AGL evolution equation

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Summary

- Introduction
- Froissart bound
- Unitarity
- GLR evolution equation
- QCD Glauber-Mueller Approach
- AGL evolution equation
- Solutions of AGL equation
- AGL with running α_s
- References

Introduction

Main breakthroughs leading to nonlinear equations:

- Gribov-Levin-Ryskin (GLR) equation. First nonlinear equation proposing the parton saturation (1981-83).
- Mueller and Qiu. Consider GLR for double logarithm approximation in perturbative QCD (1986).
- Ayala, Gay Ducati, Levin. Nonlinear equation in DLA QCD. The main degree of freedom is the gluon distribution and there are two DLA contributions $(\alpha_s(\ln 1/x)\ln(Q^2/\Lambda))^n$, $(\alpha_s(\ln 1/x)\ln(Q_s^2(x)/Q^2))^n$ (1997).
- Balitsky. Generalized BFKL Operator Product Expansion. High Energy scattering in QCD, infinite hierarchy of coupled equations for n-point Wilson line operators decouple for large N_c (1996).
- Kovchekov. Generalized BFKL Operator Product Expansion. High Energy scattering in QCD, infinite hierarchy of coupled equations for n-point Wilson line operators decouple for large N_c (1996).

Froissart bound

- The Froissart bound is a limit for the cross section for the scattering of two hadrons.
- It is derived using Mandelstam representation and is based on two hypothesis:
- First Froissart hypothesis: the strong interaction has finite range.
- In the days of Froissart, this range is determined by the pion mass m_π

$$R \sim \frac{1}{m_\pi}. \tag{1}$$

- This scale is **nonperturbative**.

Froissart bound

- Second Froissart hypothesis: S-matrix is unitary

$$SS^\dagger \;=\; S^\dagger S \;=\; 1 \; . \tag{2}$$

- A complete set of particle states $|m\rangle$ satisfies completeness relation

$$\sum_m |m\rangle\langle m| \;=\; 1 \; . \tag{3}$$

- If the initial state is $|i\rangle$, the probability to find the final state $|f\rangle$ is given by:

$$P_{fi} = |\langle f|S|i\rangle|^2 \; . \tag{4}$$

- The probability that an initial state evolves into any final state is 1: then:

$$\sum_f P_{fi} \;=\; \sum_f \langle i|S|f\rangle\langle f|S|i\rangle \;=\; \langle i|S^\dagger S|i\rangle \;=\; 1 \; , \tag{5}$$

therefore, it is understood that the S-matrix is unitary.

Froissart bound

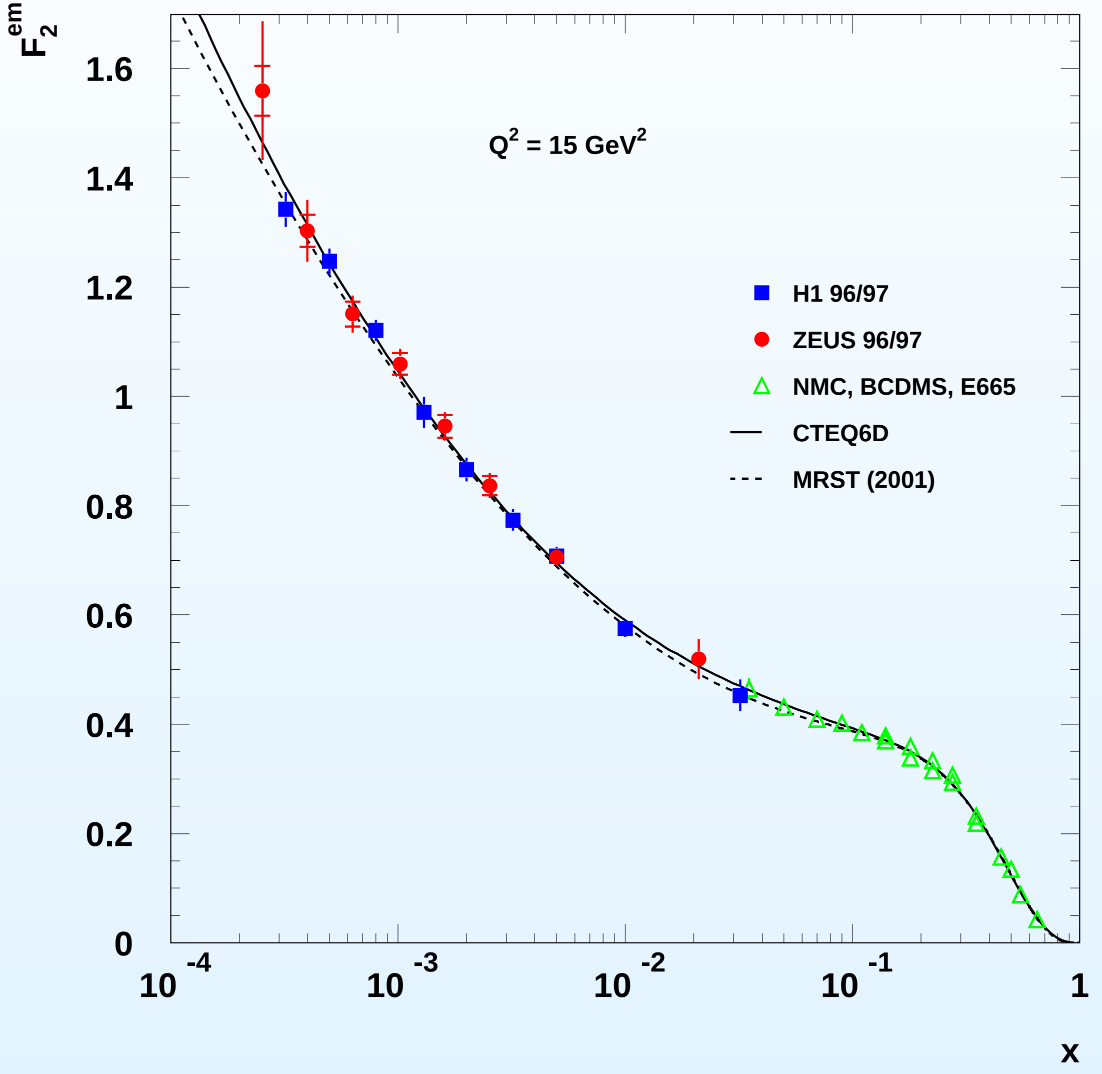
- The Froissart bound limits the total cross section for scattering of two hadrons:

$$\sigma_{\mathrm{TOT}} \leq \frac{\pi}{R^2} (\ln s)^2 \; . \tag{6}$$

- If the pion mass is used, $\sigma_{\mathrm{TOT}} \leq (62\mathrm{mb})(\ln s)^2$.
- COMPETE experimental data says that $\sigma_{\mathrm{TOT}} \leq (0.3152\mathrm{mb})(\ln s)^2$.
- It was derived for all regions of QCD (including pQCD and npQCD).
- However, the available data show no sign that Froissart bound is valid (or invalid).
- At high photon virtualities, the DIS structure function appears to increase very fast for a logarithm dependence.
- One hopes that with more exclusive processes (maybe diffraction) saturation can be observed.

Froissart bound

F_2 structure function data from HERA collider and fixed target experiments.



Unitarity

- High parton density in Deep Inelastic Scattering is going to be studied.
- Small x region (high energy) is an interface between non-perturbative QCD (npQCD) and perturbative QCD. In this pQCD frontier the coupling constant α_s is still small.
- It is observed that F_2 increases for small values of x, suggesting violation of unitarity.
- The unitarity limit is the Froissart bound, which shows that the cross section cannot be larger than $\sigma \leq B \ln^2 s$, where s is the squared center of mass energy.
- This unitarity shall be restored in the theory, not implied in both DGLAP and BFKL evolution equations.

Emphasizing: H.E. Behavior of QCD Amplitudes

Problem:

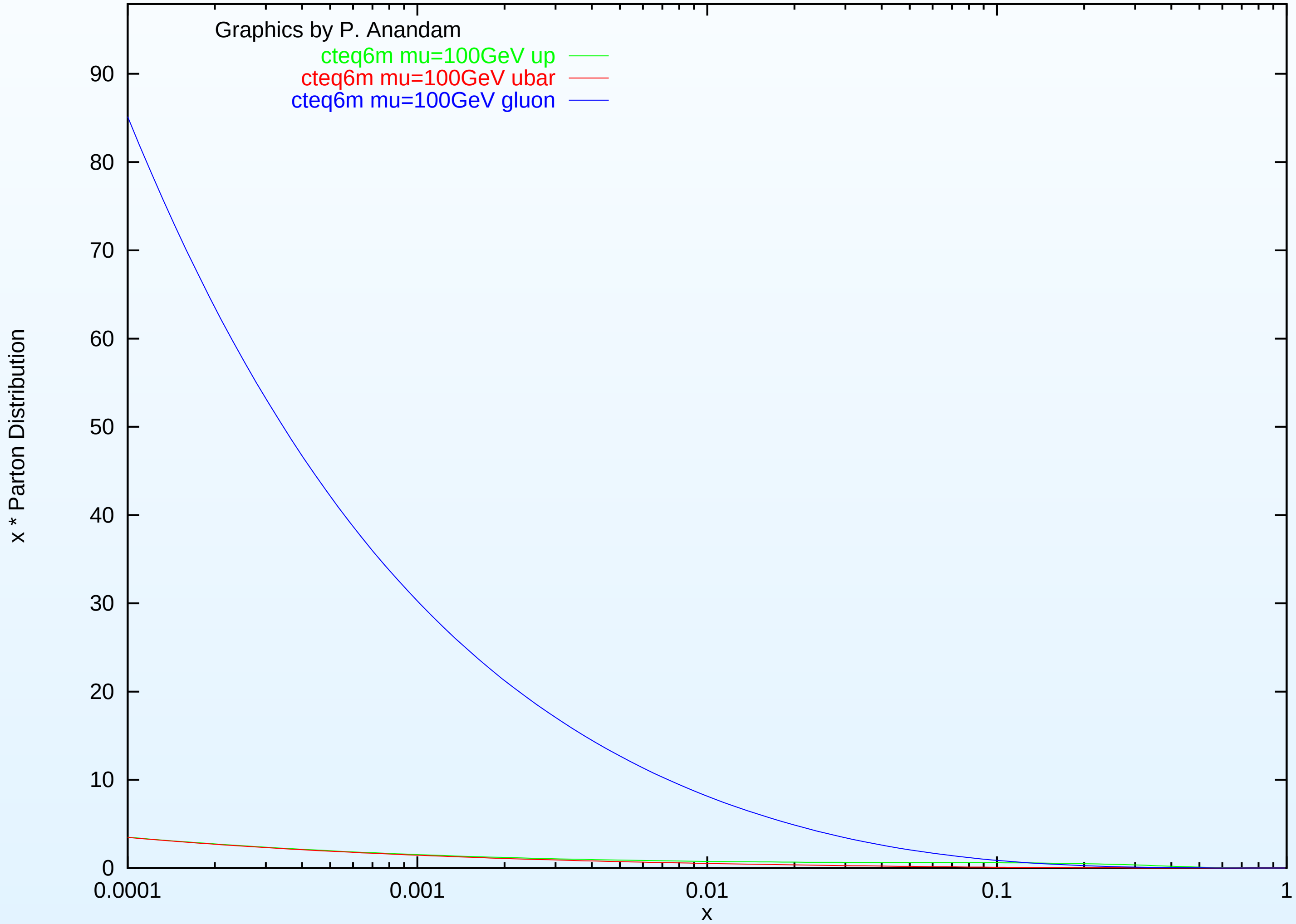
- Analytically separate.
- Small and large distances.
- Contributions to high energy
- Amplitudes in a properly gauge invariant formalism.

Questions:

- Unitarity corrections.
- Signature of UC in the observables.
- Predictions from different models.
- Common limit among different formalisms.
- Analytic solutions $\Rightarrow g(x, Q^2)$ all range.

Gluon importance in small-x physics

http://zebu.uoregon.edu/˜parton/partongraph.html



DGLAP evolution equation

- The DGLAP evolution equation:

$$\frac{dq_i(x,Q^2)}{d\ln Q^2} = \int_x^1 \frac{dy}{y}\left[q_i(y,Q^2)\mathcal{P}_{qq}\left(\frac{x}{y},\alpha_s(Q^2)\right) + G(y,Q^2)\mathcal{P}_{qG}\left(\frac{x}{y},\alpha_s(Q^2)\right)\right] \tag{7}$$

$$\frac{dG(x,Q^2)}{d\ln Q^2} = \int_x^1 \frac{dy}{y}\sum_i\left[q_i(y,Q^2)\mathcal{P}_{Gq}\left(\frac{x}{y},\alpha_s(Q^2)\right) + G(y,Q^2)\mathcal{P}_{GG}\left(\frac{x}{y},\alpha_s(Q^2)\right)\right] \tag{8}$$

  summs all diagrams whithin LL(Q^2)A:

$$\alpha_s \ln Q^2/Q_0^2 \approx 1 \qquad \alpha_s \ln 1/x \ll 1 \quad \alpha_s \ll 1.$$

- Also, only longitudinal momenta in the parton cascade are (strongly) ordered:

$$x < x_i < \cdots < x_1 < 1 \tag{9}$$

$$Q \approx k_{\perp i} \gg \cdots \gg k_{\perp 1} \approx Q_0 \tag{10}$$

BFKL evolution equation

- The BFKL evolution equation:

$$\frac{\partial\phi(x,k^2)}{\partial\ln(1/x)} = \frac{3\alpha_s}{\pi}\int_0^\infty \frac{dk'^2}{k'^2}\left\{\frac{\phi(x,k'^2)+\phi(x,k^2)}{|k'^2-k^2|} + \frac{\phi(x,k^2)}{\sqrt{4k'^4+k^4}}\right\} \tag{11}$$

with

$$xG(x,Q^2) = \int^{Q^2}\frac{dk^2}{k^2}\phi(x,k^2) \tag{12}$$

summs all diagrams whithin LL($1/x$)A:

$$\alpha_s\ln 1/x \approx 1 \qquad \alpha_s\ln Q^2/Q_0^2 \ll 1 \quad \alpha_s \ll 1.$$

- Also, only longitudinal momenta in the parton cascade are ordered:

$$x \ll x_i \ll \cdots \ll x_1 \ll 1 \tag{13}$$

$$Q \approx k_{\perp i} \approx \cdots \approx k_{\perp 1} \approx Q_0 \tag{14}$$

- That is why BFKL is an equation in the unintegrated gluon distribution.

BFKL evolution equation

BFKL Pomeron problems in pQCD:

- Lack of unitarity;
- Diffusion into infrared region of gluon virtualities. (Perturbative theory goes to nonpertubartive region.)

The cross section at high energy center of momentum in pQCD:

- LLA $\alpha_a \ll 1$ and $\alpha_s \log s \sim 1$. Leading singularity in complex angular momentum plane (which corresponds to the vacuum quantum number exchange):

$$j = 1 + 4N_c\alpha_s \ln\frac{2}{\pi} \tag{15}$$

- For QCD the BFKL Pomeron provides that

$$s \to \infty\,, \quad \sigma^{TOT} \sim s^{j-1} = s^{0.5} \tag{16}$$

for $\alpha_s \approx 0.2$.

Double logarithm approximation

- We consider the double logarithm approximation (DLA) of pQCD. (For example, DGLAP at low x.) The kinematic region of interest is:

$$\begin{aligned}
\alpha_s \ln 1/x \ln Q^2/Q_0^2 &\sim 1 \\
\alpha_s \ln 1/x &\ll 1 \\
\alpha_s \ln Q^2/Q_0^2 &\ll 1 \\
\alpha_s &\ll 1
\end{aligned}$$

- This is the same as to state that $\alpha_s \ll \gamma \ll 1$, where the anomalous dimension is $\gamma = \frac{N_c \alpha_s}{\pi}$.
- Also, a strong ordering in both transverse and longitudinal gluon momenta occurs:

$$x \ll x_i \ll \cdots \ll x_1 \ll 1 \tag{17}$$

$$Q \gg k_{\perp i} \gg \cdots \gg k_{\perp 1} \gg Q_0 \tag{18}$$

DGLAP at small x

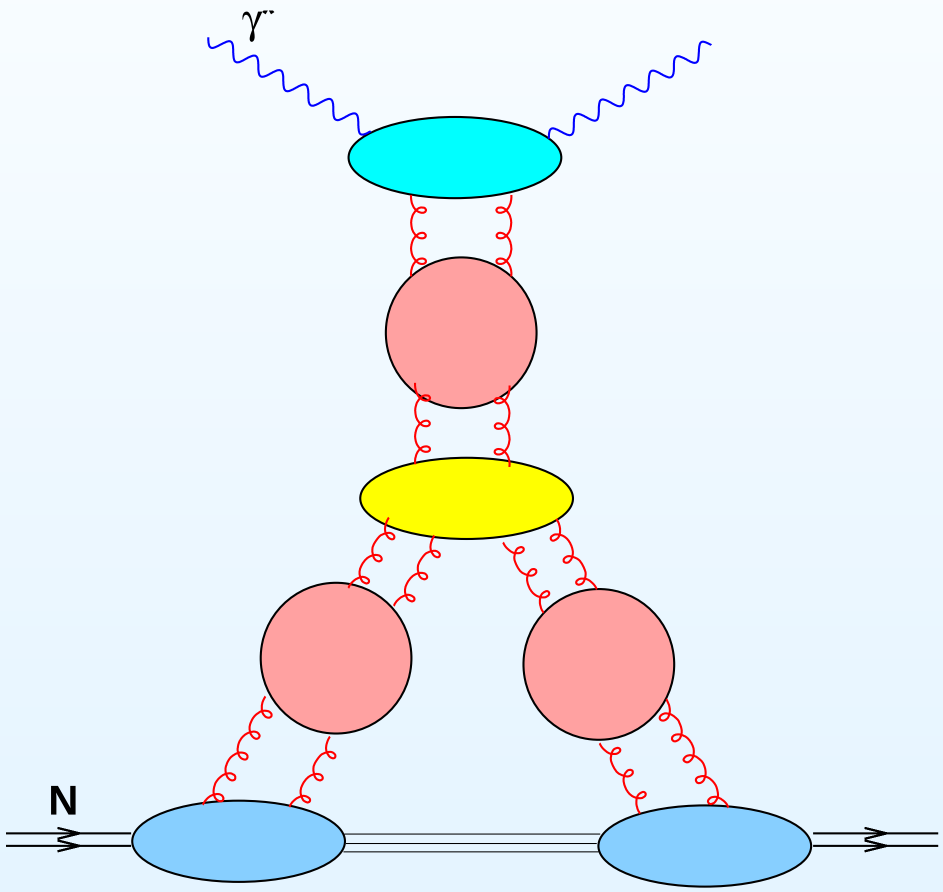
- In DGLAP, for the evolution of gluon distribution, the vertices considered are quark splitting into quark and gluon and gluon splitting into two gluons.
- In other words, only gluon emission diagrams are considered.
- In the limit of small x (DLA), DGLAP can be written as:

$$\frac{\partial^2 xg(x,Q^2)}{\partial \ln(1/x) \partial \ln Q^2} = \frac{\alpha_s N_c}{\pi} xg(x,Q^2). \tag{19}$$

- At some point, with a large gluon distribution, the emitted gluons can start to interact.
- Then, the vertex $gg \to g$ cannot be neglected.
- Both DGLAP and BFKL do not consider gluon recombination.
- Therefore, a modified DGLAP equation must be considered.

$gg \longrightarrow g$ vertex

- This vertex allows gluon cascade merging (also called *fan* diagrams):



GLR evolution equation

- The GLR evolution equation is a formalism in DLA derived by Gribov, Levin and Ryskin.
- It considers gluon recombination ($gg \to g$ vertex).
- The GLR equation can be written,

$$\frac{\partial^2 xg(x,Q^2)}{\partial \ln \frac{1}{x} \partial \ln Q^2} = \frac{\alpha_s N_c}{\pi} xg(x,Q^2) \quad (20)$$

$$- \frac{4\pi^3 N_c^2}{(N_c^2-1)} \frac{1}{Q^2} \frac{\alpha_s^2(Q^2)}{\pi^2} x^2 g^{(2)}(x,Q^2), \quad (21)$$

where $g^{(2)}(x,Q^2)$ is the two-gluon correlation function.

- The coeficient of the two-gluon correlation function was calculated by Mueller and Qiu in 1986.

Model for the two-gluon correlation function

- Mueller and Qiu also proposed a model for the two-gluon correlation function.
- They considered gluons homogeneously distributed in the target (in this case, a nucleon):

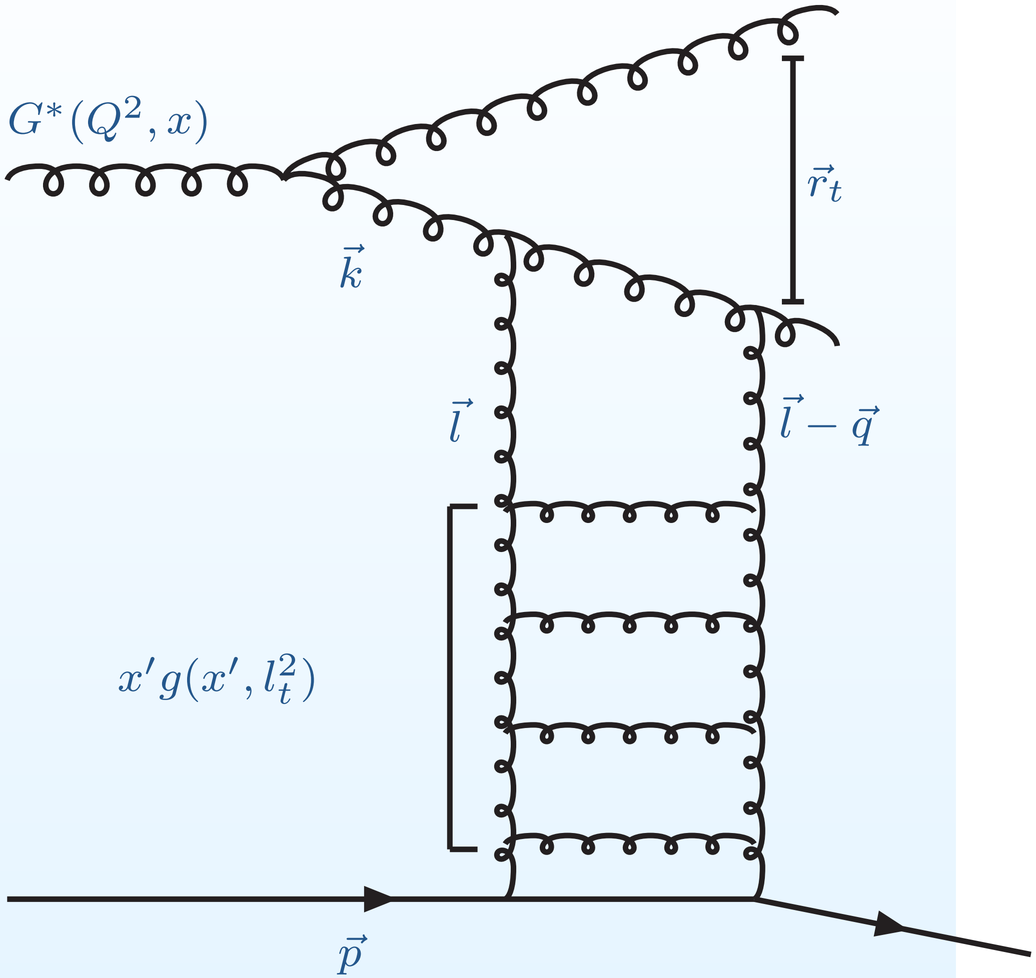
$$x^2 g^{(2)}(x,Q^2) = \frac{9}{8}\frac{1}{\pi R^2}[xg(x,Q^2)]^2, \quad (22)$$

  in which R^2 is not necessarily the nucleon radius, but a distance related with the region where gluons are spread and dependent on the model used.
- Then, it is seen clear that GLR is a nonlinear equation (using $N_c = 3$):

$$\frac{\partial^2 xg(x,Q^2)}{\partial \ln \frac{1}{x} \partial \ln Q^2} = \frac{\alpha_s N_c}{\pi} xg(x,Q^2) - \frac{81}{16}\frac{\alpha_s^2(Q^2)}{Q^2}[xg(x,Q^2)]^2. \quad (23)$$

QCD Glauber-Mueller Approach

- For simplicity, DIS is considered in the target rest frame.
- Q^2 is the photon (or gluon) virtuality.
- m is the proton mass.
- $x = x_{Bj} \equiv \frac{Q^2}{s}$, where $\sqrt{s} = W$ is the center of mass energy of the incoming particle plus the target.
- $\vec{k}_t$ is the quark (or gluon) transverse momentum.
- $\vec{r}_t$ is the quark-antiquark (or gluon-gluon) transverse separation.
- $\vec{b}_t$ is the impact parameter of the reaction (conjugated to $\vec{q}_t$).
- $\vec{l}_t$ is the transverse momentum of the gluon attached to the pair $q\bar{q}$ (gg).
- z and $1 - z$ are the energy fractions carried by the quark-antiquark (or gluon-gluon) pair.



QCD Glauber-Mueller Approach

- The variation of transverse distance r_t is

$$\Delta r_\perp \propto R \frac{k_\perp}{E} \tag{24}$$

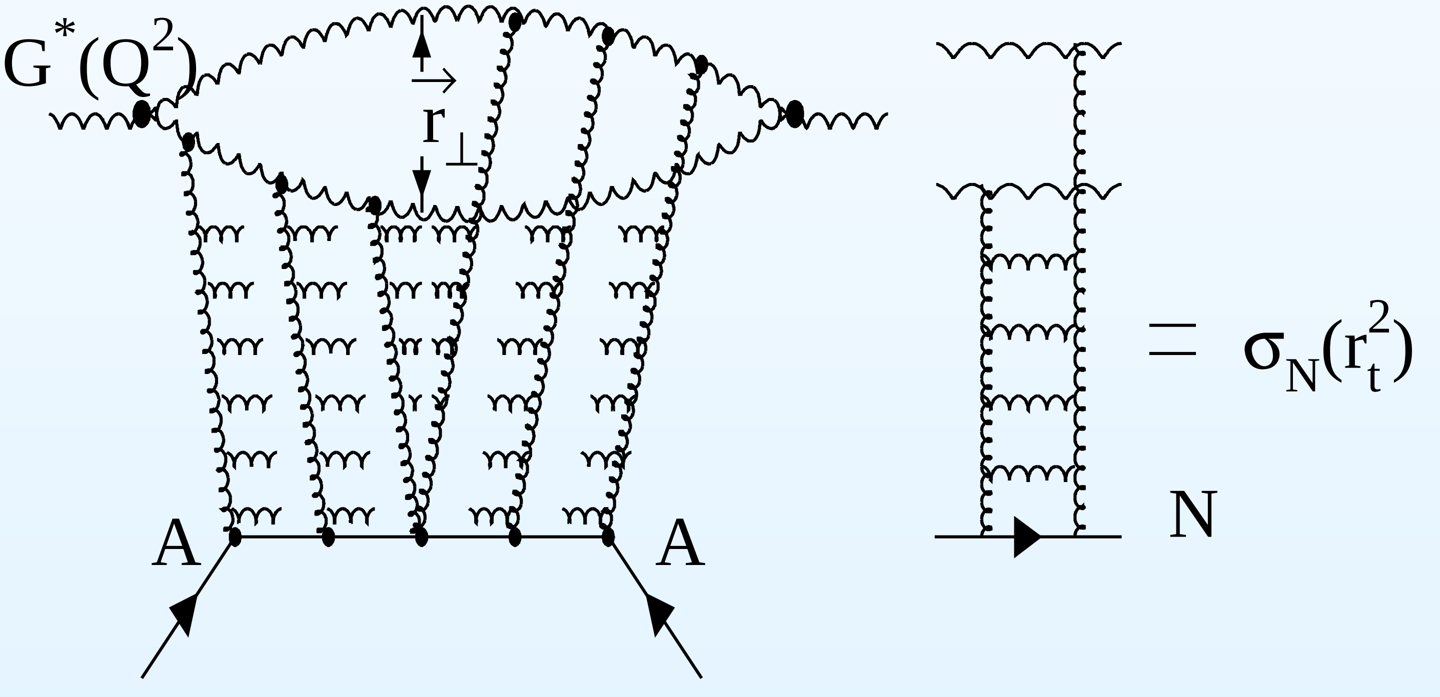
where R is the target size, k_t is the parton transverse momentum and E is the pair energy in the target rest frame.

- Considering that $k_t \propto r_\perp$ It can be showed that

$$x \ll \frac{1}{2mR} \Rightarrow \frac{\Delta r_\perp}{r_\perp} \ll 1, \tag{25}$$

therefore, the parton pair transverse separation holds during the soft radiation and r_t is a good degree of freedom.

QCD Glauber-Mueller Approach



QCD Glauber-Mueller Approach

- The cross section between the virtual particle and the nucleon is given by:

$$\sigma(G^*) = \int_0^1 dz \int \frac{d^2 r_\perp}{\pi} |\Psi_\perp^{G^*}(Q^2, r_\perp, x, z)|^2 \sigma_{tot}^{GG+nucleon}(x, r_\perp^2). \qquad (26)$$

- $\Psi_\perp^{G^*}$ is the transversely polarized gluon wave function.
- $\sigma_{tot}^{GG+nucleon}$ is the cross section between the gluon pair and the nucleon.
- Changes in z during the interaction are neglected.

QCD Glauber-Mueller Approach

- The cross section can be written as:

$$\sigma_{tot}(s) = 2 \int d^2b \; \text{Im} \; a(s,b). \tag{27}$$

  where $a(s,b)$ is the elastic amplitude.

- The unitarity constraint is then:

$$2 \; \text{Im} \; a(s,b) = |a(s,b)|^2 + G_{in}(s,b) \tag{28}$$

  where the term G_{in} represents the sum of all inelastic processes and the elastic amplitude is given by $\sigma_{el} = \int d^2b|a|^2$.

- In the limit of high energy ($x \rightarrow 0$), the real part of a can be neglected and the solution to the above equation is:

$$a(s,b) = \imath(1 - \mathrm{e}^{-\frac{1}{2}\Omega}) \tag{29}$$

$$G_{in} = 1 - \mathrm{e}^{-\Omega} \tag{30}$$

Opacity function

- The opacity function $\Omega = \Omega(x, r_\perp, b)$ is an arbitrary real function, determined by the model used for the interaction.
- $\mathrm{e}^{-\Omega}$ represents the probability of the gluon pair not being inelastically scattered by the target.
- For large Q^2, $\Omega \ll 1$.
- It will be also considered that $\Omega(x, r_\perp, b) = \tilde{\Omega}(x, r_\perp) S(b)$.
- Therefore, it can be showed from last equations that (for $\Omega \ll 1$):

$$\tilde{\Omega} = \sigma_{tot}^{GG+nucleon} . \tag{31}$$

- Remembering that the cross section is dominated by gluon distributions, one can obtain an expression for $\tilde{\Omega}$:

$$\sigma_{tot}^{GG+nucleon} = \frac{3\pi^2 \alpha_s}{4} r_\perp^2 xG\left(x, \frac{4}{r_\perp^2}\right) . \tag{32}$$

QCD Glauber-Mueller Approach

Putting it all together:

$$xG(x,Q^2) = \frac{2}{\pi^2}\int_0^1 dz \int \frac{d^2 r_t}{\pi}\int \frac{d^2 b_t}{\pi}|\Psi_\perp^{G^*}|^2 \int_x^1 \frac{dx'}{x'} \\ 2\left\{1 - e^{-\frac{1}{2}\sigma_{tot}^{GG+nucleon}(x',r_t^2)S(b_t^2)}\right\} \quad (33)$$

Using the approximation

$$|\Psi_\perp^{G^*}|^2 = \frac{2}{z(1-z)r_\perp^4}, \quad (34)$$

$$xG(x,Q^2) = \frac{4}{\pi^2}\int_x^1 \frac{dx'}{x'}\int_0^1 dz \int_{\frac{4}{Q^2}}^\infty \frac{d^2 r_t}{\pi r_\perp^4}\int_0^\infty \frac{d^2 b}{\pi} \\ 2\left\{1 - e^{-\frac{1}{2}\sigma_{tot}^{GG+nucleon}(x',\frac{r_t^2}{4})S(b)}\right\} \quad (35)$$

QCD Glauber-Mueller Approach

The function $S(b^2)$ will be parametrized as a Gaussian function:

$$S(b^2) = \frac{1}{\pi R^2} \mathrm{e}^{-\frac{b^2}{R^2}} . \tag{36}$$

Hence, integrating over b

$$xG(x,Q^2) = \frac{2R^2}{\pi} \int_x^1 \frac{dx'}{x'} \int_{1/Q^2}^{1/Q_0^2} \frac{dr_t^2}{r_t^4} \{C + \ln \kappa_G + E_1(\kappa_G)\} \tag{37}$$

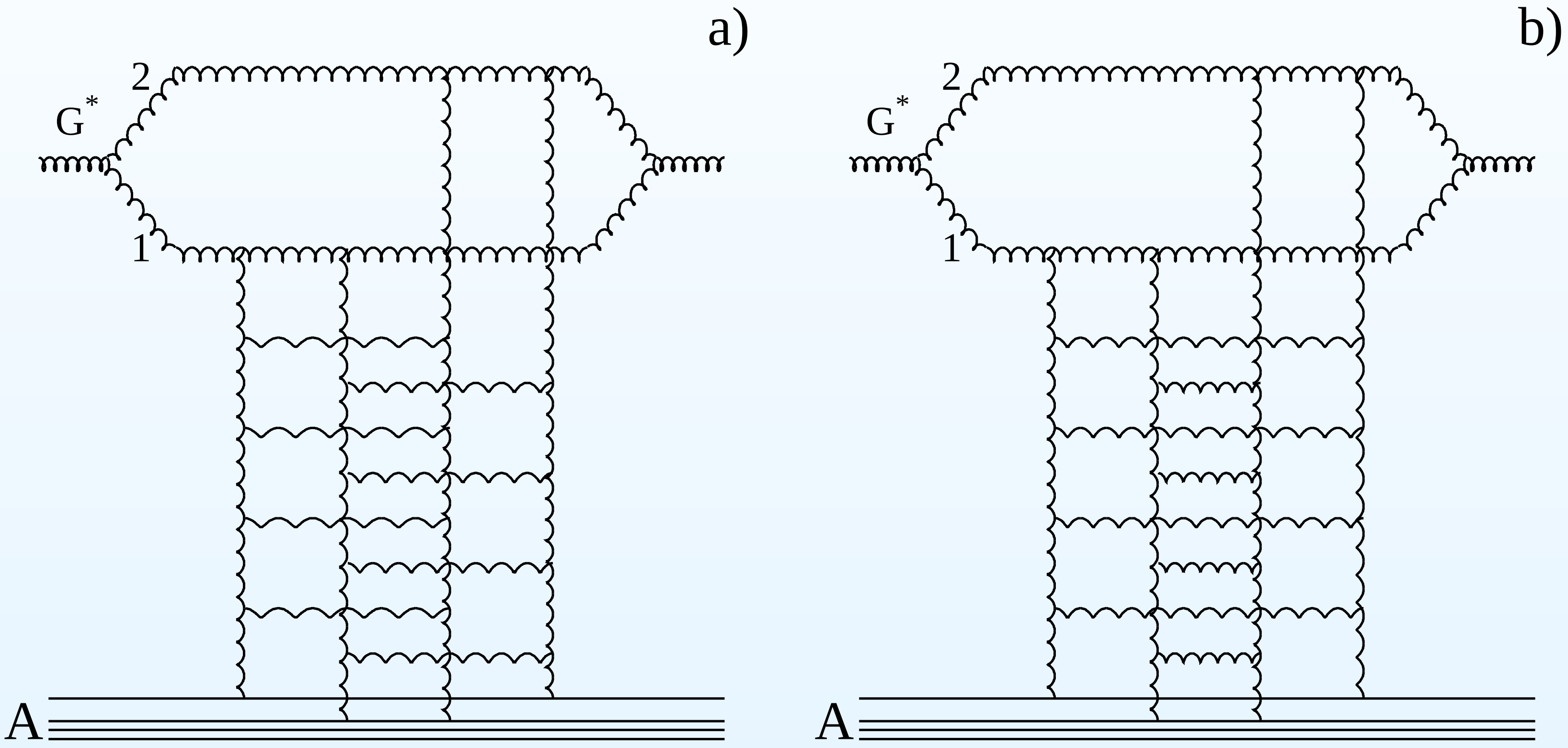
where $C = 0.577215665$ is the Euler constant, $E_1(x) = \int_x^\infty \frac{\mathrm{e}^{-u} du}{u}$ is the exponential integral and κ_G is given by

$$\kappa_G(x, r_t^2) = \frac{3\alpha_s \pi r_t^2}{2R_A^2} xG(x, 1/r_t^2) \tag{38}$$

The probability interpretation of κ_G is related to the **density** of gluons.

QCD Glauber-Mueller Approach

Partons from different parton cascades



a) Not in Mueller Formalism (MF)

b) MF and GLR

Gluon structure function for a nucleon

Testing the model:

$$R_1^N = \frac{xG^{A=1}(x,Q^2)}{xG^{GRV}(x,Q^2)} \qquad (39)$$

Screening correction $\rightarrow$ sizable contribution at very low x.
The average anomalous dimension

$$< \gamma >= \frac{\partial \ln(xG^N(x,Q^2))}{\partial \ln(Q^2/Q_0^2)} \quad \rightarrow \quad Q^2 \ \text{dependence} \qquad (40)$$

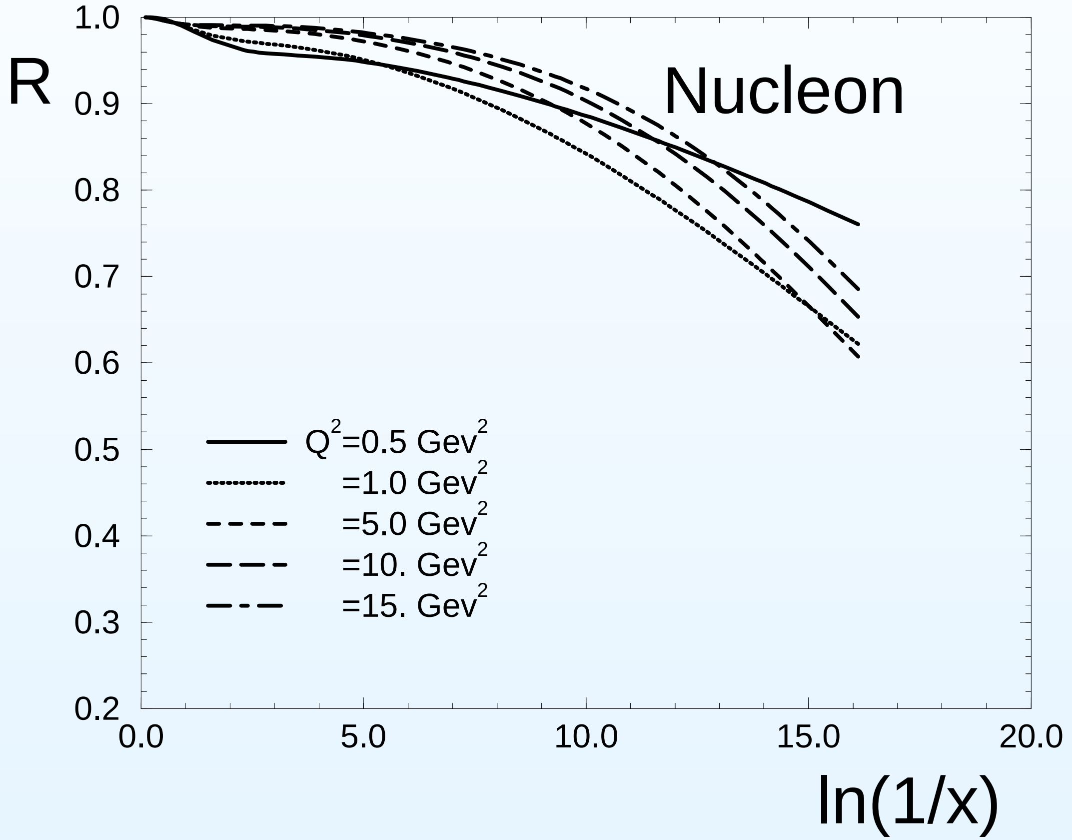
the average effective power

$$< \omega >= \frac{\partial \ln(xG^N(x,Q^2))}{\partial \ln(1/x)} \quad \rightarrow \quad x \ \text{dependence} \qquad (41)$$

In semiclassical approach

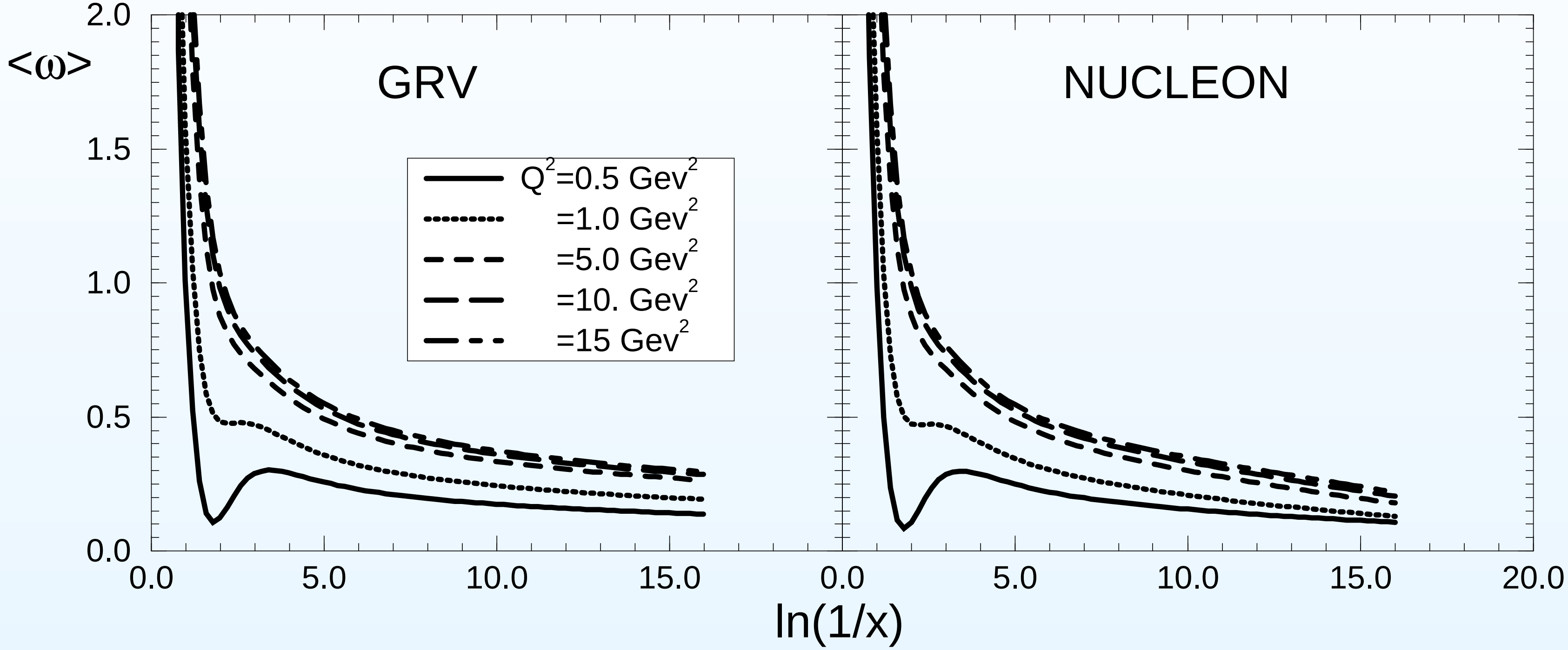
$$xG^N(x,Q^2) \propto \{Q^2\}^{<\gamma>} \left(\frac{1}{x}\right)^{<\omega>} . \qquad (42)$$

QCD Glauber-Mueller Approach



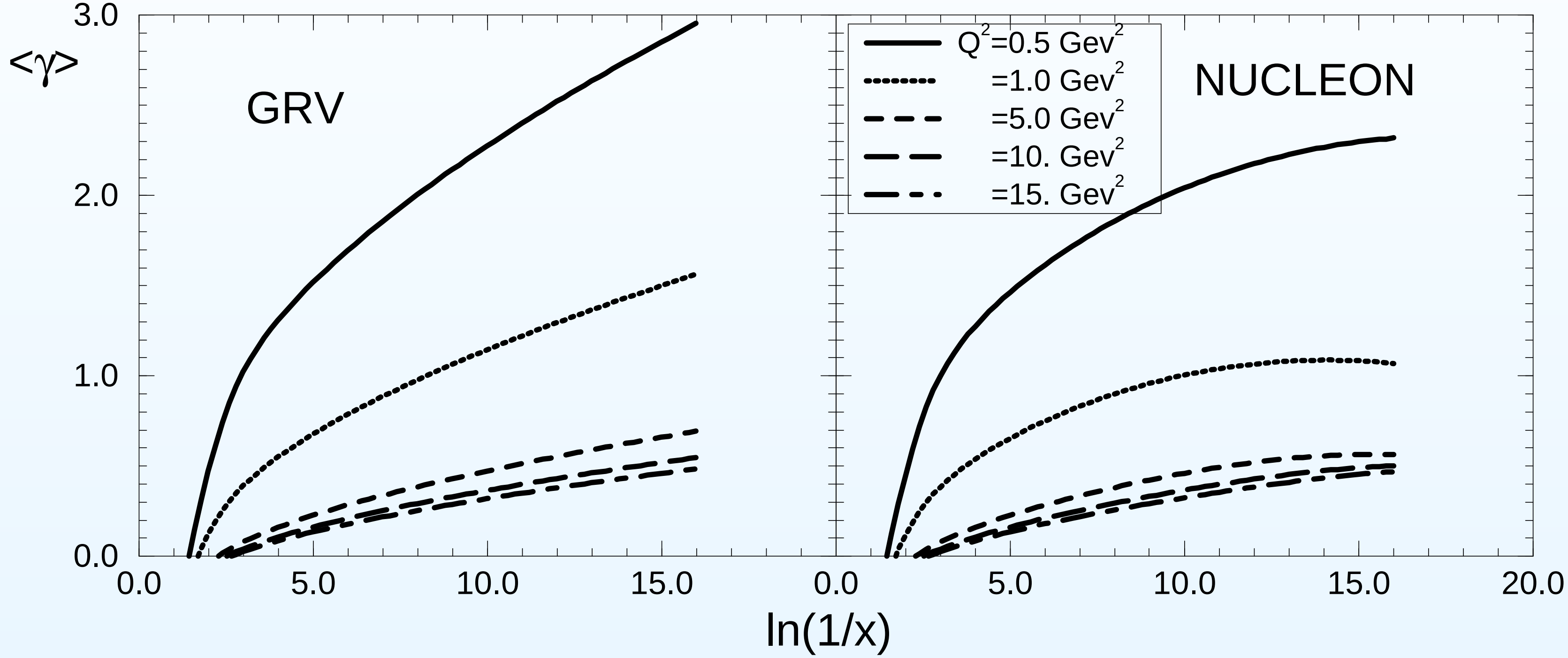
- Screening correction sizable at small x values (no data avaliable).
- For small Q^2, the model is not applicable.

QCD Glauber-Mueller Approach



- Roughly, GRV and MF gluon distributions present the same behavior for $<\omega>$, only that it is slightly suppressed for small x values by the corrections.

QCD Glauber-Mueller Approach



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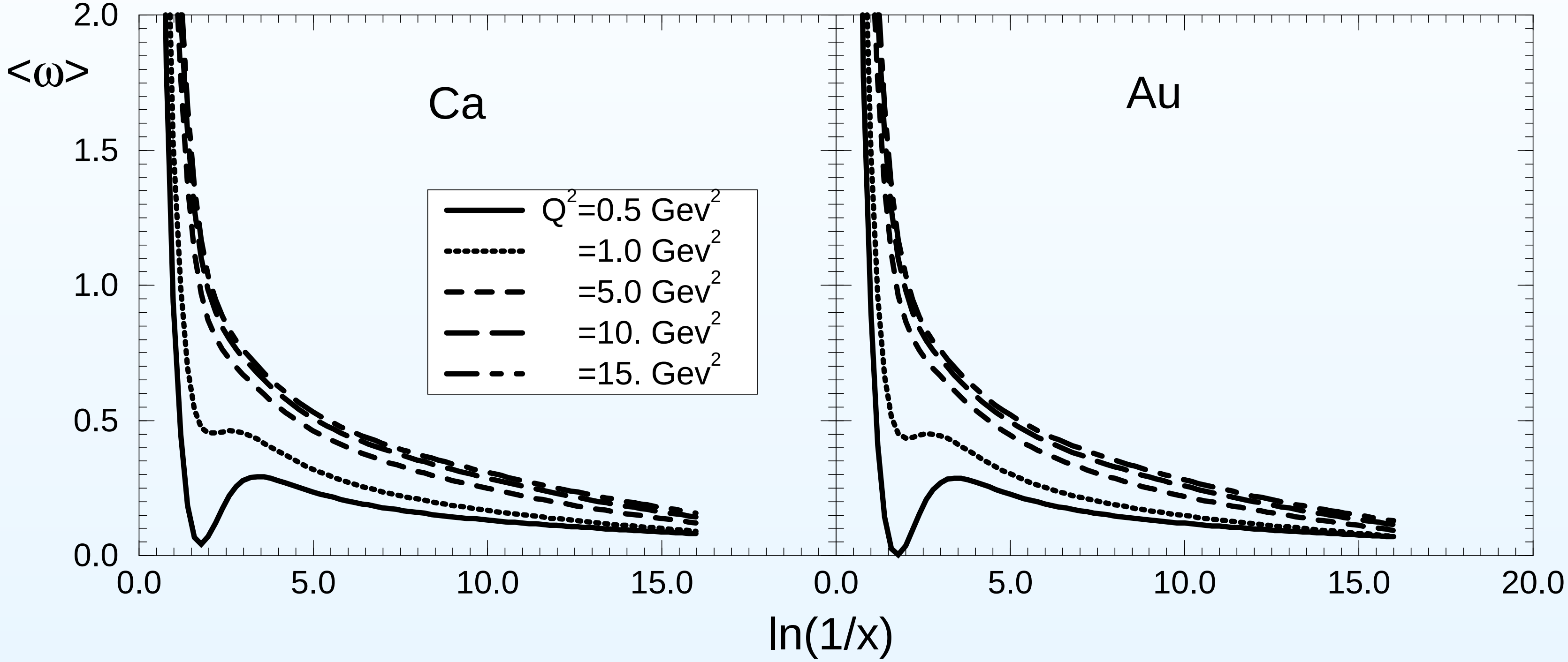
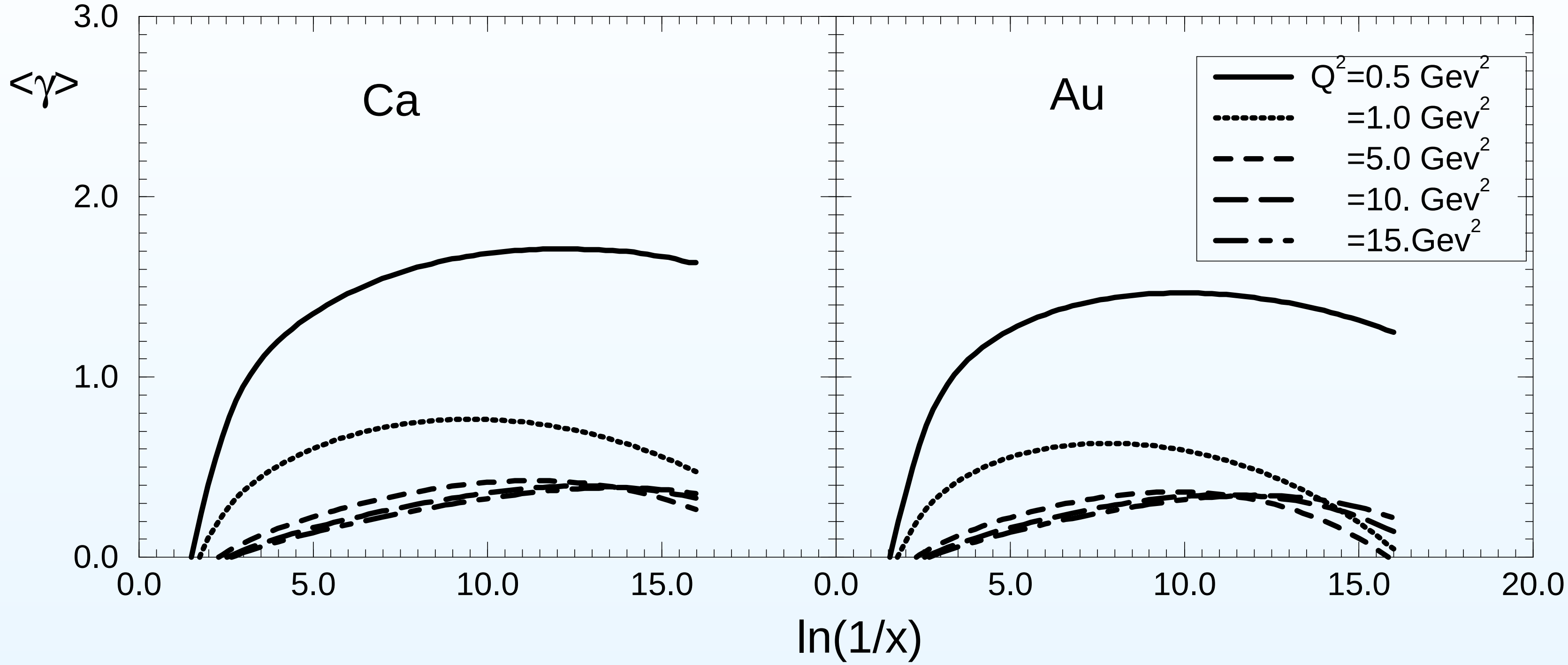
QCD Glauber-Mueller Approach



- For nucleus, $<\omega>$ is more suppresed.
- However, it is never smaller than 0.08, the soft pomeron value.

QCD Glauber-Mueller Approach



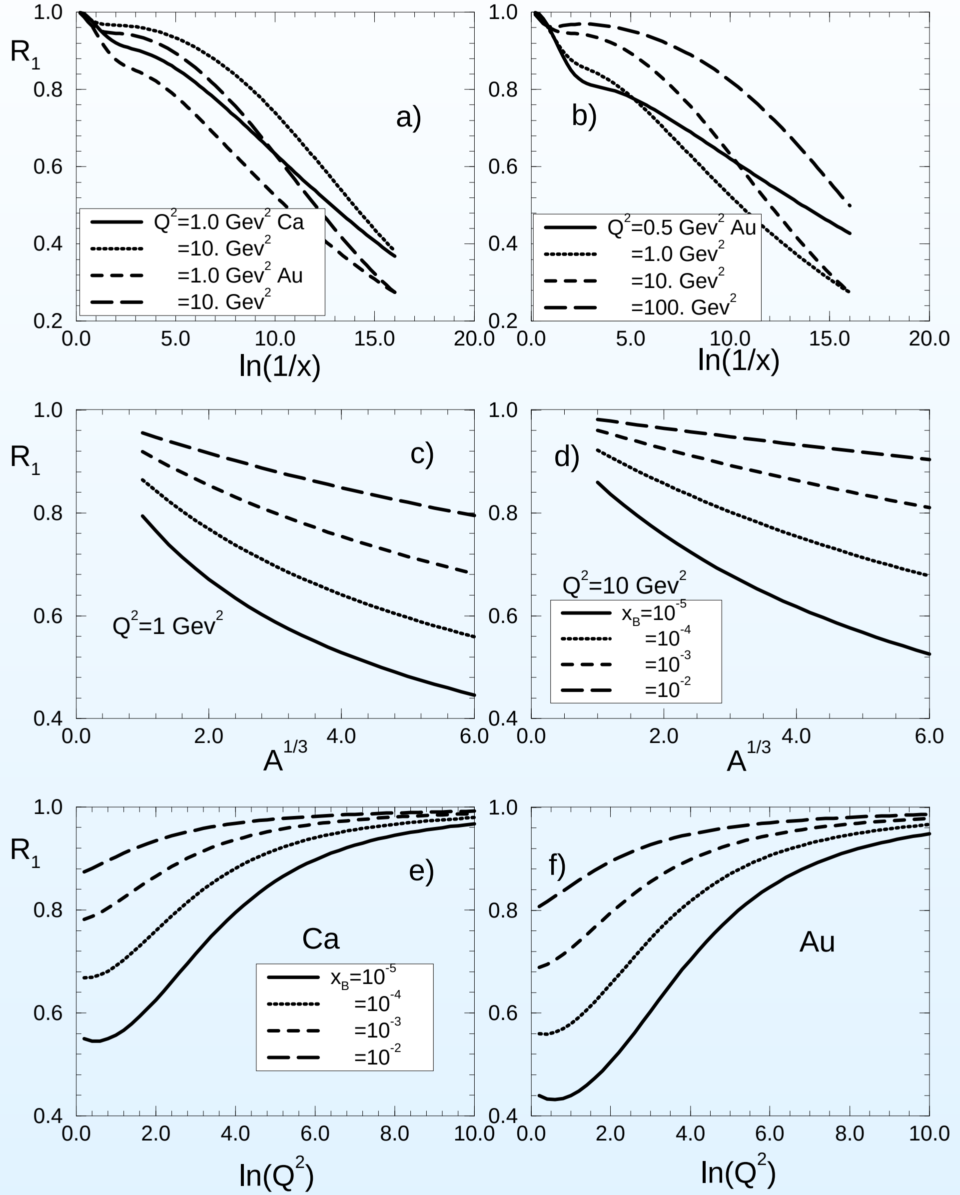
- For nucleus, $< \gamma >$ does not change very much for $\ln(1/x) < 5$.
- However, at smaller values of x, the anomalous dimension presents a sizeable reduction, which increases with A.
- For $Q^2 = 1.0 GeV^2$, the anomalous dimension is close to 1/2, and for $Q^2 > 5.0 GeV^2$ it is always smaller than 1/2.
- For $\ln(1/x) > 15$, $< \gamma >$ tends to zero, unlike DGLAP solutions.

QCD Glauber-Mueller Approach

- The gluon structure function for nucleus: R_1 as a function of the variables $ln(1/x)$, lnQ^2 and $A^{1/3}$;

$$R_1 = \frac{xG^A(x,Q^2)}{AxG_N(x,Q^2)}.$$

- For large nucleus R_1 behaves as a straight line.
- The suppression increases with $ln(1/x)$ and A and decreases with Q^2.
- The gluon structure function is far away from the asymptotic ($R_1 \rightarrow 1$) one.



κ parameter

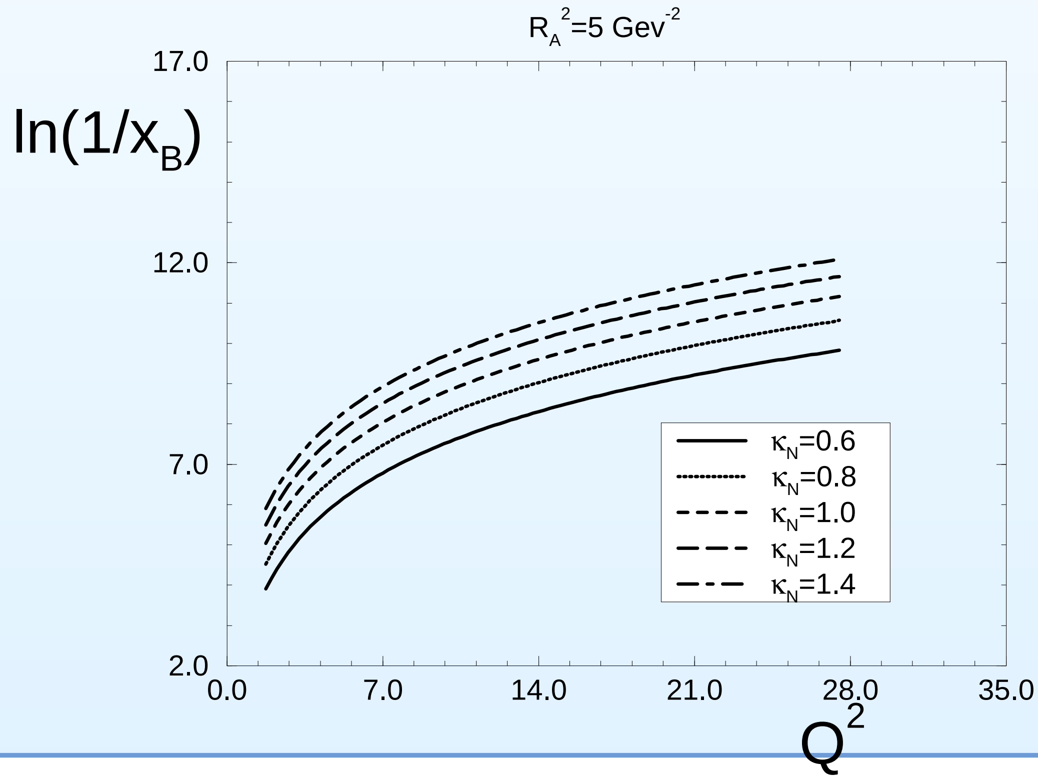
- κ is the strength of the screening corrections, given by:

$$\kappa = \frac{3\alpha_s \pi A}{2Q^2 R_A^2} xg^{DGLAP}(x, Q^2) \quad (43)$$

$$\kappa \gg 1 \Rightarrow \text{Large Screening corrections}$$

$$\kappa \ll 1 \Rightarrow \text{DGLAP holds}$$

Contour for κ = cte. for Nucleon.



QCD Glauber-Mueller Approach

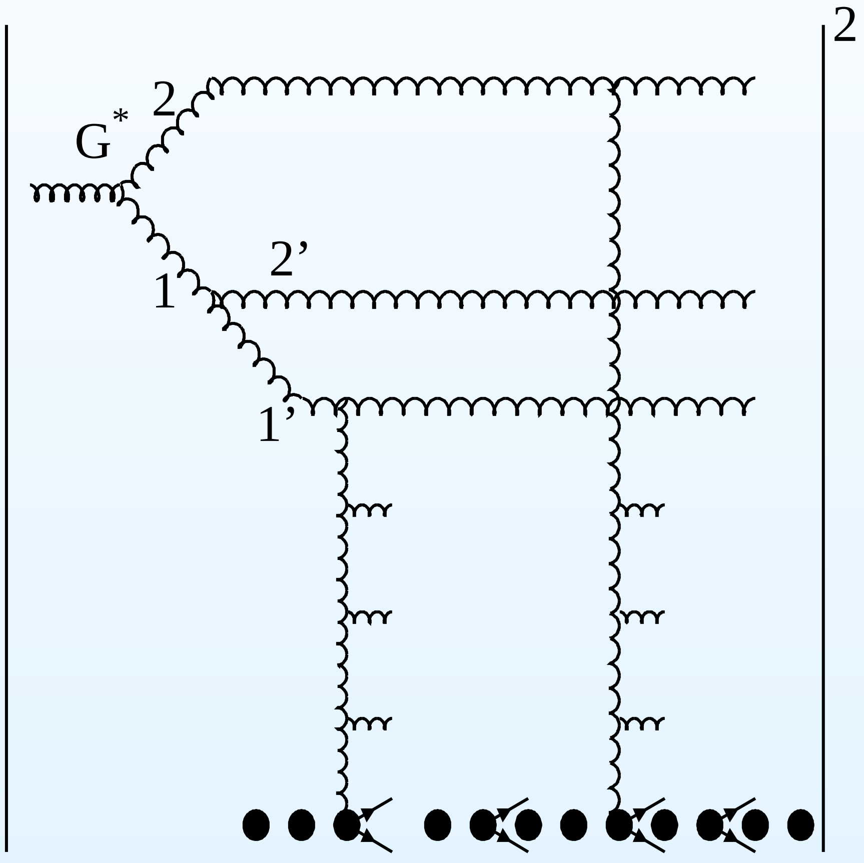
- Beyond the Glauber formula
- Second interaction of MF
- corrections to Glauber approach
- towards a complete theory for DIS off a nucleus
- SIMF $\Rightarrow$ takes into account the rescattering of the next fastest gluon
- Ordering in the parton cascade in leading $\ln(1/x)$

$$x_B < x_n < ... < x_1 < 1 \;\Rightarrow\; \text{fastest parton in the cascade} \tag{44}$$

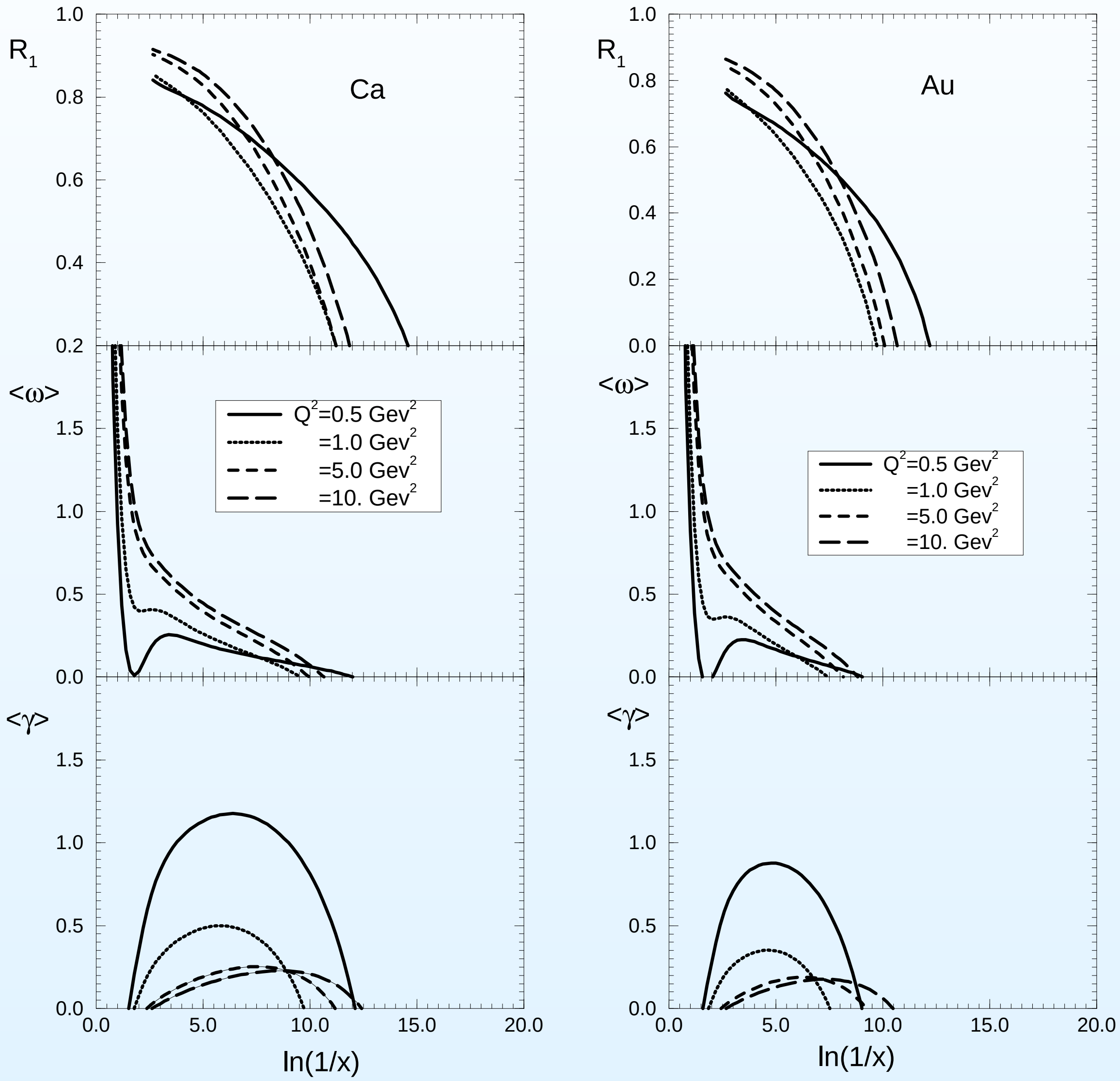
$$\text{In MF} \begin{cases} 1^{\text{st}} \text{ interaction} \Rightarrow & G_N(x,Q^2) = G_N^{GRV}(x,Q^2) \\ 2^{\text{nd}} \text{ interaction} \Rightarrow & G_N(x,Q^2) = \frac{xG_A(x,Q^2)}{A} - G_N(x,Q^2) \end{cases}$$

- The first term of R.H.S. in the second equation is the result of the first interaction.

QCD Glauber-Mueller Approach



QCD Glauber-Mueller Approach



AGL Evolution Equation

- It was show how to use the Mueller formula to sum the screening corrections.
- Then, why an evolution equation is needed?
- It is not sure that the iteration of Mueller equation converges. (The solution can be unstable.)
- The iteration itself is a cumbersome process.
- Then, when the answer is given by a large number of or even all iterations, another mathematical framework is needed.

AGL Evolution Equation

- The Mueller formula is taken:

$$xG(x,Q^2) = \frac{2R^2}{\pi^2}\int_x^1 \frac{dx'}{x'}\int_{\frac{1}{Q^2}}^{\frac{1}{Q_0^2}} \frac{dr_\perp^2}{r_\perp^4}\left[C + \ln\kappa^{\mathrm{DGLAP}} + E_1(\kappa^{\mathrm{DGLAP}})\right] \qquad (45)$$

with

$$\kappa^{\mathrm{DGLAP}}(x,Q^2) = \frac{N_c\alpha_s\pi}{2Q^2R^2}xG^{\mathrm{DGLAP}}(x,Q^2). \qquad (46)$$

- The Mueller equation is derived with relation to $\ln(Q^2/Q_0^2)$ (it is useful to recognize that $\partial_{\ln(Q^2/Q_0^2)} = Q^2\partial_{Q^2} = -Q^{-2}\partial_{Q^{-2}}$:

$$\frac{\partial xG(x,Q^2)}{\partial\ln(Q^2/Q_0^2)} = \frac{2R^2}{\pi^2}\int_x^1 \frac{dx'}{x'}Q^2\left[C + \ln\kappa^{\mathrm{DGLAP}}(x,Q^2) + E_1(\kappa^{\mathrm{DGLAP}}(x,Q^2))\right]. \qquad (47)$$

- Then, the above equation is derived with relation to $\ln(1/x)$ (again, $\partial_{\ln(1/x)} = -x\partial_x$:

$$\frac{\partial^2 xG(x,Q^2)}{\partial\ln(1/x)\partial\ln(Q^2/Q_0^2)} = \frac{2R^2Q^2}{\pi^2}\left[C + \ln\kappa^{\mathrm{DGLAP}}(x,Q^2) + E_1(\kappa^{\mathrm{DGLAP}}(x,Q^2))\right]. \qquad (48)$$

AGL Evolution Equation

- Finaly, the crutial step: instead of using κ^{DGLAP}, simply κ is going to be used:

$$\frac{\partial^2 xG(x,Q^2)}{\partial\ln(1/x)\partial\ln(Q^2/Q_0^2)} = \frac{2R^2Q^2}{\pi^2}\left[C + \ln\kappa(x,Q^2) + E_1(\kappa(x,Q^2))\right] . \tag{49}$$

- Then, the equation can be rewritten for $\kappa(x,Q^2)$ (now that $xG(x,Q^2) = \frac{2Q^2R^2}{N_c\alpha_s\pi}\kappa(x,Q^2)$:

$$\frac{2R^2}{N_c\alpha_s\pi}\left(\frac{\partial^2 Q^2\kappa(x,Q^2)}{\partial\ln(1/x)\partial\ln(Q^2/Q_0^2)}\right) = \frac{2R^2Q^2}{\pi^2}\left[C + \ln\kappa(x,Q^2) + E_1(\kappa(x,Q^2))\right] \tag{50}$$

$$\frac{\partial^2 \kappa}{\partial\ln(1/x)\partial\ln(Q^2/Q_0^2)} + \frac{\partial\kappa(x,Q^2)}{\partial\ln(1/x)} = \frac{N_c\alpha_s}{\pi}\left[C + \ln\kappa(x,Q^2) + E_1(\kappa(x,Q^2))\right] . \tag{51}$$

- These equations are the double-differential AGL evolution equation for the gluon density 49 and for the $\kappa(x,Q^2)$ parameter 51.

AGL Evolution Equation

- A generalized evolution equation was derived.
- The purpose of this equation is to sum all screening corrections (all iterations of Mueller formula).

$$\frac{\partial^2 xG(x,Q^2)}{\partial \ln(1/x)\partial \ln(Q^2/Q_0^2)} = \frac{2R^2Q^2}{\pi^2}\left[C + \ln \kappa(x,Q^2) + E_1(\kappa(x,Q^2))\right] . \qquad (52)$$

- All functions depend on the same Q^2. With a convenient choice of Q^2 (for small distances), only perturbative effects (theoretical known) can be present.
- In spite of that, nonperturbative effects (large distances) cannot be discarded. They enter in boundary and initial conditions.
- Good equation to separate known (pQCD) and unknown (npQCD).

GLR from AGL Evolution Equation

- The evolution equation derived must be consistent with previous GLR evolution equation.
- The GLR dynamics occurs when κ is small. Then;

$$E_1(\kappa) \approx -C - \ln\kappa + \kappa - \kappa^2/4 \tag{53}$$

and the equation is:

$$\frac{\partial^2 xG(x,Q^2)}{\partial\ln(1/x)\partial\ln(Q^2/Q_0^2)} = \frac{2R^2Q^2}{\pi^2}\left[\kappa(x,Q^2) - \frac{\kappa(x,Q^2)^2}{4}\right] \tag{54}$$

$$\frac{\partial^2 xG(x,Q^2)}{\partial\ln(1/x)\partial\ln(Q^2/Q_0^2)} = \frac{2R^2Q^2}{\pi^2}\frac{N_c\alpha_s\pi}{2Q^2R^2}\left[xG(x,Q^2) - \frac{N_c\alpha_s\pi}{2Q^2R^2}\frac{[xG(x,Q^2)]^2}{4}\right] \tag{55}$$

$$\frac{\partial^2 xG(x,Q^2)}{\partial\ln(1/x)\partial\ln(Q^2/Q_0^2)} = \frac{N_c\alpha_s}{\pi}xG(x,Q^2) - \frac{N_c^2\alpha_s^2}{8Q^2R^2}[xG(x,Q^2)]^2. \tag{56}$$

- Since GLR equation is itself a generalization of DGLAP equation in he DLLA, AGL equation also reproduces DGLAP evolution (smaller κ) in this region.

AGL Evolution Equation Properties

- The AGL evolution equation summs all diagrams of order $\left(\left[\alpha \ln(1/x) \ln(Q^2/Q_0^2)\right]^n\right)$ into the gluon function.
- Takes into account the interaction of all partons in a parton cascade with the target.
- All corrections κ^n are taken in account.
- In the limit $N_c \to \infty$, the equation completely describes the screening corrections.
- For $\alpha \ln(1/x) \ln(Q^2/Q_0^2) \approx 1$, this equation is equivalent to the Mueller equation (DGLAP and GLR eq. are not).
- For nuclear gluon distribution, all it has to be done is to redefine κ:

$$\kappa_A(x, Q^2) = \frac{N_c \alpha_s \pi}{2Q^2 R_A^2} x G_A(x, Q^2), \quad (57)$$

in which the factor A is included in $G_A(x, Q^2)$.

b_t-dependent AGL Evolution Equation

- AGL equation can be written with a b_t dependence. The starting point is the b_t(impact parameter)-dependent Mueller formula:

$$xG(x,Q^2) = \frac{4}{\pi^2}\int_x^1 \frac{dx'}{x'}\int_{\frac{1}{Q^2}}^{\frac{1}{Q_0^2}} \frac{dr_\perp^2}{r_\perp^4}\int_0^\infty 2\,[1-\exp(-\kappa)] \tag{58}$$

$$\frac{\partial^2 \kappa(y,\xi,b_t)}{\partial y \partial \xi} + \frac{\partial \kappa}{\partial y} = \frac{N_c \alpha_s}{\pi}\left\{1 - e^{-\kappa(x,Q^2,b_t)}\right\} \equiv F_{b_t}(\kappa) \tag{59}$$

- $\sum$ Feynman diagrams at

$$\left.\begin{aligned} &\alpha_s y \xi = 1 \\ &\alpha_s y < 1\,,\ \ \alpha_s \xi < 1\,,\ \ \alpha_s \ll 1 \\ &\alpha_s \kappa \leq 1 \end{aligned}\right\} DLA\ pQCD \tag{60}$$

κ considers the parton-parton interaction in the perturbative cascade

Solutions of AGL Equation

- AGL equation is a second order hiperbolic partial differential equation.
- Therefore, two initial conditions are required.
- At fixed x and $Q^2 \to \infty$, the gluon distribution must be te same as the one given by DGLAP equation:

$$\kappa(x, Q^2) = \frac{N_c \alpha_s \pi}{2Q^2 R^2} xG^{\rm DGLAP}(x, Q^2). \tag{61}$$

- At small $x = x_0$ ($y = y_0$, $\alpha_s y \xi \leq 1$):

$$\kappa \to \kappa_{in} = \frac{N_c \alpha_s \pi}{2Q^2 R^2} xG(x, Q^2) \tag{62}$$

  where the xG is given by the modified Mueller formula.
- For $x_0 = 10^{-2}$ and $Q^2 \geq 1$ GeV^2 the pQCD calculations are valid.
- For simplicity, the variables $y = \ln(1/x)$ (rapidity) and $\xi = \ln(Q^2/Q_0^2)$ are going to be used.

Asymptotic solution of AGL Equation

- For small x, one can suppose that the solution does not depend on ξ:

$$\lim_{y\to\infty} \kappa(y,\xi) = \kappa_{\rm a}(y). \tag{63}$$

- The AGL equation for the asymptotic solution is then:

$$\frac{d\kappa_{\rm a}}{dy} = F(\kappa_{\rm a}). \tag{64}$$

- The equation above can be solved analitically:

$$y - y_0 = \int_{\kappa_{\rm a}(y-y_0)}^{\kappa_{\rm a}(y)} \frac{d\kappa'}{F(\kappa')} \tag{65}$$

- If $\bar{\alpha}_s y > 1$, then $\kappa_{\rm a} \to \bar{\alpha}_s y \ln \bar{\alpha}_s y$.
- If $\bar{\alpha}_s (y - y_0) < 1$, then $\kappa_{\rm a} \to \kappa(y_0) {\rm e}^{\bar{\alpha}_s (y-y_0)}$.

Solutions of AGL Equation

- To see if the asymptotic solution exists, one has to show that the correction goes to zero for large y.
- Look for solution in the form

$$\kappa(y,\xi) = \kappa_{\rm a}(y) + \Delta\kappa(y,\xi-\xi_0) \tag{66}$$

where $\Delta\kappa \ll \kappa_{\rm a}$.

$$\frac{\partial^2 \Delta\kappa(y,\xi)}{\partial y \partial \xi} + \frac{\partial \Delta\kappa(y,\xi)}{\partial y} = \left.\frac{dF(\kappa)}{d\kappa}\right|_{\kappa=\kappa_{\rm a}} \Delta\kappa(y,\xi) \tag{67}$$

$$\frac{dF(\kappa)}{d\kappa} \to 0, \qquad \text{at large y,} \qquad \frac{\Delta\kappa}{\kappa} \to 0. \tag{68}$$

Solutions of AGL Equation

$$\frac{\partial^2\kappa(y,\xi,b_t)}{\partial y\partial\xi} + \frac{\partial\kappa}{\partial y} = \frac{N_c\alpha_s}{\pi}\left\{1 - e^{-\kappa(x,Q^2,b_t)}\right\} \equiv F_{b_t}(\kappa) \quad (69)$$

- The b_t-dependent equation also has an asymptotic solution

$$\kappa = \kappa_{\mathrm{a}}(y,b_t) \Rightarrow \frac{d\kappa_{\mathrm{a}}}{dy} = F_{b_t}(\kappa_{\mathrm{a}}) \quad (70)$$

$$\int_{\kappa_{\mathrm{a}}(y_0,b_t)}^{\kappa_{\mathrm{a}}(y,b_t)} \frac{d\kappa'}{F_{b_t}(\kappa')} = y - y_0 \quad (71)$$

then

$$\kappa_{\mathrm{a}}(y,b_t) = \ln\left\{1 + \left(e^{\kappa_{\mathrm{a}}(y_0,b_t)} - 1\right)\mathrm{e}^{\bar{\alpha}_s(y-y_0)}\right\} \quad (72)$$

- If $\bar{\alpha}_s y \gg 1$, then $\kappa_{\mathrm{a}}(y,b_t) \to \alpha_s y$
- If $\bar{\alpha}_s(y-y_0) < 1$, then $\kappa_{\mathrm{a}}(y,b_t) \to \kappa_{\mathrm{a}}(y,b_t)\mathrm{e}^{\bar{\alpha}_s(y-y_0)}$.

Solutions of AGL Equation

- Semiclassical approach: $\kappa = \mathrm{e}^S$ (with $\frac{\partial S}{\partial y} = \omega$ and $\frac{\partial S}{\partial \xi} = \gamma$).
- Suppose that $\frac{\partial^2 S}{\partial y \partial \xi} \ll \omega\gamma$ and then:

$$\frac{\partial S}{\partial y}\frac{\partial S}{\partial \xi} + \frac{\partial S}{\partial y} = e^{-S} F(e^S) = \phi(S) \tag{73}$$

$$\omega(\gamma + 1) = \phi(S) \tag{74}$$

  and for b_t one would have $F(\kappa) \to F_{b_t}(\kappa)$.
- Introducing a set of characteristics $\xi(t)$, $y(t)$, $S(t)$, $\omega(t)$, $\gamma(t)$ for $F(\xi, y, S, \omega, \gamma) = \omega(\gamma + 1) - \phi(S) = 0$, one is left with the following equations:

$$\frac{d\xi}{dt} = F_\gamma, \quad \frac{dy}{dt} = F_\omega, \quad \frac{dS}{dt} = \gamma F_\gamma + \omega F_\omega, \tag{75}$$

$$\frac{d\gamma}{dt} = -(F_\xi + \gamma F_S), \quad \frac{d\omega}{dt} = -(F_y + \omega F_S) \tag{76}$$

Solutions of AGL Equation

$$\frac{d\xi}{dt} = \omega, \quad \frac{dy}{dt} = \gamma + 1, \quad \frac{dS}{dt} = \omega(2\gamma + 1), \tag{77}$$

$$\frac{d\gamma}{dt} = \phi'(S)\gamma, \quad \frac{d\omega}{dt} = \phi'(S)\omega. \tag{78}$$

- Eliminating the dependence in t by the second equation above and the dependence in ω by $\omega(\gamma + 1) = \phi(S)$:

$$\frac{d\xi}{dy} = \frac{\phi(S)}{(\gamma + 1)^2}, \quad \frac{dS}{dy} = \frac{2\gamma + 1}{(\gamma + 1)^2}\phi, \quad \frac{d\gamma}{dy} = \phi\frac{\gamma}{\gamma + 1} \tag{79}$$

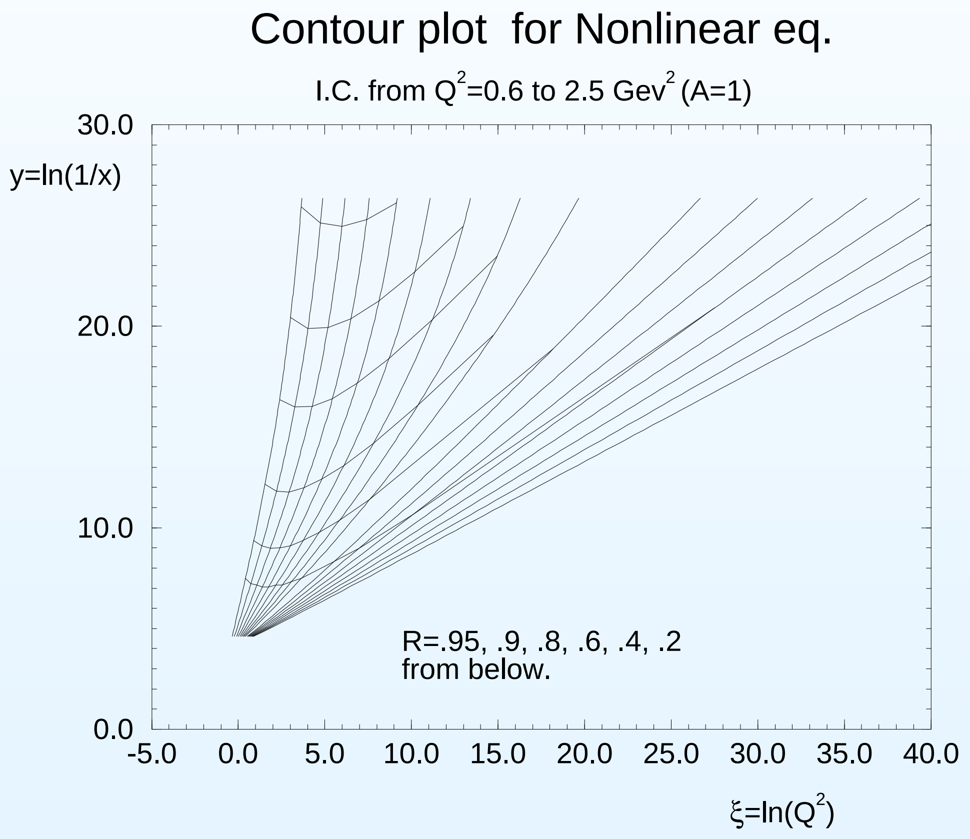
- Initial conditions

$$S_0 = \ln \kappa_{in}(y_0, \xi_0), \qquad \gamma_0 = \left.\frac{\partial \ln \kappa_{in}(y_0, \xi)}{\partial \xi}\right|_{\xi = \xi_0} \tag{80}$$

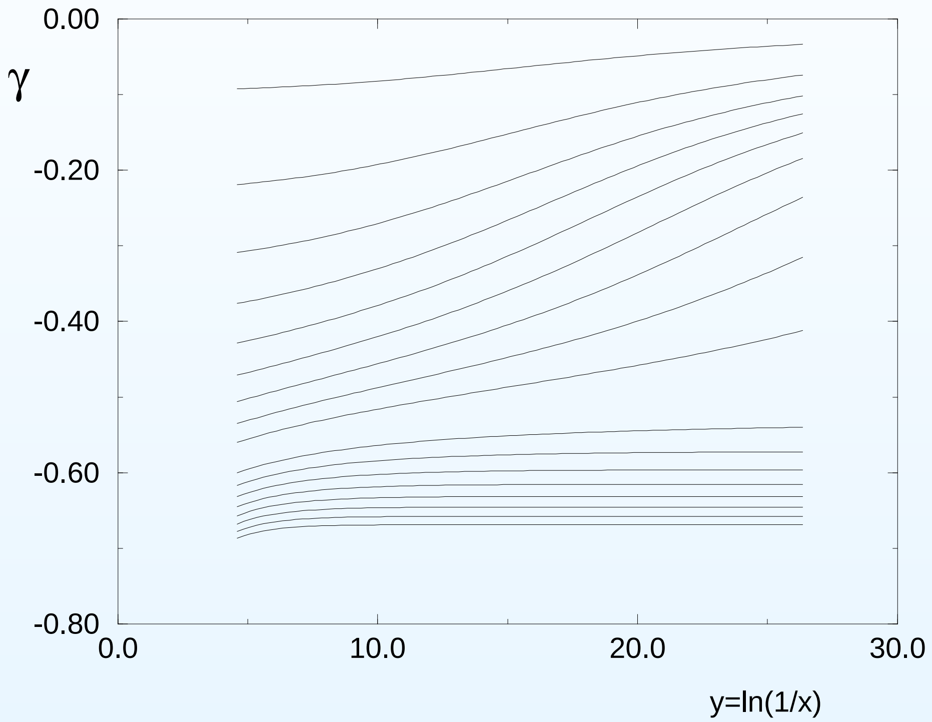
- y_0 is fixed but ξ_0 is not.
- Numerical solution?

Solutions of AGL Equation

- The trajectories of nonlinear equation approach DGLAP for $\gamma_0 < -1/2$
- For $\gamma_0 > -1/2$, $\frac{dS}{dy} > 0$ and $\frac{d\gamma}{dy} > 0$

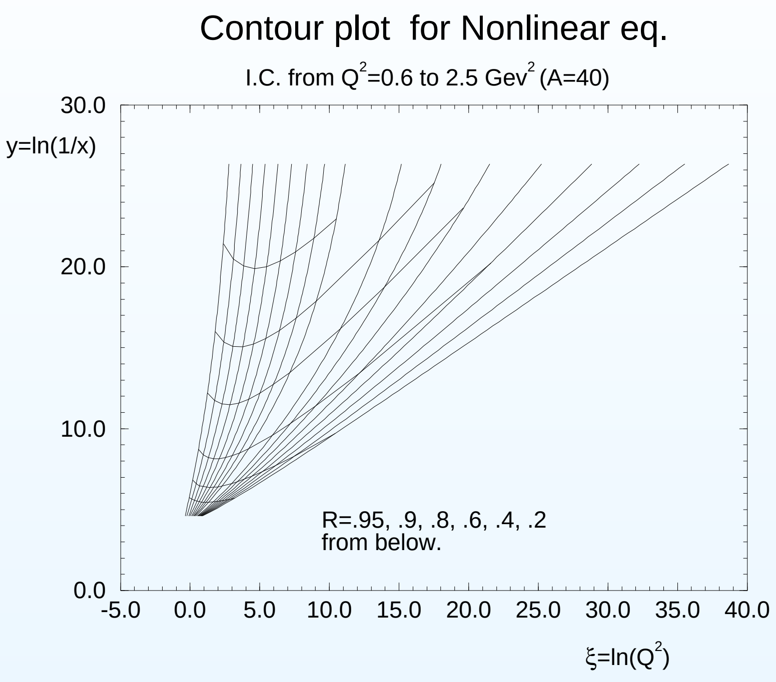


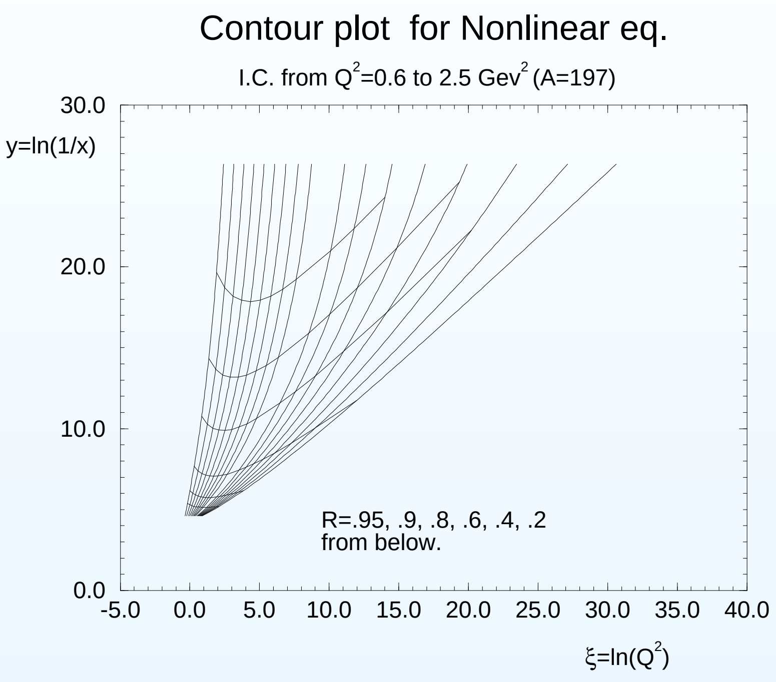
Solutions of AGL Equation



- γ evolution: when γ goes to zero as y grows, the nonlinear effects play an important role. The respective trajectory tends to a vertical line, and the AGL solution tends to the asymptotic one.
- When γ goes to a constant, the AGL solution tends to the DGLAP one.

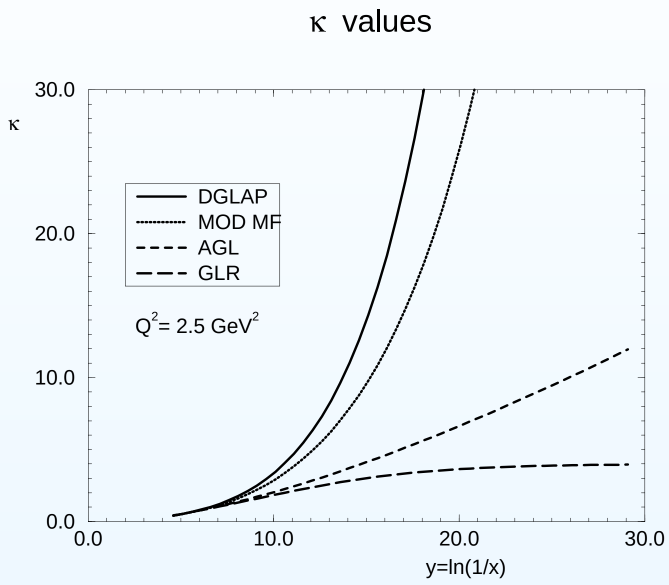
Solutions of AGL Equation

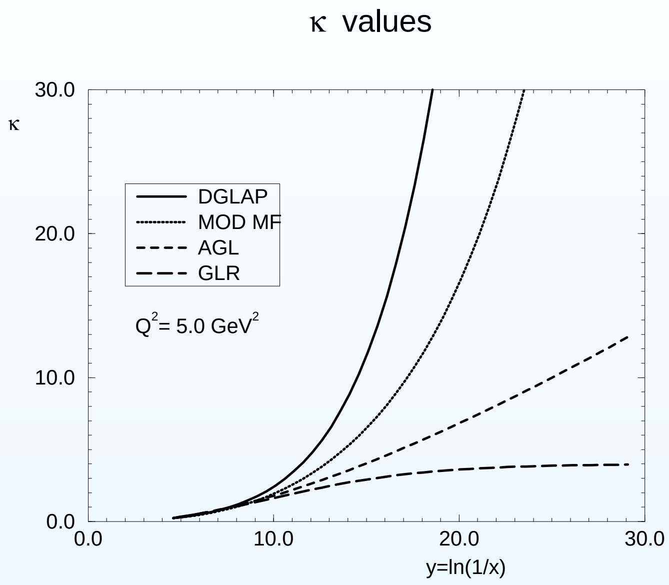


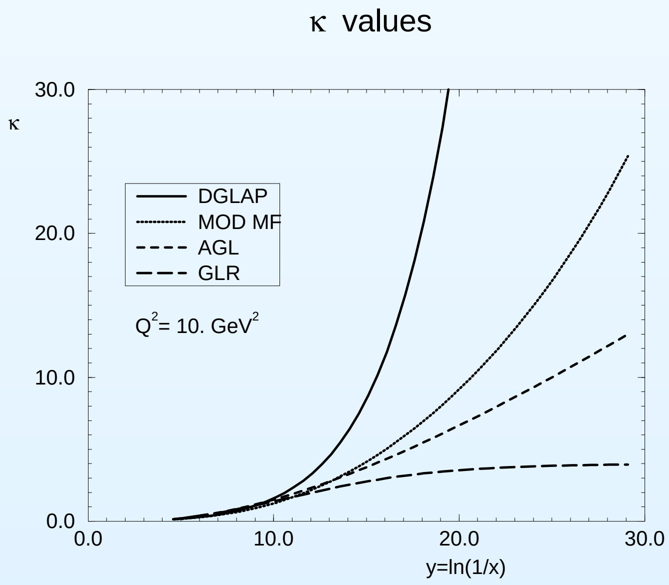


- Left plot: Ca. Right plot: Au.
- R lines show that the screening corrections are big.
- Trajectories and contour plot for the solutions of the generalized equation.

Solutions of AGL Equation







AGL with running α_s

$$\alpha_s(Q^2) = \frac{4\pi}{\beta_0 \varepsilon} \; , \qquad \beta_0 = 11 - \frac{2}{3} n_f \; , \qquad \varepsilon = \ln \left(\frac{Q^2}{\Lambda^2_{QCD}} \right) \tag{81}$$

$G \leftrightarrow \kappa_G$

$$xG(x, Q^2) = \frac{2Q^2 R^2}{N_c \pi} \frac{\beta_0}{4\pi} \varepsilon \kappa_G(x, Q^2) \tag{82}$$

from that

$$\frac{\partial^2 \kappa_G(y, \varepsilon)}{\partial y \partial \xi} + \left(\frac{1}{\varepsilon} + 1 \right) \frac{\partial \kappa_G(y, \varepsilon)}{\partial y} = \frac{N_c \alpha_s(Q^2)}{\pi} \{ C + \ln \kappa_G + E_1(\kappa_G) \} \tag{83}$$

limit large $\varepsilon \Rightarrow$ AGL (fixed α_s)

AGL with running α_s

Solution asymptotic case,

$$\frac{\partial \kappa_G^{asymp}(y,\varepsilon)}{\partial y} = \frac{\varepsilon}{1+\varepsilon} H(\kappa_G) \tag{84}$$

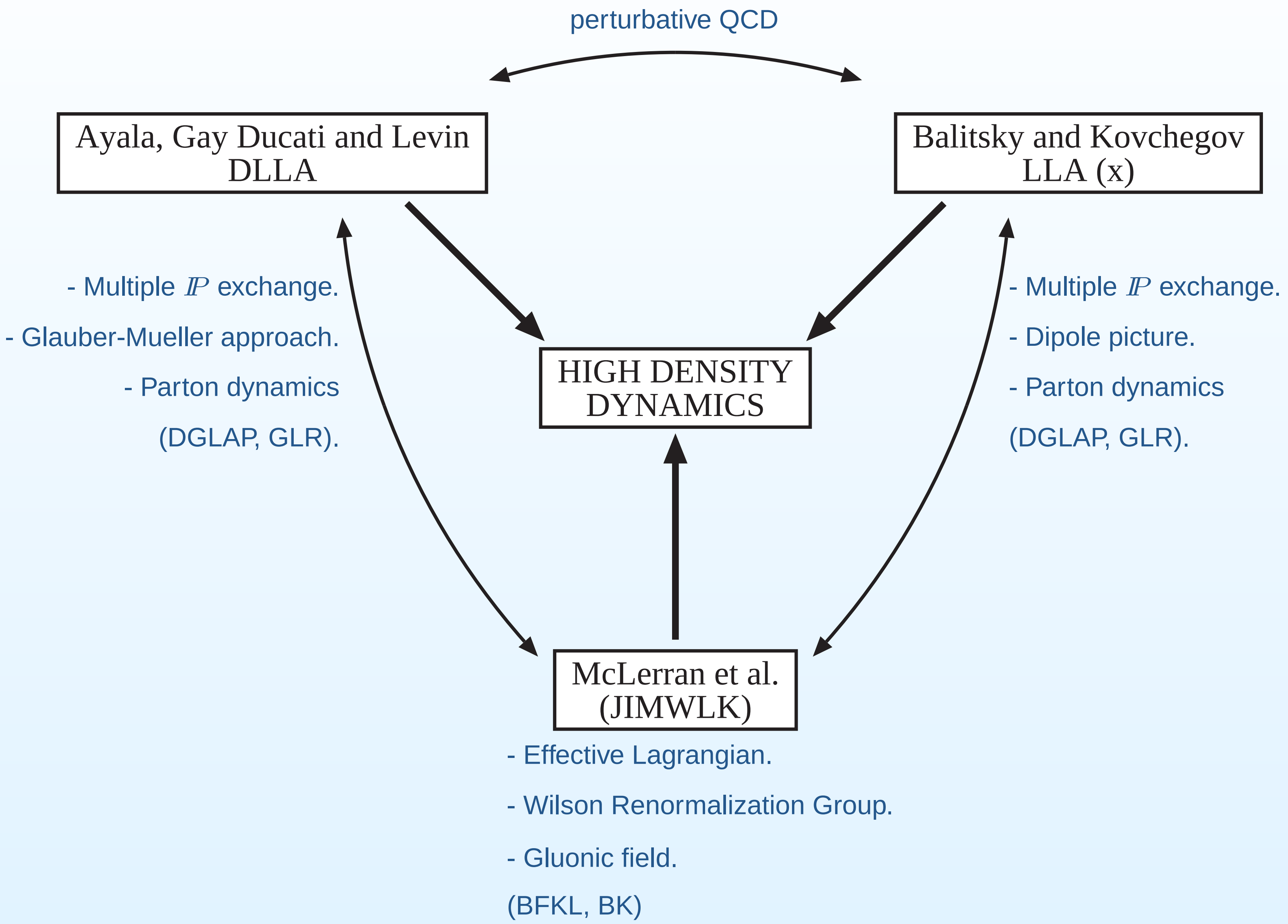
$$\int_{\kappa_G^{asymp}(y-y_0)}^{\kappa_G^{asymp}(y)} \frac{d\kappa'_G}{H(\kappa'_G)} = \frac{\varepsilon}{1+\varepsilon}(y-y_0) \tag{85}$$

Same steps as before,

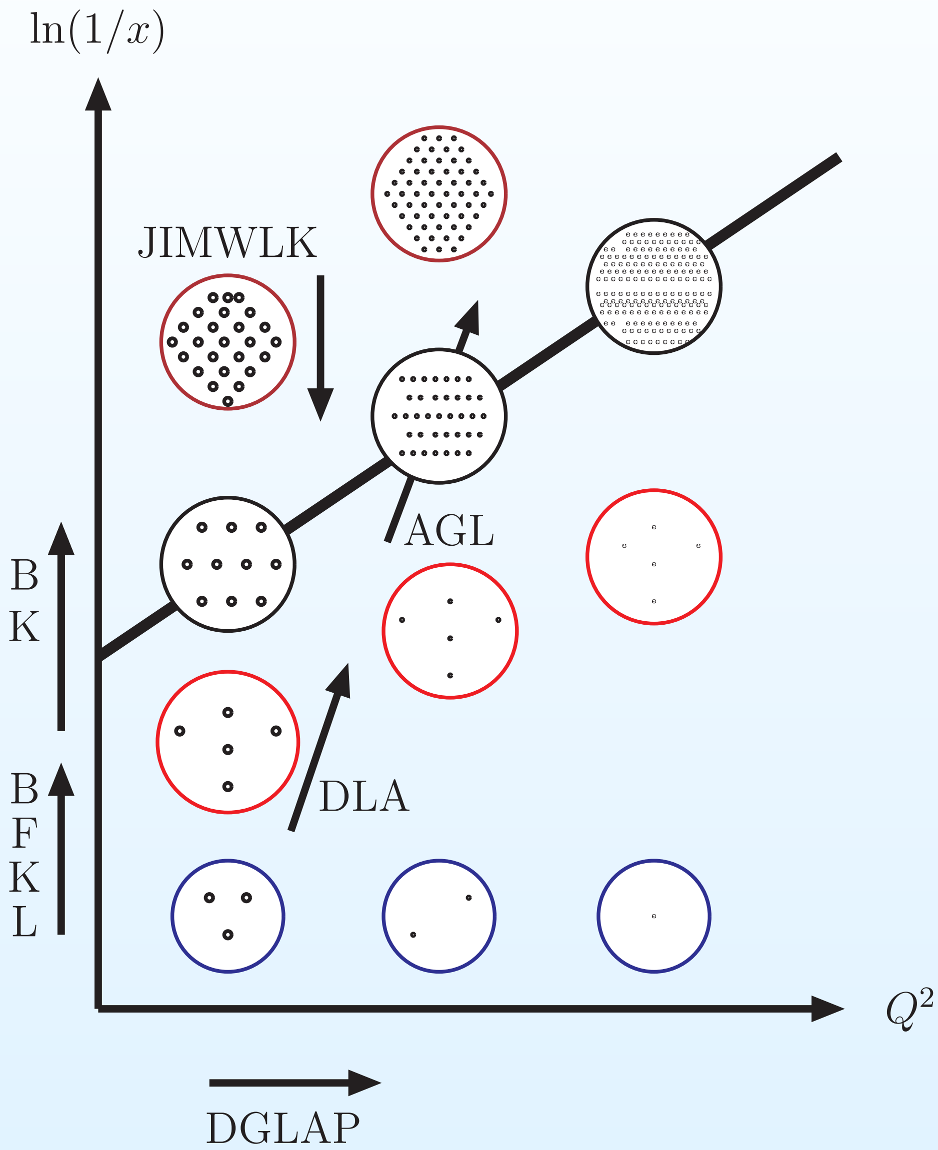
$$xG(x,Q^2) = \frac{\varepsilon}{1+\varepsilon} \frac{2N_c Q^2 R^2}{3\pi^2} \ln(1/x) \quad \Rightarrow \quad \alpha_s \ln s \tag{86}$$

Large $\varepsilon(Q^2) \Rightarrow$ solution fixed α_s, partial saturation is not modified $\Rightarrow$ unitarity correction before NLO.

Evolution equations



Evolutions equations



Evolution equations

- **Linear evolution**
  - DGLAP ($\sim$1977) evolves quark and gluon distributions in Q^2.

$$\frac{dg(x,Q^2)}{d\ln Q^2} = \frac{\alpha_s(Q^2)}{2\pi}\int_x^1 \frac{dy}{y}\left[P_{gq}\left(\frac{x}{y}\right)q_i^S(y,Q^2) + P_{gg}\left(\frac{x}{y}\right)g(y,Q^2)\right],$$

(Dokshitzer, Gribov, Lipatov, Altarelli, Parisi)

  - BFKL ($\sim$1977) evolves non-integrated gluon distribution in x.

$$\frac{\partial\phi(x,k_\perp^2)}{\partial\ln(1/x)} = \frac{3\alpha_s}{\pi}k_\perp^2\int_0^\infty \frac{dk_\perp'^2}{k_\perp'^2}\left\{\frac{\phi(x,k_\perp'^2)+\phi(x,k_\perp^2)}{|k_\perp'^2-k_\perp^2|} + \frac{\phi(x,k_\perp^2)}{\sqrt{4k_\perp'^4+k^4}}\right\},$$

(Balitsky, Fadin, Kuraev, Lipatov)

- **Non-linear evolution**
  - GLR (1983) evolves $xg(x,Q^2)$ in x and Q^2.

$$\frac{\partial^2 xg(x,Q^2)}{\partial\ln Q^2\,\partial\ln 1/x} = \frac{\alpha_s N_c}{\pi}xg(x,Q^2) - \frac{\alpha_s^2\gamma}{Q^2R^2}[xg(x,Q^2)]^2$$

(Gribov-Levin-Ryskin)

  - AGL (1997) evolves $\kappa_G(x,Q^2) = \frac{N_c\alpha_s\pi}{2Q^2R^2}xg(x,Q^2)$ in x and Q^2.

$$\frac{\partial^2\kappa_G(x,Q^2)}{\partial(\ln 1/x)\partial(\ln Q^2)} + \frac{\partial\kappa_G(x,Q^2)}{\partial(\ln 1/x)} = \frac{N_c\alpha_s}{\pi}[C + \ln(\kappa_G) + E_1(\kappa_G)]$$

(Ayala-MBGD-Levin)

  - BK (1996-1999) evolves the the dipole density (N) in $Y = \ln(1/x)$.

$$\frac{\partial^2 N(\vec{x}_{01},\vec{b}_0,Y)}{\partial Y\,\partial\ln(1/x_{01}^2\Lambda_{QCD}^2)} = \frac{\alpha_s C_F}{\pi}[2 - N(\vec{x}_{01},\vec{b}_0,Y)]N(\vec{x}_{01},\vec{b}_0,Y)$$

(Balitsky-1996; Kovchegov-1999)

  - JIMWLK ($\sim$1997-01) evolves the color charge sources correlation in $Y = \ln(1/x)$. $\frac{\partial W_Y[\rho]}{\partial Y} = \frac{1}{2}\int \frac{\delta}{\delta\rho_Y^a(x_\perp)}\chi_{ab}(x_\perp,y_\perp)[\rho]\frac{\delta}{\delta\rho_Y^b(y_\perp)}\mathcal{W}_Y[\rho]$,

(Jalilian-Marian, Kovner, Leonidov, Weigert, Iancu, McLerran)

References

- M. Froissart, Phys. Rev. 123, 1053 (1961).
- L.V. Gribov, E.M. Levin, M.G. Ryskin, Nucl. Phys. B **188**, 555 (1981)
- L.V. Gribov, E.M. Levin, M.G. Ryskin, Phys. Rept. 100, 1 (1983).
- A.H. Mueller, J. Qiu. Nucl. Phys. B **268** (1986) 427.
- A.L. Ayala, M.B. Gay Ducati and E.M. Levin. Nucl. Phys. B **493** (1997) 305.
- A.L. Ayala, M.B. Gay Ducati and E.M. Levin. Nucl. Phys. B **511** (1998) 355.
- A.L. Ayala, M.B. Gay Ducati and E.M. Levin. Eur. Phys. J. C **8** (1999), p. 115.