$$s \gg \mathbf{k}_1^2 \simeq \mathbf{k}_2^2 \simeq \ldots \simeq \mathbf{k}_n^2 \simeq \mathbf{k}_{n+1}^2 \simeq \mathbf{q}^2 \tag{75}$$

  - **Strong ordering** of the longitudinal momenta

$$1 \gg \alpha_1 \gg \alpha_2 \gg \quad \ldots \quad \gg \alpha_{n+1} \gg \frac{\mathbf{q}^2}{s}$$

$$1 \gg |\beta_{n+1}| \gg |\beta_n| \gg \quad \ldots \quad \gg \beta_2 \gg |\beta_1| \gg \frac{\mathbf{q}^2}{s}$$

- These <u>two properties</u> will be important in featuring the ladder further on.

Tree Amplitude

- From the gluon exchange one can obtain the amplitude for n gluons emitted

$$
\begin{aligned}
iA^{2\to n+2}_{\rho_1\,\dots\,\rho_n} &= (-2ig_s)p_1^{\mu_1}t^{a_1}_{ij}\left(-\frac{i}{k_1^2}\right) \\
&\times \quad g_s\, f_{a_1 a_2 b_1}\Gamma^{\rho_1}_{\mu_1\mu_2}(k_1,\, k_2)\left(-\frac{i}{k_2^2}\right) \\
&\times \quad g_s\, f_{a_2 a_3 b_2}\Gamma^{\mu_2\rho_2}_{\mu_3}(k_2,\, k_3)\left(-\frac{i}{k_3^2}\right) \\
&\vdots \\
&\times \quad g_s\, f_{a_n a_{n+1} b_n}\Gamma^{\mu_n\rho_n}_{\mu_{n+1}}(k_n,\, k_{n+1})\left(-\frac{i}{k_{n+1}^2}\right) \\
&\times \quad (-2ig_2)p_2^{\mu_{n+1}}t^{a_{n+1}}_{kl}
\end{aligned}
$$

- Note that we are treating the process through n **effective diagrams** attached to each other.

C Vector

- Using the relation between the Lipatov vertices

$$
\begin{aligned}
p_1^{\mu_1}\,\Gamma^{\rho_1}_{\mu_1\mu_2}(k_1,\,k_2)\,\Gamma^{\mu_2\,\rho_2}_{\mu_3}(k_2,\,k_3) \quad \ldots \quad \Gamma^{\mu_n\,\rho_n}_{\mu_{n+1}}(k_n,\,k_{n+1})\,p_2^{\mu_{n+1}} &= \\
&= \left(\frac{s}{2}\right) C^{\rho_1}(k_1,\,k_2)\,C^{\rho_2}(k_2,\,k_3)\,\ldots\,C^{\rho_n}(k_n,\,k_{n+1}) \\
&= \left(\frac{s}{2}\right) \prod_{i=1}^{n} C^{\rho_i}(k_i, k_{i+1})
\end{aligned}
$$

where it was defined the C vector as

$$
C^{\rho}(k_i,\,k_{i+1}) = \left(\alpha_i + \frac{2\mathbf{k}_i^2}{\beta_i s}\right) p_1^{\rho} + \left(\beta_{i+1} + \frac{2\mathbf{k}_{i+1}^2}{\alpha_i s}\right) p_2^{\rho} - (\mathbf{k}_i^{\rho} + \mathbf{k}_{i+1}^{\rho}) \tag{76}
$$

and it is related to the **Lipatov vertex** through the relation $\Gamma^{\rho}_{\mu\nu} = \left(\frac{2}{s}\right) p_{2\mu}\,p_{1\nu}\,C^{\rho}$

Scattering Amplitude

- So that, the amplitude can be rewritten as

$$
\begin{aligned}
A_{2\rightarrow n+2}^{\rho_1 \cdots \rho_n} &= 2\, i\, s\, g_s\, t_{ij}^{a_1} \left(\frac{i}{\mathbf{k}_1^2}\right) \\
&\times \quad g_s f_{a_1 a_2 b_1} C^{\rho_1}(k_1,\, k_2) \left(\frac{i}{\mathbf{k}_2^2}\right) \\
&\times \quad g_s f_{a_2 a_3 b_2} C^{\rho_2}(k_2,\, k_3) \left(\frac{i}{\mathbf{k}_3^2}\right) \\
&\vdots \\
&\times \quad g_s f_{a_n a_{n+1} b_n} C^{\rho_n}(k_n,\, k_{n+1}) \left(\frac{i}{\mathbf{k}_{n+1}^2}\right) \\
&\times \quad g_s\, t_{kl}^{a_{n+1}}
\end{aligned}
$$

- However this is only the tree amplitude!
- It does not take into account **virtual radiative corrections** in the t-channel.

Radiative Corrections

- It was proposed an <u>ansatz</u> by **Lipatov**, **Kuraev** and **Fadin**
  - A *modification* in the gluon propagator in the t-channel to account for these corrections in all orders in α_s
  - Modification proposed inspired in Regge Theory

$$-\frac{i}{k_i^2} \to -\frac{i}{k_i^2}\left(-\frac{s_i}{k_i^2}\right)^{\epsilon(k_i^2)} \simeq -\frac{i}{k_i^2}\left(\frac{\alpha_i - 1}{\alpha_i}\right)^{\epsilon(k_i^2)} \qquad (77)$$

where $s_i = (k_{i-1} - k_{i+1})^2 \simeq \left(\frac{\alpha_i - 1}{\alpha_i}\right)(\mathbf{k}_i - \mathbf{k}_{i+1})^2$

is the center-of-mass energy in the i-th section of the ladder, and

$$\epsilon(k_i^2) = \frac{N_c \alpha_s}{4\pi^2} \int d^2\mathbf{h} \left[\frac{-\mathbf{k}_i^2}{\mathbf{h}^2(\mathbf{h} - \mathbf{k}_i)^2}\right] \qquad (78)$$

is the dimensionless function already seen before with an auxiliary vector $\mathbf{h}$.

Reggeized Gluon

- Then, in the Feynman Gauge, the modified gluon propagator is

$$D_{\mu\nu}(s_i, k_i^2) = -\frac{i\, g_{\mu\nu}}{k_i^2} \left(\frac{s}{\mathbf{k}^2}\right)^{\epsilon(t)} \tag{79}$$

$\Rightarrow$ **Radiative Corrections are directly included in the propagator.**

- Exemplifying for the elastic qq scattering one obtains

$$A(s,t) = 8\pi\alpha_s (t^a_{ij} t^a_{kl}) \left(\frac{s}{t}\right) \left(\frac{s}{|t|}\right)^{\epsilon(t)}$$

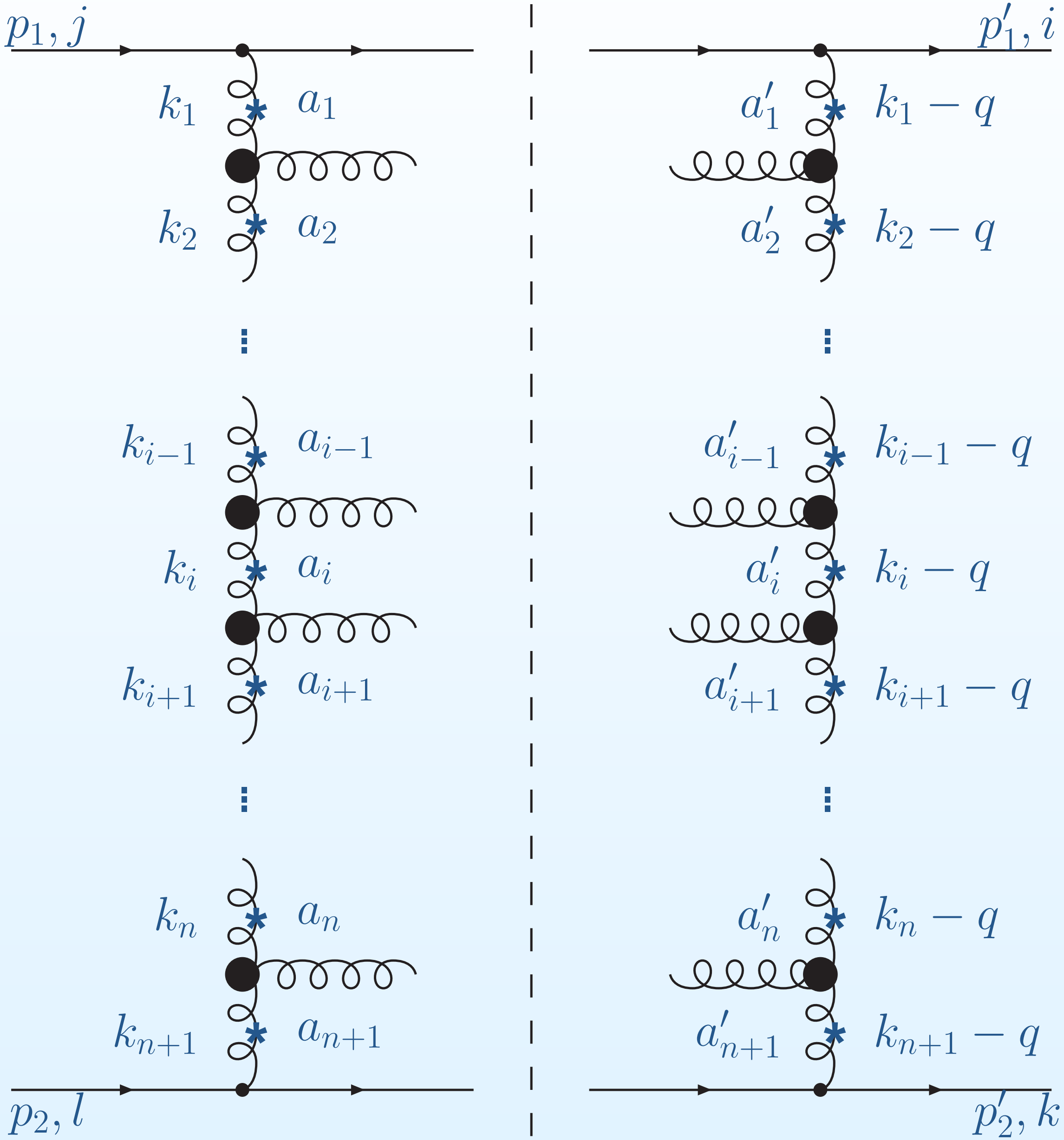
which coincides with the result obtained in the LLA amplitude expansion.

'Reggeized' BFKL Ladder

- Rewriting the tree amplitude with the modified propagator one gets

$$
\begin{aligned}
A_{2\to n+2}^{\rho_1\cdots\rho_n} &= 2i\,s\,g_s\,t_{ij}^{a_1}\left(\frac{i}{\mathbf{k}_1^2}\right)\left(\frac{1}{\alpha_1}\right)^{\epsilon(k_1^2)} \\
&\times\; g_s f_{a_1 a_2 b_1} C^{\rho_1}(k_1,\,k_2)\left(\frac{i}{\mathbf{k}_2^2}\right)\left(\frac{\alpha_1}{\alpha_2}\right)^{\epsilon(k_2^2)} \\
&\vdots \\
&\times\; g_s f_{a_n a_{n+1} b_n} C^{\rho_n}(k_n,\,k_{n+1})\left(\frac{i}{\mathbf{k}_{n+1}^2}\right)\left(\frac{\alpha_n}{\alpha_{n+1}}\right)^{\epsilon(k_{n+1}^2)}\cdot g_s\,t_{kl}^{a_{n+1}} \\
&= 2i\,s\,g_s^2\,(t_{ij}^{a_1}t_{kl}^{a_{n+1}})\left(\frac{i}{\mathbf{k}_1^2}\right)\left(\frac{1}{\alpha_1}\right)^{\epsilon(k_1^2)} \\
&\times\; \prod_{i=1}^{n}\left\{g_s\,f_{a_i a_{i+1} b_i} C^{\rho_i}(k_i,\,k_{i+1})\left(\frac{i}{\mathbf{k}_{i+1}^2}\right)\left(\frac{\alpha_i}{\alpha_{i+1}}\right)^{\epsilon(k_{i+1}^2)}\right\}
\end{aligned}
$$

Gluon Ladder



Imaginary Ladder Amplitude

- Carrying out the contraction of the C's vectors one gets the imaginary part of the scattering amplitude of the Gluon Ladder

$$
\begin{aligned}
\mathrm{Im}\,\mathcal{A}_R(s,t) &= \frac{1}{2}\sum_{n=0}^{\infty} 4s^2 g_s^4\, \mathcal{G}_R \int d\Pi_{n+2} \left[\frac{1}{\mathbf{k}_1^2(\mathbf{k}_1-\mathbf{q})^2}\right]\left(\frac{1}{\alpha_1}\right)^{\epsilon(k_1^2)+\epsilon([k_1-q]^2)} \\
&\times \prod_{i=1}^{n}\left\{\left[\frac{g_s^2}{\mathbf{k}_{i+1}^2(\mathbf{k}_{i+1}-\mathbf{q})^2}\right](-2\eta_R)\,K(\mathbf{k}_i,\,\mathbf{k}_{i+1})\right. \\
&\times \left.\left(\frac{\alpha_i}{\alpha_{i+1}}\right)^{\epsilon(\mathbf{k}_{i+1}^2)+\epsilon([\mathbf{k}_{i+1}-\mathbf{q}]^2)}\right\}, \qquad (80)
\end{aligned}
$$

where

$$
\begin{cases}
\mathcal{G}_{\underline{1}} = \frac{N_c^2-1}{4N_c} & \mathcal{G}_{\underline{8}} = -\frac{N_c}{8} \\
\eta_{\underline{1}} = \frac{N_c}{2} & \eta_{\underline{8}} = N_c
\end{cases} \qquad (81)
$$

Mellin Transform

- The full Amplitude can be obtained using the Dispersion Relation as made before;
- We'll adopt a new proposal that suggests work in the **complex angular momentum plane**!
- Instead of working directly with $\mathcal{A}_R$, it will be convenient to calculate its Mellin transform

$$f_R = \int_1^\infty d\left(\frac{s}{|t|}\right)\left(\frac{s}{|t|}\right)^{-\omega-1}\frac{\text{Im}\mathcal{A}_R(s,t)}{s} \quad (82)$$

in the **Froissart-Gribov** representation of the partial-wave amplitude.

- The inverse Mellin transform is

$$\frac{\text{Im}\mathcal{A}_R(s,t)}{s} = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} d\omega \left(\frac{s}{|t|}\right)^{\omega} f_R(\omega,t) \quad (83)$$

Watson-Sommerfeld Transform

- One can take the u-channel contribution using the property

$$\text{Im}\mathcal{A}_R(s,t) = -\xi_R\,\text{Im}\mathcal{A}_R(u,t) \tag{84}$$

- The quantities ξ_R are the signatures defined as

$$\xi_{\underline{1}} = +1 \quad \xi_{\underline{8}} = -1 \tag{85}$$

- Since $u \simeq -s$, the u-channel term is taken into account by the replacement

$$f_R(\omega,t) \longrightarrow (1 + \xi_R\, e^{-i\pi\omega})\; f_R(\omega,t) \tag{86}$$

- The partial-wave amplitude $f_R(\omega,t)$ is related to the amplitude $\mathcal{A}_R$ by the relation

$$A_R(s,t) = -\frac{1}{4\pi i}\int_{c-i\infty}^{c+i\infty} d\omega\,\left(\frac{s}{|t|}\right)^{\omega+1}\left[\frac{\xi_R - e^{-i\pi\omega}}{\sin\pi\omega}\right] f_R(\omega,t) \tag{87}$$

which is called the **Watson-Sommerfeld Transform** .

Phase Space

- Starting the calculation of the BFKL Equation, one takes the $(n+2)-$body phase space

$$
\begin{aligned}
d\Pi_{n+2} &= \frac{s^{n+1}}{2^{n+1}(2\pi)^{3n+2}} \int \prod_{i=1}^{n+1} d\alpha_i \, d\beta_i \, d^2\mathbf{k}_i \\
&\times \delta(-\beta_1[1-\alpha_1]s - \mathbf{k}_1^2)\, \delta(\alpha_{n+1}[1+\beta_{n+1}]s - \mathbf{k}_{n+1}^2) \\
&\times \prod_{j=1}^{n} \delta([\alpha_j - \alpha_{j+1}][\beta_j - \beta_{j+1}]s - [\mathbf{k}_j - \mathbf{k}_{j+1}]^2)
\end{aligned} \tag{88}
$$

which in the Multi-Regge kinematics is simplified to

$$
\begin{aligned}
d\Pi_{n+2} &= \frac{1}{2^{n+1}(2\pi)^{3n+2}} \prod_{i=1}^{n} \int_{\alpha_{i+1}}^{1} \frac{d\alpha_i}{\alpha_i} \int_0^1 d\alpha_{n+1} \\
&\times \prod_{j=1}^{n+1} \int d^2\mathbf{k}_j \, \delta(\alpha_{n+1}s - \mathbf{k}^2)
\end{aligned} \tag{89}
$$

Partial-Wave Amplitude

- Computing the amplitude using the Mellin transform one can find

$$
\begin{aligned}
f_R(\omega, \mathbf{q}^2) &= (4\pi\alpha_s)^2\, \mathcal{G}_R \sum_{n=0}^{\infty} \prod_{i=1}^{n+1} \frac{d^2\mathbf{k}_i}{(2\pi)^2} \\
&\times \frac{1}{\mathbf{k}_1^2(\mathbf{k}_1-\mathbf{q})^2} \frac{1}{\omega - \epsilon(\mathbf{k}_1^2) - \epsilon([\mathbf{k}_1-\mathbf{q}]^2)} \\
&\times (-2\alpha_s\eta_R)\, K(\mathbf{k}_1, \mathbf{k}_2) \\
&\times \frac{1}{\mathbf{k}_2^2(\mathbf{k}_2-\mathbf{q})^2} \frac{1}{\omega - \epsilon(\mathbf{k}_2^2) - \epsilon([\mathbf{k}_2-\mathbf{q}]^2)} \\
&\vdots \\
&\times (-2\alpha_s\eta_R)\, K(\mathbf{k}_n, \mathbf{k}_{n+1}) \\
&\times \frac{1}{\mathbf{k}_{n+1}^2(\mathbf{k}_{n+1}-\mathbf{q})^2} \frac{1}{\omega - \epsilon(\mathbf{k}_{n+1}^2) - \epsilon([\mathbf{k}_{n+1}-\mathbf{q}]^2)} \qquad (90)
\end{aligned}
$$

The BFKL Equation

- Writting the amplitude in the recursive form

$$f_R(\omega, \mathbf{q}^2) = (4\pi\alpha_s)^2 \, \mathcal{G}_R \int \frac{d^2\mathbf{k}}{(2\pi)^2} \frac{\mathcal{F}_R(\omega, \mathbf{k}, \mathbf{q})}{\mathbf{k}^2(\mathbf{k}-\mathbf{q})^2} \tag{91}$$

that is

$$[\omega - \epsilon(-\mathbf{k}^2) - \epsilon(-[\mathbf{k}-\mathbf{q}]^2)] \, \mathcal{F}_R(\omega, \mathbf{k}, \mathbf{q}) = 1 - \frac{2\alpha_s N_c}{4\pi^2} \int d^2\mathbf{h} \left[\frac{K(\mathbf{k}, \mathbf{h})}{\mathbf{h}^2(\mathbf{x}-\mathbf{q})^2} \right] \mathcal{F}_R(\omega, \mathbf{h}, \mathbf{q}) \tag{92}$$

- This is the general form of the **BFKL Equation**:

This equation describes the evolution of the gluon ladder in the LL$_x$A limit.

Color-Octet from BFKL Equation

- Using explicitly the expressions for the reggeized gluon trajectories as seen before

$$\epsilon(-\mathbf{k}^2) = -\frac{N_c\alpha_s}{4\pi^2}\int d^2\mathbf{h}\left[\frac{-\mathbf{k}^2}{\mathbf{h}^2(\mathbf{h}-\mathbf{k})^2}\right] \quad (93a)$$

$$\epsilon(-[\mathbf{k}-\mathbf{q}]^2) = -\frac{N_c\alpha_s}{4\pi^2}\int d^2\mathbf{h}\left[\frac{(\mathbf{k}-\mathbf{q})^2}{(\mathbf{h}-\mathbf{q})^2(\mathbf{k}-\mathbf{h})^2}\right] \quad (93b)$$

this terms will cancel with those of the expression $K(\mathbf{k},\mathbf{h})$ related to the virtual corrections (ϵ's terms) and yielding

$$\omega\,\mathcal{F}_{\underline{8}}(\omega,\mathbf{k},\mathbf{q}) = 1 - \frac{N_c\,\alpha_s}{4\pi^2}\int d^2\mathbf{h}\left[\frac{\mathbf{q}^2}{\mathbf{h}^2\,(\mathbf{h}-\mathbf{q})^2}\right]\mathcal{F}_{\underline{8}}(\omega,\mathbf{h},\mathbf{q}) \quad (94)$$

which admits the $\mathbf{k}$-independent solution

$$\mathcal{F}_{\underline{8}} = \frac{1}{\omega-\epsilon(-\mathbf{q}^2)} \quad (95)$$

Octet Partial-Wave Amplitude

- From the recursive relation we get

$$f_{\underline{8}}(\omega, \mathbf{q}^2) = 2\pi^2 \alpha_s \left[\frac{\epsilon(-\mathbf{q}^2)}{\mathbf{q}^2} \right] \frac{1}{\omega - \epsilon(-\mathbf{q}^2)} \tag{96}$$

- In terms of the complex angular momentum $\ell \equiv \omega + 1$, the octet partial-wave amplitude behaves as

$$f_{\underline{8}}(\ell, t) \sim \frac{1}{\ell - \alpha_g(t)} \tag{97}$$

where the $\alpha_g(t) = 1 + \epsilon(t)$.

- We can see that $f_{\underline{8}}(\ell, t)$ has a **pole singularity** as $\ell = \alpha_g(t)$.

- Computing the inverse Mellin transform one gets the imaginary part of the amplitude

$$\mathrm{Im}\mathcal{A}_{\underline{8}}(s, t) = 2\pi^2 \alpha_s \epsilon(t) \left(\frac{s}{|t|} \right)^{1+\epsilon(t)} \tag{98}$$

Color-Octet Amplitude

- Taking the total amplitude from dispersion relations and adding the u-channel contribution we obtain the full amplitude for the color-octet exchange

$$A_{\underline{8}}(s,t) = -4\pi\,\alpha_s\,(t^a_{ij}t^a_{kl})\left[1 - e^{-i\pi\alpha_g(t)}\right]\left(\frac{s}{|t|}\right)^{\alpha_g(t)} \qquad (99)$$

  which is the **Regge-type amplitude** for the qq elastic scattering.

- In the Multi-Regge regime one can approximate $\alpha_g(t) \simeq 1$

$$A_{\underline{8}}(s,t) \simeq -8\pi\,\alpha_s\,(t^a_{ij}t^a_{kl})\left(\frac{s}{|t|}\right)^{\alpha_g(t)} \qquad (100)$$

  which coincides with the result obtained from *one-loop exchange* and justifies the **ansatz** proposed previously.

Color-Singlet from BFKL Equation

- The Gluon ladder in color-singlet configuration contributes directly to the **QCD Pomeron**!

- For this configuration we can rewrite the BFKL equation as

$$
\begin{aligned}
[\omega - \epsilon(-\mathbf{k}^2) \quad &- \quad \epsilon(-[\mathbf{k}-\mathbf{q}]^2)]\; F(\omega, \mathbf{k}, \mathbf{k}', \mathbf{q}) = \\
&= \quad \delta^2(\mathbf{k}-\mathbf{k}') - \frac{\alpha_s N_c}{2\pi^2} \int d^2\mathbf{h} \left[\frac{K(\mathbf{k},\mathbf{h})}{\mathbf{h}^2(\mathbf{h}-\mathbf{q})^2} \right] F(\omega, \mathbf{h}, \mathbf{k}', \mathbf{q})
\end{aligned}
$$

- We can introduce the function $F(\omega, \mathbf{k}, \mathbf{k}', \mathbf{q})$ related to $\mathcal{F}_{\underline{1}}(\omega, \mathbf{k}, \mathbf{q})$ by

$$
\mathcal{F}_{\underline{1}}(\omega, \mathbf{k}, \mathbf{q}) = \int \frac{d^2\mathbf{k}'}{\mathbf{k}'^2}\, \mathbf{k}^2\, F(\omega, \mathbf{k}, \mathbf{k}', \mathbf{q}) \tag{101}
$$

The Color-Singlet BFKL Equation

- With some algebra we get

$$
\begin{aligned}
\omega\ F(\omega,\mathbf{k},\mathbf{k}',\mathbf{q}) &= \delta^2(\mathbf{k}-\mathbf{k}') \\
&+ \frac{\alpha_s N_c}{2\pi^2}\int d^2\mathbf{h}\left\{\left(\frac{-\mathbf{q}^2}{(\mathbf{h}-\mathbf{q})^2\,\mathbf{k}^2}\right)F(\omega,\mathbf{h},\mathbf{k}',\mathbf{q})\right. \\
&+ \frac{1}{(\mathbf{h}-\mathbf{k})^2}\left[F(\omega,\mathbf{h},\mathbf{k}',\mathbf{q}) - \frac{\mathbf{k}^2 F(\omega,\mathbf{h},\mathbf{k}',\mathbf{q})}{\mathbf{h}^2+(\mathbf{k}-\mathbf{h})^2}\right] \\
&+ \left.\frac{1}{(\mathbf{h}-\mathbf{k})^2}\left[\frac{(\mathbf{k}-\mathbf{q})^2\,\mathbf{h}^2\,F(\omega,\mathbf{h},\mathbf{k}',\mathbf{q})}{(\mathbf{h}-\mathbf{q})^2\mathbf{k}^2} - \frac{(\mathbf{k}-\mathbf{q})^2\,F(\omega,\mathbf{h},\mathbf{k}',\mathbf{q})}{(\mathbf{h}-\mathbf{q})^2+(\mathbf{k}-\mathbf{h})^2}\right]\right\}
\end{aligned}
$$

- This is the standard form of the **color-singlet BFKL equation**.
- From this solution one can find the inverse Mellin transform as

$$
F(s,\mathbf{k},\mathbf{k}',\mathbf{q}) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} d\omega\left(\frac{s}{|t|}\right)^{\omega} F(\omega,\mathbf{k},\mathbf{k}',\mathbf{q}) \tag{102}
$$

Some Properties

- Analyzing the BFKL equation for the color-singlet case we see:
  - **Ultraviolet** finite in the limits $\mathbf{h}^2 \to \infty$ and $\mathbf{k}^2 \to \infty$;
  - **Infrared** divergences:
    - Regular **infrared** behavior for $\mathbf{h} \to 0$ and $\mathbf{k} = \mathbf{h}$;
      - The singularities that arise from $1/(\mathbf{h} - \mathbf{k})^2$ are cancelled by the other terms!
    - Problem in the infrared case:
      - Singularities from the virtual-gluon terms when $\mathbf{k}^2 \to 0$;
    - **Answer** to this problem (thanks to **Lipatov**)
    - A Colorless particle has quarks and gluons confined and it regulates the divergences!
      - The confinement limits the quarks and gluons to be **on-mass shell**!

The Integro-Differential Equation

- We can write the BFKL Equation for zero momentum transfer, so

$$\omega\, F(\omega, \mathbf{k}, \mathbf{k}', \mathbf{q}) = \delta^2(\mathbf{k} - \mathbf{k}') + \int d^2\mathbf{h}\, \mathcal{K}(\mathbf{k}, \mathbf{h})\, F(\omega, \mathbf{h}, \mathbf{k}') \tag{103}$$

where the function $\mathcal{K}$ is called "**BFKL kernel**“ and has the form

$$\begin{aligned} \mathcal{K}(\mathbf{k}, \mathbf{h}) &= \mathcal{K}_{\text{virtual}}(\mathbf{k}, \mathbf{h}) + \mathcal{K}_{\text{real}}(\mathbf{k}, \mathbf{h}) \\ &= 2\epsilon(-\mathbf{k}^2)\, \delta^2(\mathbf{k} - \mathbf{h}) + \left(\frac{N_c \alpha_s}{\pi^2}\right) \frac{1}{(\mathbf{k} - \mathbf{h})^2} \end{aligned} \tag{104}$$

- Expressing the BFKL Equation with the inverse Mellin transform we get

$$\begin{aligned} \frac{\partial F(s, \mathbf{k}, \mathbf{k}')}{\partial\, \ell n(s/\mathbf{k}^2)} &= \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} d\omega \left(\frac{s}{\mathbf{k}^2}\right)^{\omega} \omega\, F(\omega, \mathbf{k}, \mathbf{k}') \\ &= \frac{N_c\, \alpha_s}{\pi^2} \int \frac{d^2\mathbf{h}}{(\mathbf{k} - \mathbf{h})^2} \left[F(s, \mathbf{h}, \mathbf{k}') - \left(\frac{\mathbf{k}^2}{\mathbf{h}^2 + (\mathbf{k} - \mathbf{h})^2}\right) F(s, \mathbf{k}, \mathbf{k}') \right] \end{aligned} \tag{105}$$

which describes the evolution of the BFKL amplitude $F(s, \mathbf{k}, \mathbf{k}')$.

Eigenvalues of $\mathcal{K}$

- In order to solve the BFKL equation for zero momentum transfer, it can be done rewriting

$$\omega F = 1 + \mathcal{K} \otimes F \tag{106}$$

- Solving this equation, one founds the eigenfunctions ϕ_α of $\mathcal{K}$

$$\mathcal{K} \otimes \phi_\alpha = \omega_\alpha \, \phi_\alpha . \tag{107}$$

- Some algebra leads to an expression for the eigenfunctions of $\mathcal{K}$

$$\phi_{n\nu}(|\mathbf{k}|, \vartheta) = \frac{1}{\pi\sqrt{2}} (\mathbf{k}^2)^{-\frac{1}{2}+i\nu} e^{-n\vartheta} \tag{108}$$

and the eigenvalues can be obtained from this expression as

$$\omega_n(\nu) = \frac{2\,\alpha_s N_c}{\pi} \,\text{Re} \int_0^1 dx \left[\frac{x^{\frac{|n|+1}{2}-i\nu} - 1}{1-x} \right] = -\frac{2\,\alpha_s N_c}{\pi} \,\text{Re} \left[\psi\left(\frac{|n|+1}{2} + i\nu \right) - \psi(1) \right] \tag{109}$$

where the function ψ is the Digamma function such that $\psi(1) = -\gamma_E = -0.577215\ldots$.

Solution for $t = 0$

- The solution of the BFKL equation for zero momentum transfer reads

$$F(\omega,\mathbf{k},\mathbf{k}') = \frac{1}{2\pi^2(\mathbf{k}^2\mathbf{k}'^2)^{\frac{1}{2}}} \sum_{n=0}^{\infty} e^{in(\vartheta-\vartheta')} \int_{-\infty}^{+\infty} d\nu \left[\frac{e^{i\nu\,\ell n\left(\frac{\mathbf{k}^2}{\mathbf{k}'^2}\right)}}{\omega-\omega_n(\nu)} \right] \tag{110}$$

- The leading $\ell n\, s$ behavior of $F(s,\mathbf{k},\mathbf{k}',\mathbf{q})$ retain only the contribution from $n = 0$

$$\omega_0(\nu) \simeq \lambda - \frac{1}{2}\lambda'\nu^2 \tag{111}$$

- This result lead us to the LLA pomeron solution of the **BFKL Equation**

$$F(s,\mathbf{k},\mathbf{k}') = \frac{1}{\sqrt{2\pi^3\lambda'\mathbf{k}^2\mathbf{k}'^2}} \left(\frac{1}{\sqrt{ln(s/\mathbf{k}^2)}} \right) \left(\frac{s}{\mathbf{k}^2}\right)^{\lambda} exp\left[\frac{\ell n^2(\mathbf{k}^2/\mathbf{k}'^2)}{2\lambda'\,\ell n(s/\mathbf{k}^2)} \right] \tag{112}$$

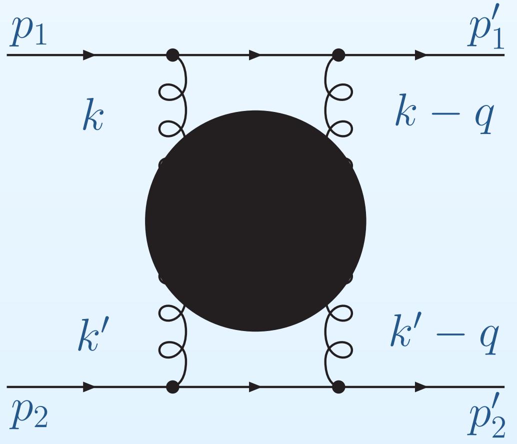
Applications: $qq \longrightarrow qq$

- Applying the result to the quark-quark scattering it gives us

$$A_{\underline{1}}^{qq}(s,t) = (8\pi^2\alpha_s)^2 \left[\frac{N_c^2-1}{4N_c}\right] \delta_{ij}\delta_{kl}\, i\, s \int \frac{d^2\mathbf{k}}{(2\pi)^2} \int \frac{d^2\mathbf{k}'}{(2\pi)^2} \frac{F(s,\mathbf{k},\mathbf{k}',\mathbf{q})}{\mathbf{k}'^2(\mathbf{k}-\mathbf{q})^2} \tag{113}$$

- The total cross section is obtained as

$$\sigma_{\text{total}}^{qq} = \frac{1}{s}\,\text{Im}\, A_{\underline{1}}(s,t=0) = 4\alpha_s^2 \left(\frac{N_c^2-1}{4N_c^2}\right) \int d^2\mathbf{k} \int d^2\mathbf{k}' \frac{F(s,\mathbf{k},\mathbf{k}')}{\mathbf{k}^2\mathbf{k}'^2} \tag{114}$$



with rapidity defined as $y = \ell n(s/\mathbf{k}_{min}^2)$ it results

$$\sigma_{\text{total}}^{qq} = \frac{\pi(N_c^2-1)}{N_c^2}\left(\frac{\alpha_s^2}{\mathbf{k}_{min}^2}\right)\frac{e^{\lambda y}}{\sqrt{\pi\lambda' y/8}} \tag{115}$$

Unitarity Violation

- The unitarity of the S-matrix

$$SS^\dagger = S^\dagger S = \mathbb{1} \tag{116}$$

which implies that

$$|f\rangle = S|i\rangle = SS^\dagger|f\rangle \tag{117}$$

- From this feature arises the **Froissart-Martin Theorem**:
  - When $s \longrightarrow \infty$

$$\sigma_{\text{total}} \leq C\,\ell n^2\, s \tag{118}$$

- In the case of quark-quark scattering we have

$$\sigma^{qq}_{\text{total}} \sim \frac{s^\lambda}{\sqrt{\ell n\, s}} \tag{119}$$

that violates asymptotically the Froissart-Martin bound, since $\lambda = N_c \alpha_s 4 \ell n\, 2/\pi > 1$.

Diffusion

- Features of **BFKL Equation** in the case of LLA Pomeron:
  - BFKL amplitude $F(s, \mathbf{k}, \mathbf{k}')$:
    - Gaussian distribution in $\ell n\,(\mathbf{k}^2/\mathbf{k}'^2)$;
    - Width growing with $y \equiv \ell n\,(s/\mathbf{k}^2)$.
  - As the energy increases, the *non-perturbative region* can be **probed**;
- Setting the LLA BFKL solution as ($N \longrightarrow$ iteration step)

$$F^{(N)}(\omega, \mathbf{k}_i) \sim (\mathbf{k}_i^2)^{-\frac{1}{2}}\,\psi_N\left(\ell n\left[\frac{\mathbf{k}_i^2}{\mathbf{k}_0^2}\right]\right) \equiv (\mathbf{k}_i^2)^{-\frac{1}{2}}\,\psi_N(\xi_i) \tag{120}$$

- Some algebra leads to a typical **diffusion equation**

$$\lambda\,\frac{\partial\psi(N,\xi)}{\partial N} = \frac{\lambda'}{2}\,\frac{\partial^2\psi(N,\xi)}{\partial\xi^2} \tag{121}$$

Perturbative Region

- Taking "time" as $N = 0$ the wave function as a solution of the Diffusion Equation is

$$\psi(0,\xi) = \frac{1}{(\pi\sigma^2)^{\frac{1}{4}}} \exp\left(-\frac{\xi^2}{2\sigma^2}\right) \tag{122}$$

and neglecting the initial width we obtain

$$\psi(N,\xi) \sim \left(\frac{\lambda}{2\lambda' N}\right)^{\frac{1}{2}} \exp\left(-\frac{\lambda\xi^2}{2\lambda' N}\right) \tag{123}$$

- With the correspondence $N/\lambda \to y = \ell n\,(s/\mathbf{k}^2)$ we see that
  - A diffusion spreading equivalent to the behavior of LLA Pomeron solution.
- **Important**:
  - As the energy grows the infrared region of transverse momenta becomes more **relevant**:
    - **The perturbative treatment fails**!

Running Coupling

- The Diffusion Phenomenon suggests the use of **running coupling**
  - From LLA the self-energy and vertex correction diagrams were neglected!
    - Which implies that the coupling constant α_s had been taken as a constant!
  - Strategy: Solution for the LLA BFKL equation with $\alpha_s \to \alpha_s(\mathbf{k}^2)$.
    - One finds a discrete spectrum to the BFKL kernel;
    - A pole series being the Pomeron amplitude the leading one.
  - Another important feature:
    - Upper and lower limits to the intersection of the Pomeron's trajectory:

$$1 + 1.2\left[\frac{N_c\alpha_s(\mathbf{k}_0^2)}{\pi}\right] \;\leq\; \alpha_{I\!P}(0) \;\leq\; 1 + 4\ell n\left[\frac{2N_c\alpha_s(\mathbf{k}_0^2)}{\pi}\right]. \qquad (124)$$

NLO BFKL Equation I

- Going to NLLA limit, the structure of the BFKL kernel has the form

$$\mathcal{K}(\mathbf{k},\mathbf{h}) = 2\epsilon(-\mathbf{k}^2)\delta^2(\mathbf{k}-\mathbf{h}) + \mathcal{K}_{\text{real}}(\mathbf{k},\mathbf{h}) \quad (125)$$

  - Reggeized gluon calculated in two-loop precision;
  - Real part receives contribution from the tree level and production of gg and $q\bar{q}$.

- The eigenvalues of the BFKL kernel $\mathcal{K}$ in NLO are

$$\omega(\gamma) = \frac{N_c\alpha_s(\mathbf{k}^2)}{\pi}\left[\chi^{(0)}(\gamma) + \left(\frac{N_c\alpha_s(\mathbf{k}^2)}{\pi}\right)\chi^{(1)}(\gamma)\right] \quad (126)$$

  where

  - $\chi^{(0)}(\gamma) = 2\psi(1) - \psi(\gamma) - \psi(1-\gamma)$ is the LLA contribution;
  - $\chi^{(1)}$ represents the NLO correction.

NLO BFKL Equation II

- The correction from $\chi^{(1)}(\gamma)$ has the form

$$\begin{aligned}
\chi^{(1)}(\gamma) &= -\frac{1}{4}\left\{\frac{1}{2}\left(\frac{11}{3}-\frac{2n_f}{3N_c}\right)\left[\left(\chi^{(0)}(\gamma)\right)^2-\psi'(\gamma)+\psi'(1-\gamma)\right]\right. \\
&- 6\zeta(3)+\frac{\pi^2\cos\pi\gamma}{(\mathrm{sen}\pi\gamma)(1-2\gamma)}\left[3+\left(1+\frac{n_f}{N_c^3}\right)\frac{2+3\gamma(1-\gamma)}{(3-2\gamma)(1+2\gamma)}\right] \\
&- \left(\frac{67}{9}-\frac{\pi^2}{3}-\frac{10}{9}\frac{n_f}{N_c}\right)\chi^{(0)}(\gamma)-\psi''(\gamma)-\psi''(1-\gamma) \\
&- \left.\frac{\pi^3}{\mathrm{sen}\pi\gamma}+4\phi(\gamma)\right\}. \qquad (127)
\end{aligned}$$

- The running coupling constant has a correction of the form

$$\alpha_s(\mathbf{k}^2)\simeq\alpha_s(\mu^2)\left[1-\frac{\alpha_s(\mu^2)}{4\pi}\left(\frac{11N_c}{3}-\frac{2n_f}{3}\right)\ell n\left(\frac{\mathbf{k}^2}{\mu^2}\right)\right]. \qquad (128)$$

NLO BFKL Equation III

- In this approach the eigenvalues have two types of corrections
  - From the derivative of the strong running coupling;
  - Energy-scale independence of the due to $\chi^{(1)}(\gamma)$.
- Thus, the eigenvalues can be expressed under these corrections as

$$\omega(\gamma) = \underbrace{\left[\bar{\alpha}_s(\mu^2)\chi_0(\gamma) + \bar{\alpha}_s^2(\mu^2)\chi_1(\gamma)\right]}_{\text{energy-scale independence}} + \underbrace{\left[\bar{\alpha}_s(\mu^2)\left(\frac{11}{12} - \frac{n_f}{6N_c}\right)\ell n\left(\frac{\mathbf{k}^2}{\mu^2}\right)\chi_0(\gamma)\right]}_{\text{running coupling}}. \quad (129)$$

- Leading eigenvalue is that with $\gamma = 1/2$: $\omega_0 = \bar{\alpha}_s\chi_0\left(\frac{1}{2}\right) = 2.77\bar{\alpha}_s.$
- At NLO this eigenvalue is

$$\bar{\alpha}_s\,\chi(\gamma)\big|_{\gamma=\frac{1}{2}} = \bar{\alpha}_s\,\chi_0(\gamma) + \bar{\alpha}_s^2\,\chi_1(\gamma)\big|_{\gamma=\frac{1}{2}} = \omega_0(1 - 6.61\bar{\alpha}_s) = 2.77\bar{\alpha}_s - 18.34\bar{\alpha}_s^2, \quad (130)$$

  - **HERA regime**: Correction $\chi^{(1)}(\gamma)$ so large that dominates over $\chi^{(0)}(\gamma)$!

BFKL Equation in DIS

- Using the integro-differential equation we obtain for $f(x, \mathbf{k}^2)$

$$\frac{\partial f(x,\mathbf{k}^2)}{\partial \ell n(1/x)} = \frac{3\alpha_s \mathbf{k}^2}{\pi} \int_0^{\infty} \frac{d^2\mathbf{h}}{\mathbf{h}^2} \left[\frac{f(x,\mathbf{h}^2) - f(x,\mathbf{k}^2)}{|\mathbf{h}^2 - \mathbf{k}^2|} + \frac{f(x,\mathbf{k}^2)}{(4\mathbf{h}^4 + \mathbf{k}^4)^{1/2}} \right] \tag{131}$$

  we obtain the **BFKL equation for DIS** in the leading $\ell n(1/x)$ approximation with a fixed coupling constant.

- The solution for this equation gives an unintegrated gluon distribution

$$f(x,\mathbf{k}^2) \sim \left(\frac{x}{x_0}\right)^{-\lambda} \left[\frac{(\mathbf{k}^2/\mathbf{k}_0^2)}{ln(x/x_0)}\right]^{1/2} exp\left[-\frac{ln^2(\mathbf{k}^2/\tilde{\mathbf{k}}^2)}{2\lambda' ln(x_0/x)}\right] \tag{132}$$

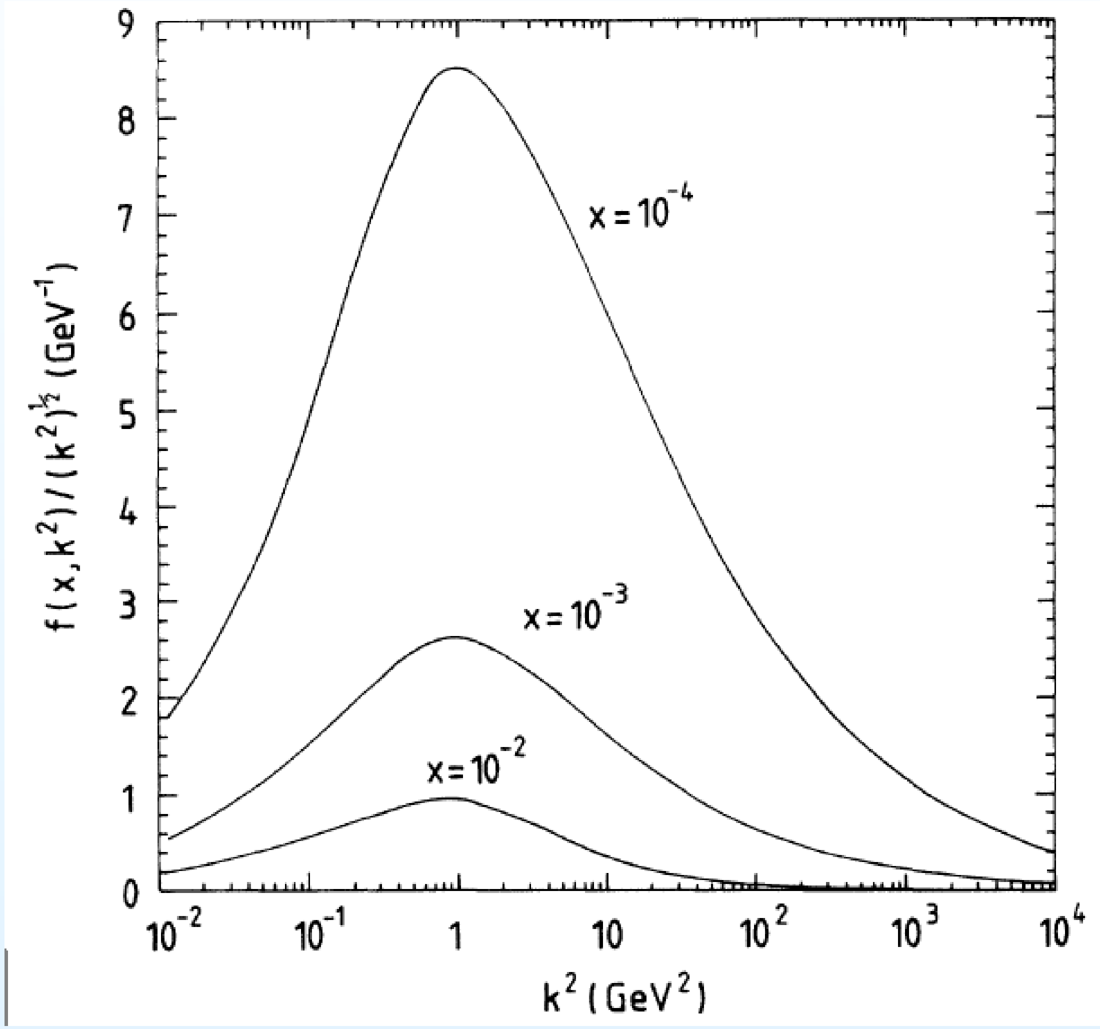
  with the leading behavior at low-x '

$$f(x,\mathbf{k}^2) \sim x^{-\lambda} \tag{133}$$

Unintegrated Gluon Distribution

- It is clearly visible:
    - The diffusion in $\mathbf{k}^2$; and
    - Growth of the type $x^{-\lambda}$.

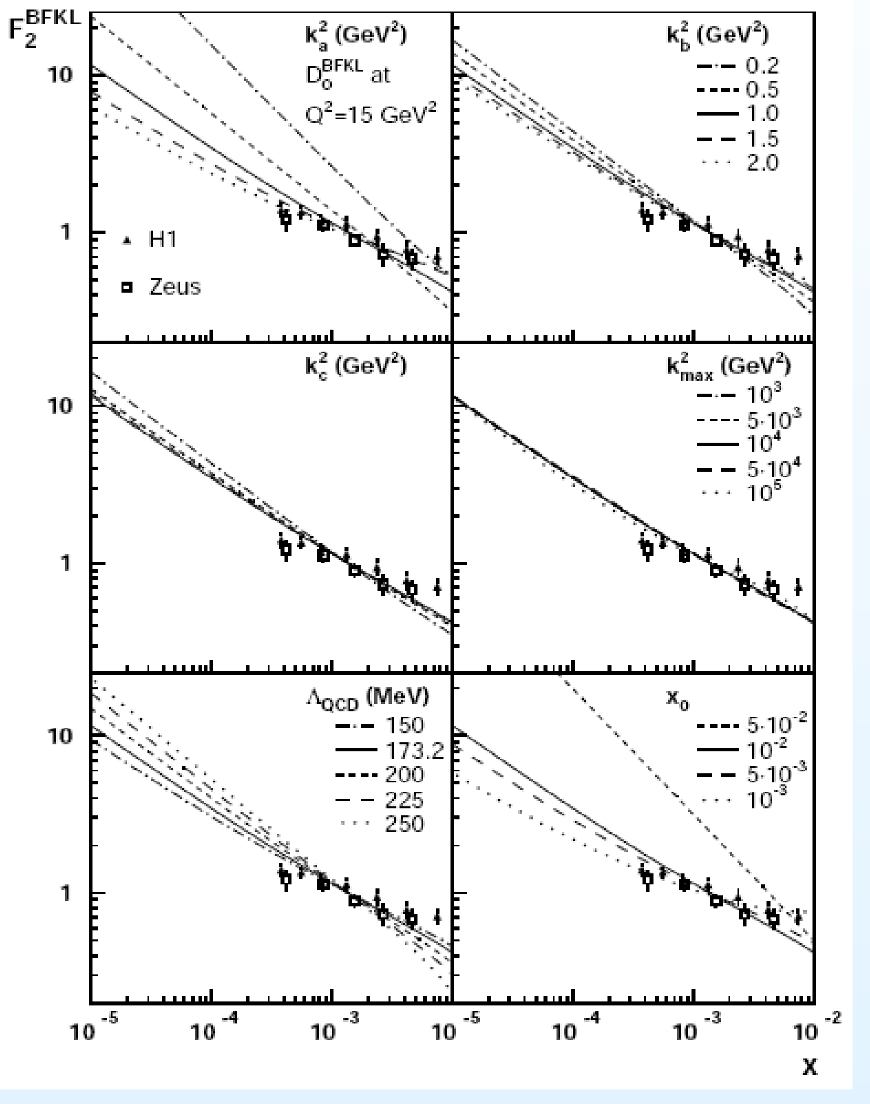
(Askew *et al*, Phys. Rev. **D49**, 4402, 1994)



BFKL evolution of $f(x, \mathbf{k}^2)/(\mathbf{k}^2)^{1/2}$.

Predictions for F_2

(Bojak and Ernst, Phys. Rev. **D53**, 80, 1996)



BFKL prescription for F_2 compared with HERA data.

Applications: Truncated BFKL Series

- Considering only the first two orders in LO perturbative theory we have for the partial-wave amplitudes

$$
\begin{aligned}
f_1(\omega, \mathbf{k}_1, \mathbf{k}_2, \mathbf{q}) &= \frac{1}{\omega}\, \delta^2(\mathbf{k}_1 - \mathbf{k}_2) \\
f_2(\omega, \mathbf{k}_1, \mathbf{k}_2, \mathbf{q}) &= -\frac{N_c \alpha_s}{2\pi^2}\frac{1}{\omega^2}\left[\frac{\mathbf{q}^2}{\mathbf{k}_1^2(\mathbf{k}_2 - \mathbf{q})^2} - \frac{1}{2}\frac{1}{(\mathbf{k}_1 - \mathbf{k}_2)^2}\left\{1 + \frac{\mathbf{k}_2^2(\mathbf{k}_1 - \mathbf{q})^2}{\mathbf{k}_1^2(\mathbf{k}_2 - \mathbf{q})^2}\right\}\right]
\end{aligned}
$$

  which corresponds to taking the **two-gluon exchange** and the **one-rung ladder** into account only.

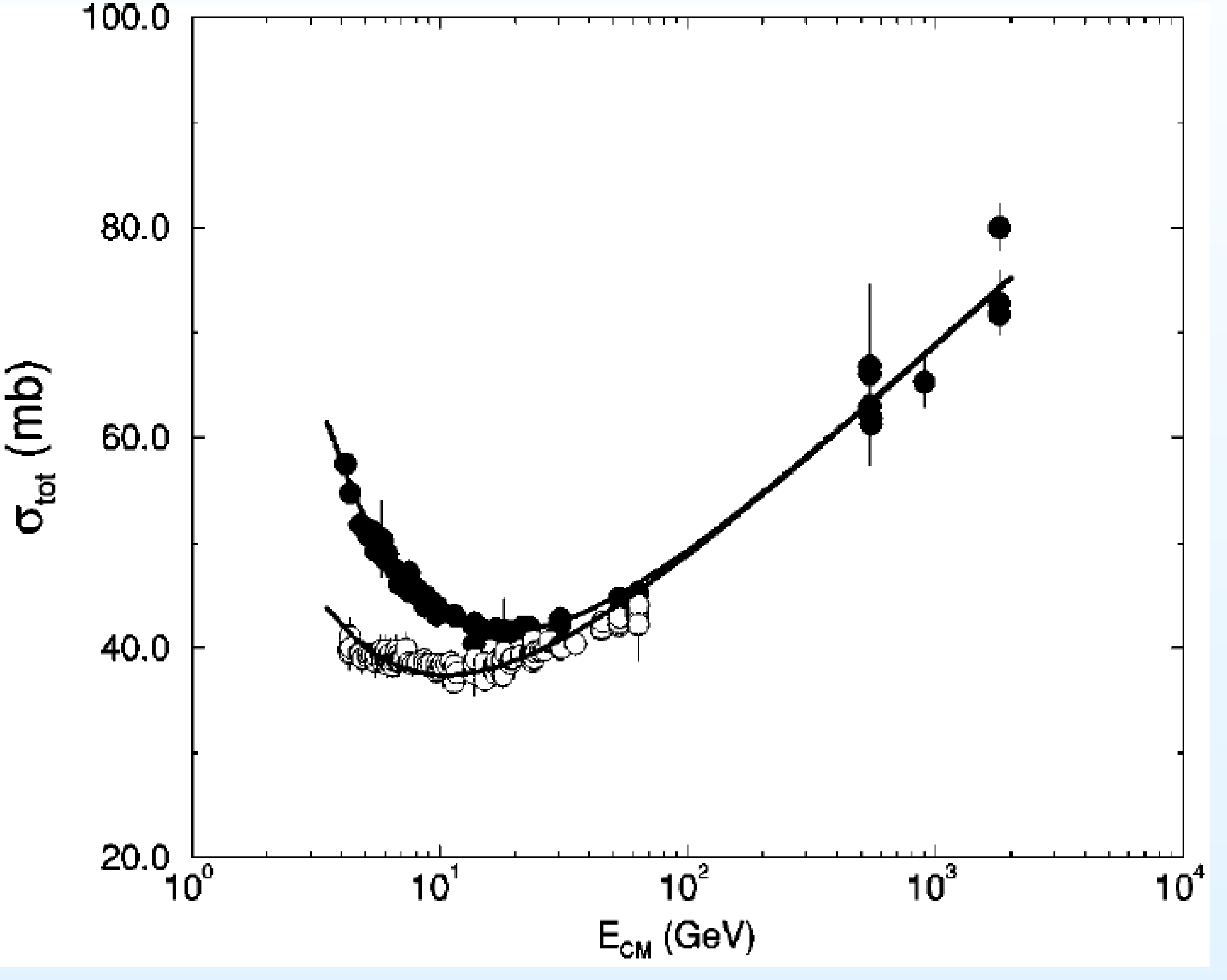
- **Truncating** the BFKL series at two orders, a parametrization is proposed to *proton-proton* and *proton-anti-proton* the total cross section goes like

$$
\sigma_{\text{total}}^{pp(p\bar{p})} = C_R (s/s_0)^{\alpha_R(0)-1} + C_{\text{Born}} + C_{\text{NO}}\, \ell n(s/s_0) \tag{134}
$$

  where $\mathbf{k}^2 = s_0 = 1\,\text{GeV}^2$.

pp and $p\bar{p}$ Total Cross Section

(Gay Ducati, Machado. Phys. Rev. **D63**, 094018, 2003)



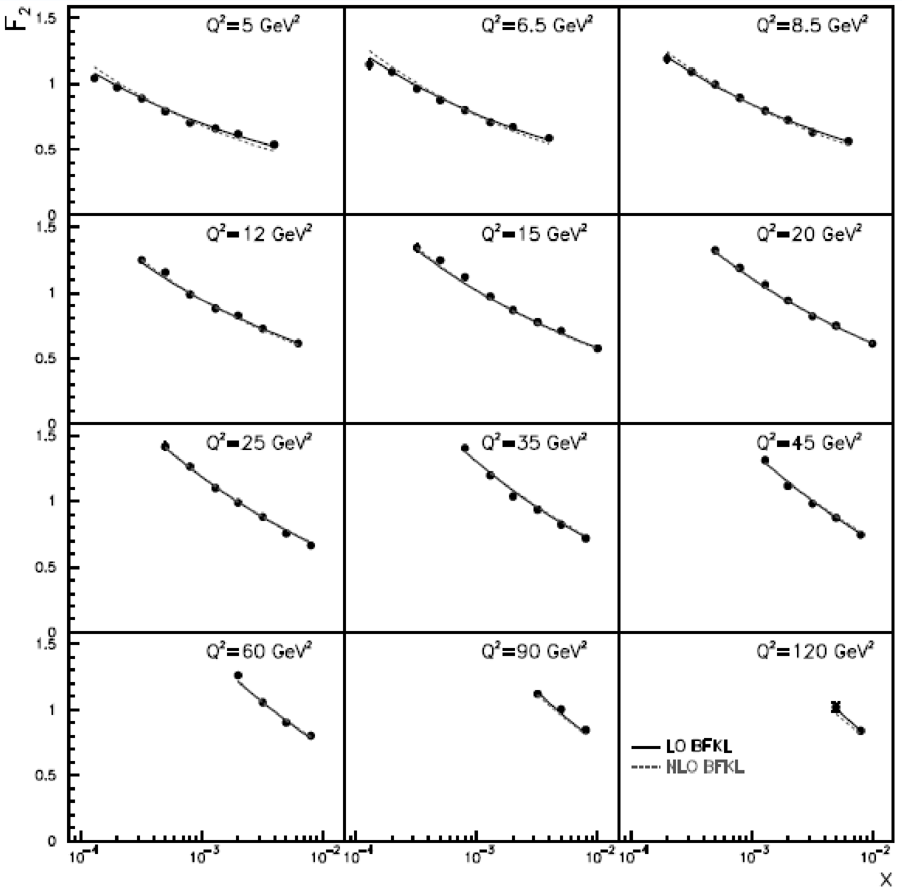
Fits for pp (lower line) and $p\bar{p}$ (upper line) total cross section from PDG data.

Application: LO *versus* NLO BFKL Equation

- Using an effective kernel and a saddle point approximation to compute F_2^p in NLO-BFKL

  $\longrightarrow$ **problem**: deviations at $Q^2 < 10$ GeV^2

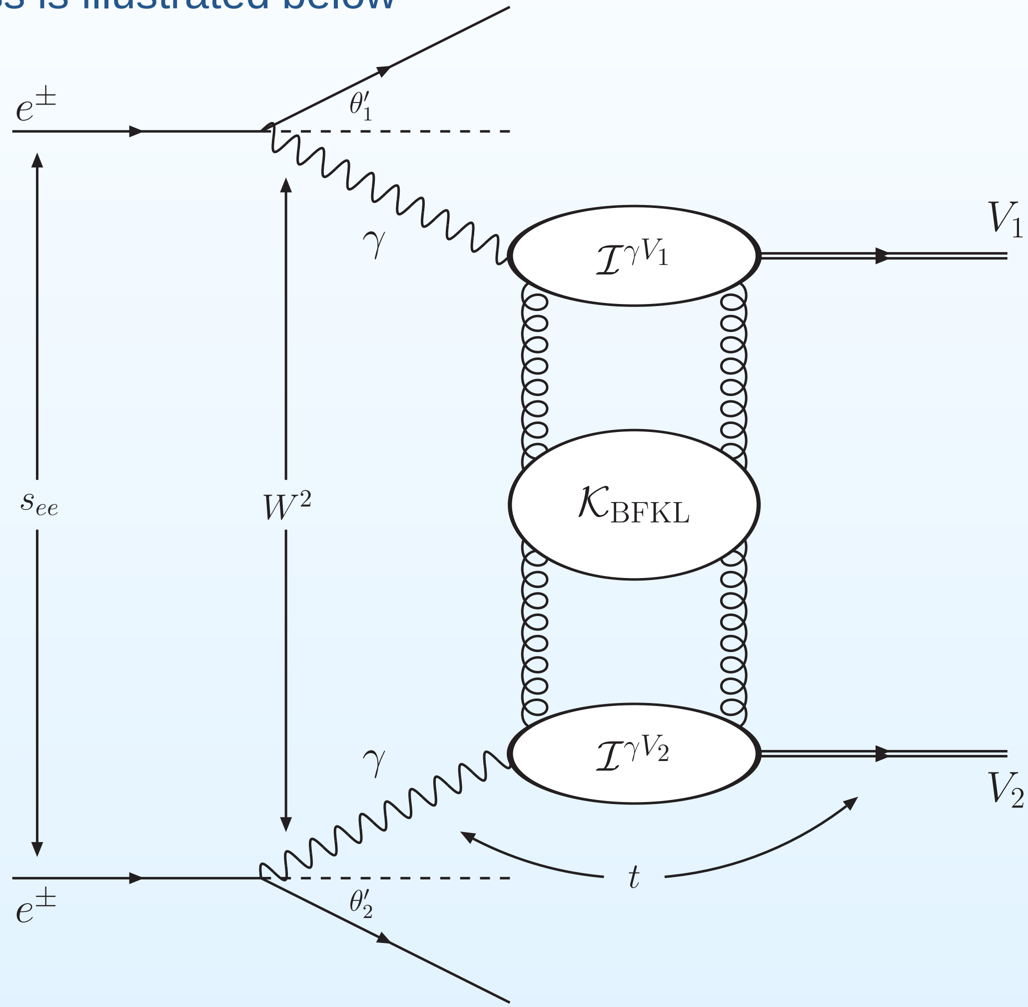
(Schoeffel, hep-ph/0505114, 2005)



Fits at LO (solid line) and NLO (dashed line) BFKL for F_2^p from H1 data.

Application: Meson Production (I)

- It can be studied the meson production via pomeron exchange in e^+e^- colliders;
- A possible process is illustrated below



Application: Vector Meson Production (II)

- The cross section is expressed in the form

$$\sigma_{e^+e^- \to e^+e^- V_1 V_2}(\sqrt{s_{ee}}) = \int dx_a \, dx_b \, f_{\gamma/e}(x_a) \, f_{\gamma/e}(x_b) \frac{d\sigma_{\gamma\gamma \to V_1 V_2}}{dt}(\hat{s}) \tag{135}$$

- The cross section of the subprocess depends on the BFKL amplitude F

$$\frac{d\sigma(\gamma\gamma \to V_1 V_2)}{dt} = \frac{16\pi}{81 t^4} |F_{BFKL}(z,\tau)|^2 \tag{136}$$

- These functions represent the incoming photons and are related to the BFKL Amplitude

$$F_{BFKL}(z,\tau) = \frac{t^2}{(2\pi)^3} \int d\nu \frac{\nu^2}{(\nu^2 + 1/4)^2} e^{\chi(\nu) z} I_\nu^{\gamma V_1}(Q_\perp) I_\nu^{\gamma V_2}(Q_\perp)^* \tag{137}$$

where the quantities $I_\nu^{\gamma V_i}$ are called *impact factors* and the quantity $\chi(\nu)$ depends of the BFKL Kernel eigenvalues

$$\chi(\nu) = 4\text{Re}\left(\psi(1) - \psi\left(\frac{1}{2} + i\nu\right)\right) \tag{138}$$

Application: Vector Meson Production (III)

- Finally, the results for the production of several mesons are described in the next table

	$\sqrt{s_{ee}} = 200$ GeV	$\sqrt{s_{ee}} = 500$ GeV	$\sqrt{s_{ee}} = 1000$ GeV	$\sqrt{s_{ee}} = 3000$ GeV
$\rho J/\Psi$	0.90 (0.015)	5.80 (0.049)	21.87 (0.097)	178.19 (0.22)
$\phi J/\Psi$	0.11 (0.0023)	0.69 (0.0073)	2.58 (0.014)	20.77 (0.033)
$\omega J/\Psi$	0.075 (0.0013)	0.48 (0.0041)	1.85 (0.0081)	15.03 (0.019)
$J/\Psi J/\Psi$	0.045 (0.0021)	0.27 (0.0066)	0.98 (0.012)	7.56 (0.031)
$\rho \Upsilon$	0.0013 (0.000055)	0.0093 (0.00017)	0.036 (0.00034)	0.31 (0.00080)
$\omega \Upsilon$	0.00011 (0.0000055)	0.00078 (0.000017)	0.0030 (0.000034)	0.026 (0.000080)
$\phi \Upsilon$	0.0002 (0.000011)	0.0013 (0.000034)	0.0050 (0.000068)	0.040 (0.00016)
$J/\Psi \Upsilon$	0.00025 (0.000027)	0.0015 (0.000086)	0.0052 (0.00017)	0.038 (0.00040)
$\Upsilon \Upsilon$	0.0000072 (0.0000014)	0.000038 (0.0000045)	0.00012 (0.0000088)	0.0008 (0.000020)

The double vector meson production cross sections in e^+e^- processes at different energies, $|t|_{min} = 0$ and $\theta_{max} = 30$ mrad, assuming the BFKL Pomeron (Two-gluon) exchange. Cross sections are given in pb.

Application: Higgs Boson Production (I)

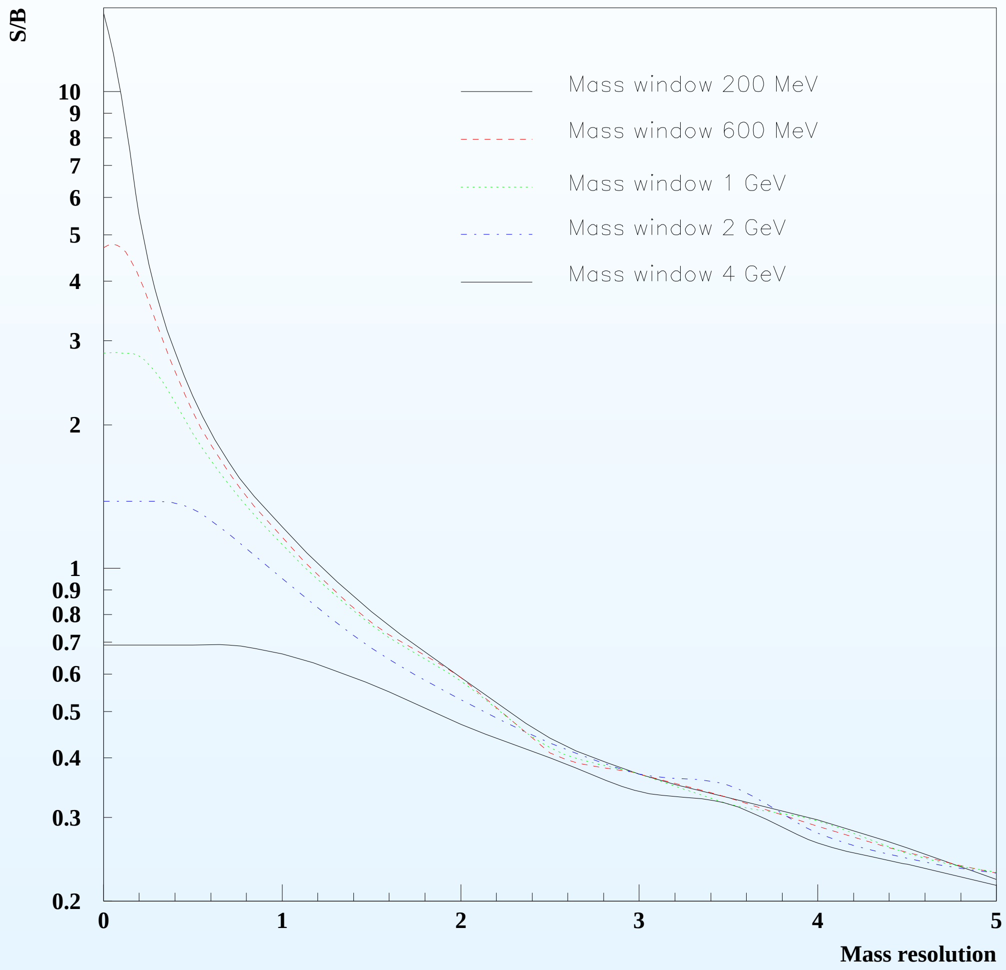
- A way to study the production of the Higgs boson is the double-pomeron exchange;
- In order to find the Higgs boson, two processes are accounted (Royon, C. hep-ph/0601226)
- Exclusive Process:

$$
\begin{aligned}
d\sigma_h^{exc}(s) &= C_h \left(\frac{s}{M_h^2}\right)^{2\epsilon} \delta\left(\xi_1\xi_2 - \frac{M_h^2}{s}\right) \\
&\times \prod_{i=1,2} \left\{ d^2v_i \frac{d\xi_i}{1-\xi_i} \; \xi_i^{2\alpha' v_i^2} \exp(-2\lambda_h v_i^2) \right\} \sigma(gg \to h)
\end{aligned} \tag{139}
$$

- Inclusive Process:

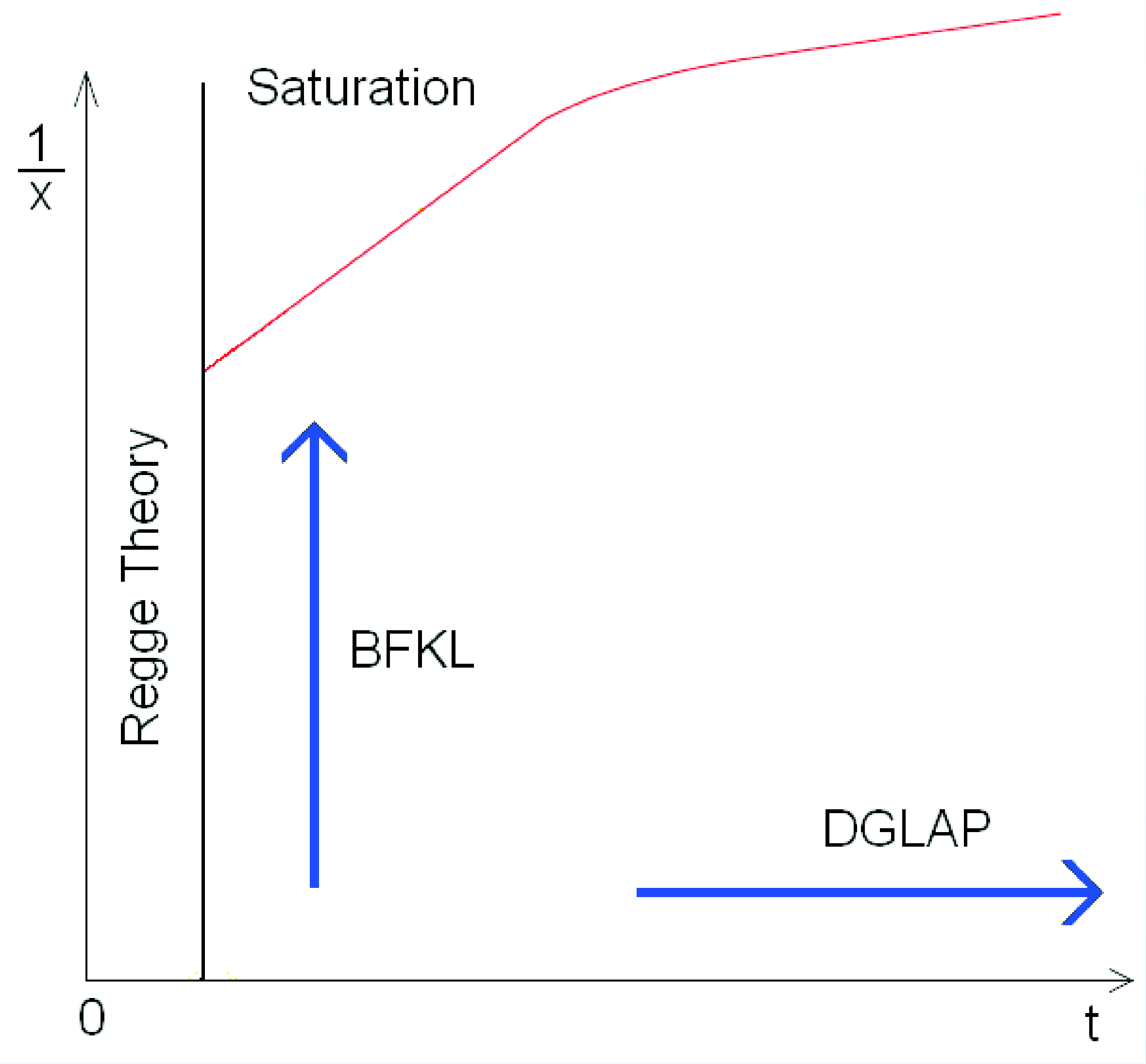
$$
\begin{aligned}
d\sigma_H^{incl} &= C_H \left(\frac{x_1^g x_2^g s}{M_H^2}\right)^{2\epsilon} \delta\left(\xi_1\xi_2 - \frac{M_H^2}{x_1^g x_2^g s}\right) \\
&\times \prod_{i=1,2} \left\{ G_P(x_i^g, \mu) \; dx_i^g d^2v_i \; \frac{d\xi_i}{1-\xi_i} \; {\xi_i}^{2\alpha' v_i^2} \; \exp\left(-2v_i^2 \lambda_H\right) \right\};
\end{aligned} \tag{140}
$$

Application: Higgs Boson Production (II)



Higgs boson signal-to-background ratio as a function of the resolution on the missing-mass, in GeV. ($m_H = 120$ GeV)

The QCD Evolution Landscape



The BFKL evolution and its limitations.

Conclusions

- Clarifies the knowledge about the High-Energy particle phenomenology;
  - Goal for Regge Theory.
- Good agreement with Low-x data beyond the DGLAP Equation;
- Require some analysis in non-perturbative region.

References

- E.A. Kuraev, L.N. Lipatov, V.S. Fadin, Phys. Lett. B 60, 50 (1975); Sov. Phys. JETP 44, 443 (1976); Sov. Phys. JETP 45, 199 (1977).
- I.I. Balitsky, L.N. Lipatov, Sov. J. Nucl. Phys. 28, 822 (1978).
- Barone V., Predazzi E., *High-Energy Particle Diffraction*, Springer-Verlag, Berlin, 2002.
- Forshaw J.R., Ross D.A., *Quantum Chromodynamics and the Pomeron*, Cambridge University Press, Cambridge, 1997.
- Ellis R.K., Stirling W.J., Webber B.R., *QCD and Collider Physics*, Cambridge University Press, Cambridge, 1996.
- Machado M.V.T., *Física de Pomeron a Altas Energias*, PhD Thesis, GFPAE, IF-UFRGS, Porto Alegre, 2002.
- Schmidt C.R., *Review of BFKL*, hep-ph/0106181, 2001.
- Bojak I., Ernst M., *BFKL versus HERA*, Phys. Rev. **D53**, 80-88, 1996.
- Marage P., *Hadronic Strucure, Low-x and Diffraction*, hep-ph/9911426, 1999.
- Gonçalves, Sauter. Phys. Rev. D **73**, 077502 (2006)