$\gamma^* - p$ cross section from dipole model in momentum space

J. T. S. Amaral

thiago.amaral@ufrgs.br

Instituto de Física, Universidade Federal do Rio Grande do Sul, Porto Alegre, Brazil

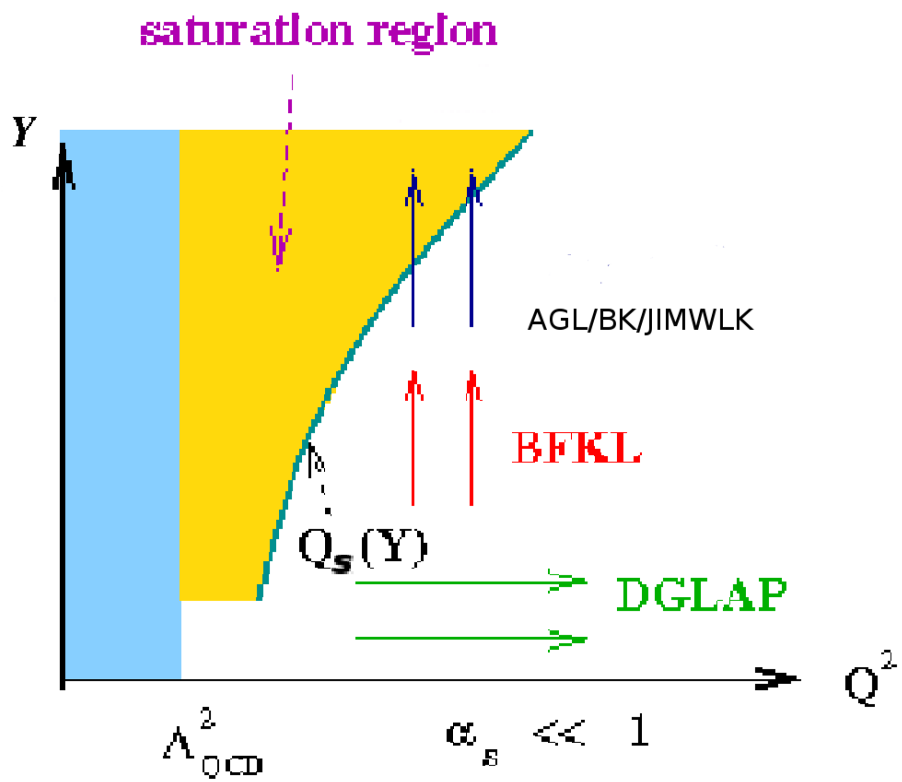
GFPAE - UFRGS

http://www.if.ufrgs.br/gfpae

Introduction

- One of the most intriguing problems in Quantum Chromodynamics is the growth of the cross sections for hadronic interactions with energy; the increase of energy causes a fast growth of the gluon density and consequently of the cross section

- It is believed that at this regime gluon recombination might be important and it would decrease the growth of the parton density; this is called saturation



- $Q_s(Y)$ is the so called saturation scale

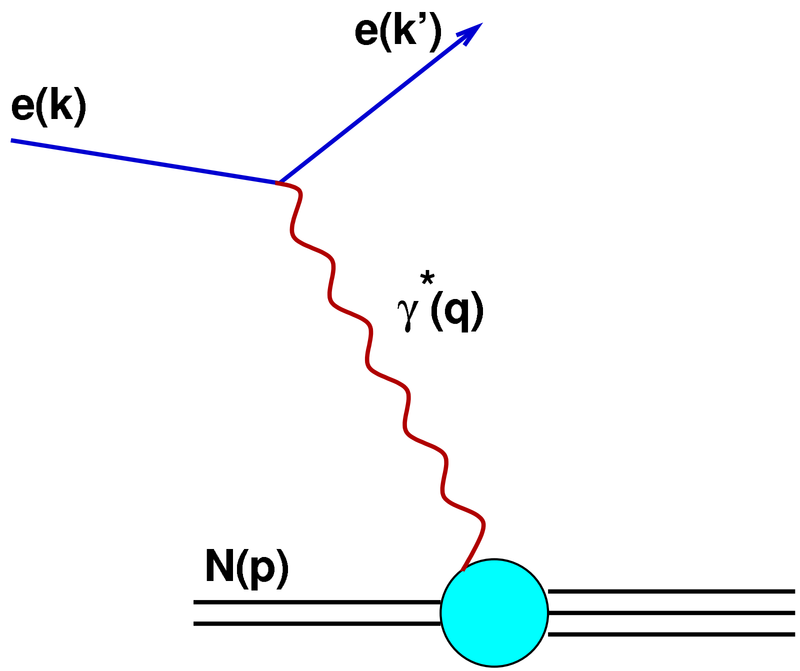
- The nonlinear saturarion effects are important for all $Q \lesssim Q_S(Y)$, which is known as saturation region

Introduction

- There has been a large amount of work devoted to the description and understanding of QCD in the high energy limit corresponding to the *saturation*
  - Theory: non-linear QCD equations describing the evolution of scattering amplitudes towards saturation - AGL, BK and JIMWLK equations
  - Phenomenology: discovery of geometric scaling in DIS at HERA
- The Balitsky-Kovchegov (BK) nonlinear equation describes the evolution in rapidity of the scattering amplitude of a dipole off a given target; this equation has been shown to lie in the same universality class as the Fisher-Kolmogorov-Petrovsky-Piscounov (FKPP) equation
- Geometric scaling has a natural explanation in terms of the so-called traveling wave solutions of BK equation
- The evolution at intermediate energies is well understood and is described by a linear equation; the deep saturation regime can also be evaluated in some models
- However, the transition between these two regimes is still a challenge

Deep Inelastic Scattering (DIS)

Kinematics and variables



- The total energy squared of the photon-nucleon system

$$s = (p+q)^2$$

- Photon virtuality

$$q^2 = (k - k') = -Q^2 < 0$$

- The Bjorken variable

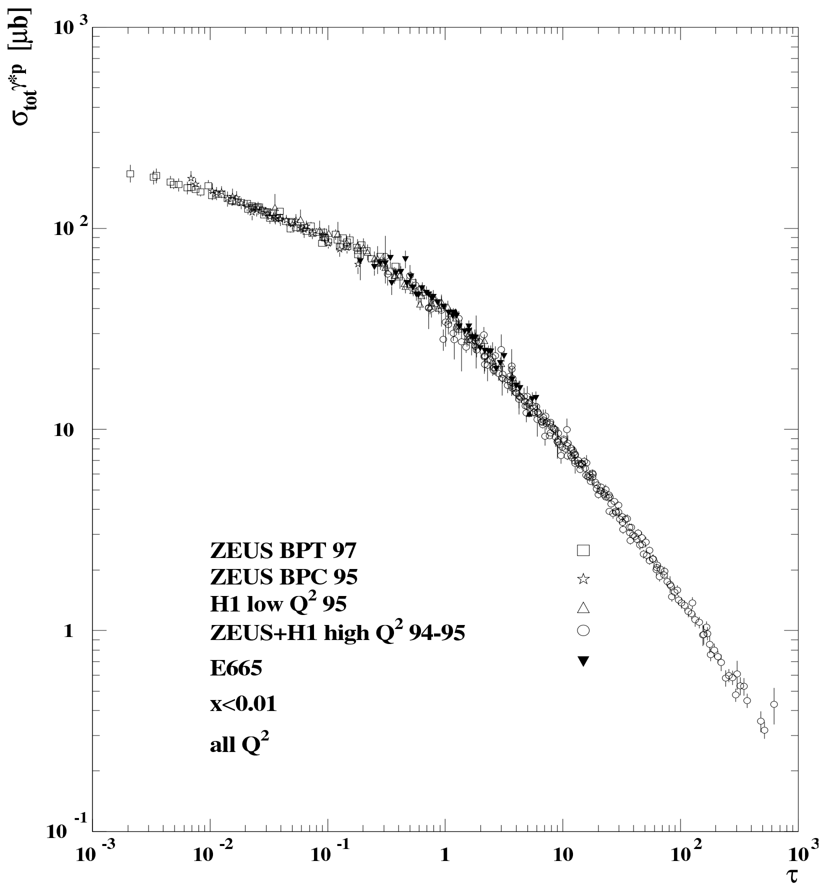
$$x \equiv x_{Bj} = \frac{Q^2}{2p \cdot q} = \frac{Q^2}{Q^2 + s}$$

- The high energy limit:

$$s \to \infty, \quad x \approx \frac{Q^2}{s} \to 0$$

Geometric Scaling

- Geometric scaling is a phenomenological feature of high energy deep inelastic scattering (DIS) which has been observed in the HERA data on inclusive $\gamma^* - p$ scattering, which is expressed as a scaling property of the virtual photon-proton cross section

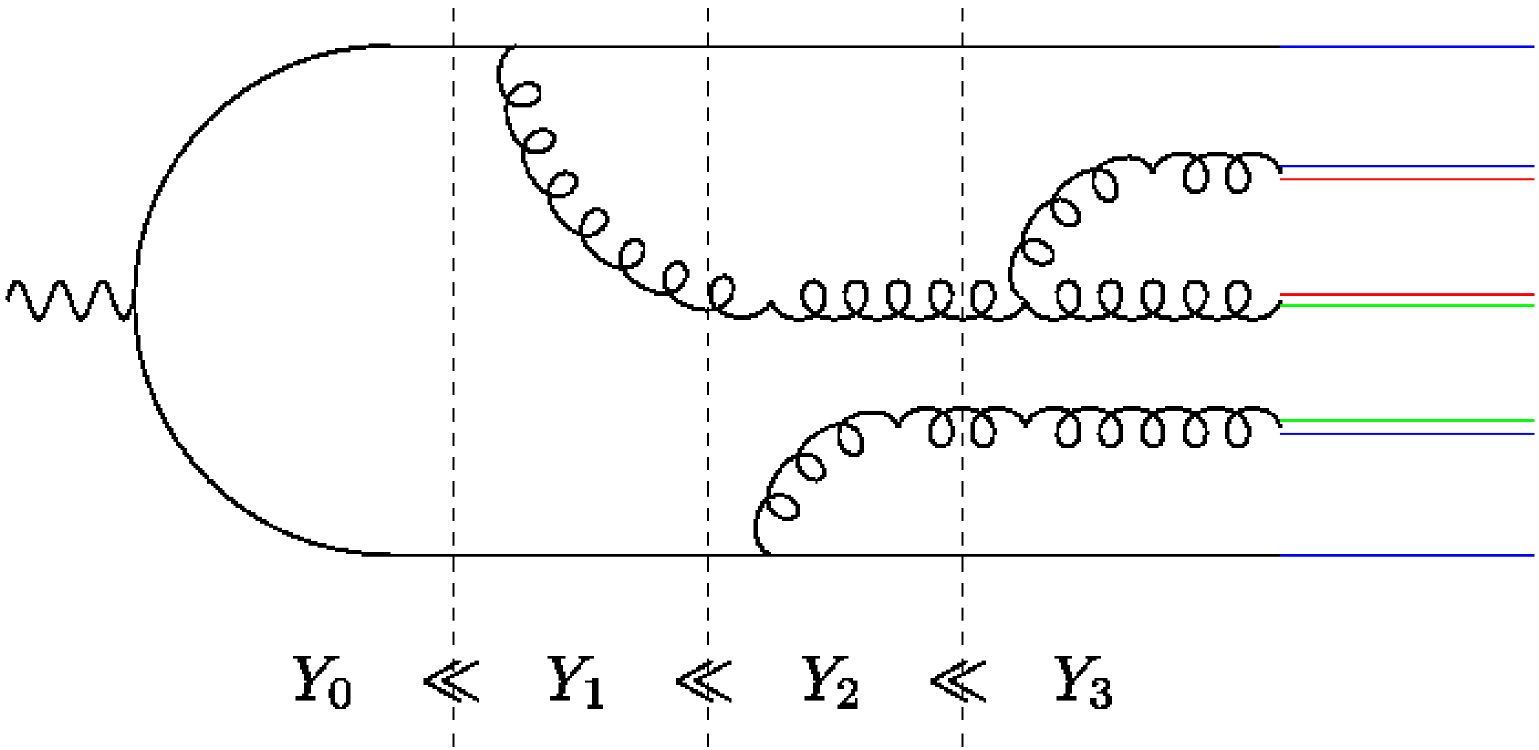


$$\sigma^{\gamma^* p}(Y, Q) = \sigma^{\gamma^* p}(\tau), \quad \tau = \frac{Q^2}{Q_s^2(Y)}$$

where Q is the virtuality of the photon, $Y = \log 1/x$ is the total rapidity and $Q_s(Y)$ is an increasing function of Y called saturation scale

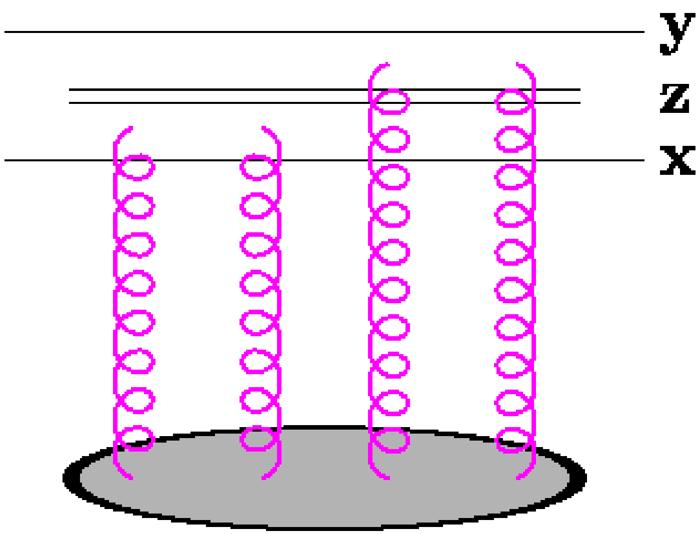
The evolution equation

- Consider a fast-moving $q\bar{q}$



- In the large N_c limit the gluons emitted can be replaced by quark-anti-quark pairs, which interact with the target via two gluon exchanges

Balitsky-Kovchegov equation



- If one considers multiple scatterings
- In the evolution, these multiple scatterings appear as a term proportional to T^2

$$\partial_Y T(\mathbf{x},\mathbf{y},Y) = \bar{\alpha}\int d^2\mathbf{z}\frac{(\mathbf{x}-\mathbf{y})^2}{(\mathbf{x}-\mathbf{z})^2(\mathbf{z}-\mathbf{y})^2}\left[T(\mathbf{x},\mathbf{z},Y)+T(\mathbf{z},\mathbf{y},Y)-T(\mathbf{x},\mathbf{y},Y)\right.$$
$$\left.-T(\mathbf{x},\mathbf{z},Y)T(\mathbf{z},\mathbf{y},Y)\right] \qquad (1)$$

- The quadratic term is important when $T \approx 1$

BK equation in momentum space

- Let us consider that the amplitude T is independent of the impact parameter $\mathbf{b} = \frac{\mathbf{x}+\mathbf{y}}{2}$ and of the direction of $\mathbf{r} = \mathbf{x} - \mathbf{y}$

- Then

$$T(\mathbf{x}, \mathbf{y}) \rightarrow T(\mathbf{r}, \mathbf{b}) \rightarrow T(r) \quad (2)$$

- The dipole cross section is proportional to the forward scattering amplitude T through the relation

$$\sigma_{dip}^{\gamma^* p}(Y, r) = 2\pi R_p^2 T(r, Y) \quad (3)$$

- We define the forward scattering amplitude in momentum space $T(k, Y)$

$$T(k, Y) = \int_0^\infty \frac{dr}{r} J_0(kr) T(r, Y) \quad (4)$$

- The BK equation then reads

$$\partial_Y T = \bar{\alpha}\chi(-\partial_L)T - \bar{\alpha}T^2, \qquad \bar{\alpha} = \frac{\alpha_s N_c}{\pi} \quad (5)$$

BK equation in momentum space

- In this equation

$$\chi(\gamma) = 2\psi(1) - \psi(\gamma) - \psi(1-\gamma) \tag{6}$$

is the characteristic function of the well known Balitsky-Fadin-Kuraev-Lipatov (BFKL) kernel, and $L = \log\left(k^2/k_0^2\right)$, where k_0 is some fixed low momentum scale

- The kernel χ is an integro-differential operator which may be defined with the help of the formal series expansion

$$\begin{aligned}\chi(-\partial_L) \quad = \quad & \chi(\gamma_0)\mathbf{1} + \chi'(\gamma_0)(-\partial_L - \gamma_0\mathbf{1}) + \frac{1}{2}\chi''(\gamma_0)(-\partial_L - \gamma_0\mathbf{1})^2 \\ & + \frac{1}{6}\chi^{(3)}(\gamma_0)(-\partial_L - \gamma_0\mathbf{1})^3 + \ldots \end{aligned} \tag{7}$$

for some γ_0 between 0 and 1

BK and FKPP equations

- The Fisher and Kolmogorov-Petrovsky-Piscounov (FKPP) equation is a known equation in non-equilibrium statistical physics, whose dynamics is called reaction-diffusion dynamics,

$$\partial_t u(x,t) = \partial_x^2 u(x,t) + u - u^2, \tag{8}$$

  where t is time and x is the coordinate.

- It has been shown that, after the change of variables

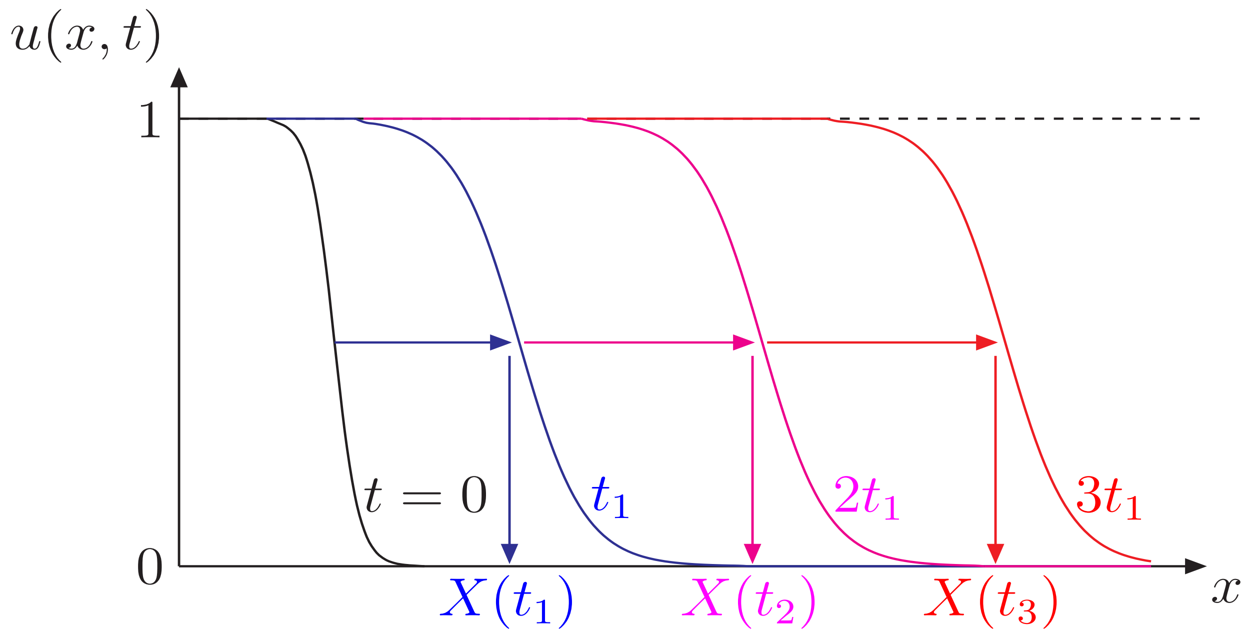
$$t \sim \bar{\alpha} Y, \quad x \sim \log(k^2/k_0^2), \quad u \sim T \tag{9}$$

  BK equation reduces to FKPP equation, when its kernel is approximated by the first three terms of the expansion, the so-called diffusive approximation or saddle point approximation

$$\chi(-\partial_L) = \chi(\gamma_c)\mathbf{1} + \chi'(\gamma_c)(-\partial_L - \gamma_c\mathbf{1}) + \frac{1}{2}\chi''(\gamma_c)(-\partial_L - \gamma_c\mathbf{1})^2, \tag{10}$$

Traveling wave solutions

- The FKPP evolution equation admits the so-called traveling wave solutions
  - For a traveling wave solution one can define the position of a wave front $x(t) = v(t)t$, irrespective of the details of the nonlinear effects
  - At larges times, the shape of a traveling wave is preserved during its propagation, and the solution becomes only a function of the scaling variable $x - vt$



Traveling waves and saturation

- In the language of saturation physics the position of the wave front is nothing but the saturation scale

$$x(t) \sim \ln Q_s^2(Y) \tag{11}$$

and the scaling corresponds to the geometric scaling

$$x - x(t) \sim \ln k^2/Q_s^2(Y) \tag{12}$$

- Summarizing:

$$\begin{aligned} \text{Time } t &\longrightarrow Y \\ \text{Space } x &\longrightarrow L \\ \text{Wave front } u(x - vt) &\longrightarrow T(L - vY) \\ \text{Traveling Waves} &\longrightarrow \text{Geometric Scaling} \end{aligned} \tag{13}$$

Scattering amplitude

- The linear part of the BK equation is solved by

$$T(k,Y) = \int \frac{d\gamma}{2\pi i} T_0(\gamma) \exp(-\gamma L + \bar{\alpha}\chi(\gamma)Y) \tag{14}$$

- The velocity of the front is given by

$$v = \min \bar{\alpha}\frac{\chi(\gamma)}{\gamma} = \bar{\alpha}\frac{\chi(\gamma_c)}{\gamma_c} = \bar{\alpha}\chi'(\gamma_c) \tag{15}$$

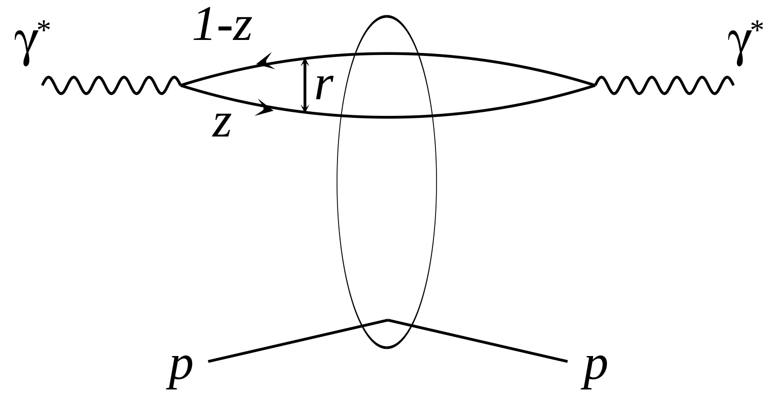
  where γ_c is the saddle point of the exponential phase factor. This fixes, for the BFKL kernel, $\gamma_c = 0.6275...$, $v = 4.88\bar{\alpha}$

- In terms of QCD variables, the dipole forward scattering amplitude in momentum space near the wave front reads

$$\begin{aligned} T(\tau,Y) &\propto \sqrt{\frac{2}{\bar{\alpha}\chi''(\gamma_c)}} \log\left(\frac{k^2}{Q_s^2(Y)}\right)\left(\frac{k^2}{Q_s^2(Y)}\right)^{-\gamma_c} \exp\left(-\frac{\log^2\left(\frac{k^2}{Q_s^2(Y)}\right)}{2\bar{\alpha}\chi''(\gamma_c)Y}\right), \\ Q_s^2(Y) &= Q_0^2 \exp\left(\bar{\alpha}\frac{\chi(\gamma_c)}{\gamma_c}Y - \frac{3}{2\gamma_c}\log Y\right). \end{aligned} \tag{16}$$

Dipole model

- It is convenient to work within the QCD dipole frame of DIS



- In the LLA of perturbative QCD (pQCD), the cross section factorizes as

$$\sigma_{T,L}^{\gamma^* p}(Y,Q) = \int d^2r \int_0^1 dz \, \left|\Psi_{T,L}(r,z;Q^2)\right|^2 \sigma_{dip}^{\gamma^* p}(r,Y), \quad (17)$$

$\sigma_{dip}^{\gamma^* p}(Y,\mathbf{r})$ is the dipole-proton cross section, z is the fraction of photon's momentum carried by the quark and $\mathbf{r}$ is the transverse separation of the quark-anti-quark pair

Dipole model

- The wavefunctions

$$|\Psi_T(r,z;Q^2)|^2 = \frac{2N_c\alpha_{em}}{4\pi^2}\sum_q e_q^2\left\{\left[z^2+(1-z)^2\right]\bar{Q}_q^2K_1^2(\bar{Q}_qr)+m_q^2K_0^2(\bar{Q}_qr)\right\} \quad (18)$$

and

$$|\Psi_L(r,z;Q^2)|^2 = \frac{2N_c\alpha_{em}}{4\pi^2}\sum_q e_q^2\left\{4Q^2z^2(1-z)^2K_0^2(\bar{Q}_qr)\right\} \quad (19)$$

where $\bar{Q}_q = z(1-z)Q^2+m_q^2$, m_q the light quark mass and $K_{0,1}$ are the Mc Donald functions of rank zero and one, respectively.

- If one treats the proton as a homogeneous disk of radius R_p

$$\sigma_{dip}^{\gamma^*p}(r,Y) = 2\pi R_p^2 T(r,Y)$$

F_2 structure function

- The proton structure function F_2 can be obtained from the $\gamma^* p$ cross section through the relation

$$F_2(x,Q^2) = \frac{Q^2}{4\pi^2\alpha_{em}}\left[\sigma_T^{\gamma^* p}(x,Q^2) + \sigma_L^{\gamma^* p}(x,Q^2)\right]. \quad (20)$$

- We can express the $\gamma^* p$ cross section in terms of $T(k,Y)$. After a bit of algebra one obtains

$$F_2(x,Q^2) = \frac{Q^2 R_p^2 N_c}{4\pi^2}\int_0^\infty \frac{dk}{k}\int_0^1 dz\,|\tilde{\Psi}(k^2,z;Q^2)|^2 T(k,Y), \quad (21)$$

where the wavefunction is now expressed in momentum space

$$|\tilde{\Psi}(k^2,z;Q^2)|^2 = \sum_q \left(\frac{4\bar{Q}_q^2}{k^2+4\bar{Q}_q^2}\right)^2 e_q^2 \left\{\left[z^2+(1-z)^2\right]\left[\frac{4(k^2+\bar{Q}_q^2)}{\sqrt{k^2(k^2+4\bar{Q}_q^2)}}\text{arcsinh}\left(\frac{k}{2\bar{Q}_q}\right) + \frac{k^2-2\bar{Q}_q^2}{2\bar{Q}_q^2}\right] + \frac{4Q^2 z^2(1-z)^2+m_q^2}{\bar{Q}_q^2}\left[\frac{k^2+\bar{Q}_q^2}{\bar{Q}_q^2} - \frac{4\bar{Q}_q^4+2\bar{Q}_q^2 k^2+k^4}{\bar{Q}_q^2\sqrt{k^2(k^2+4\bar{Q}_q^2)}}\text{arcsinh}\left(\frac{k}{2\bar{Q}_q}\right)\right]\right\}$$

Transition to saturation

- The goal of this work is to study the connection between the traveling wave solution and the saturation region

- These different domains can be parametrized as

$$T(\tau, Y) = c - \log\left(\frac{k}{Q_s(Y)}\right)$$

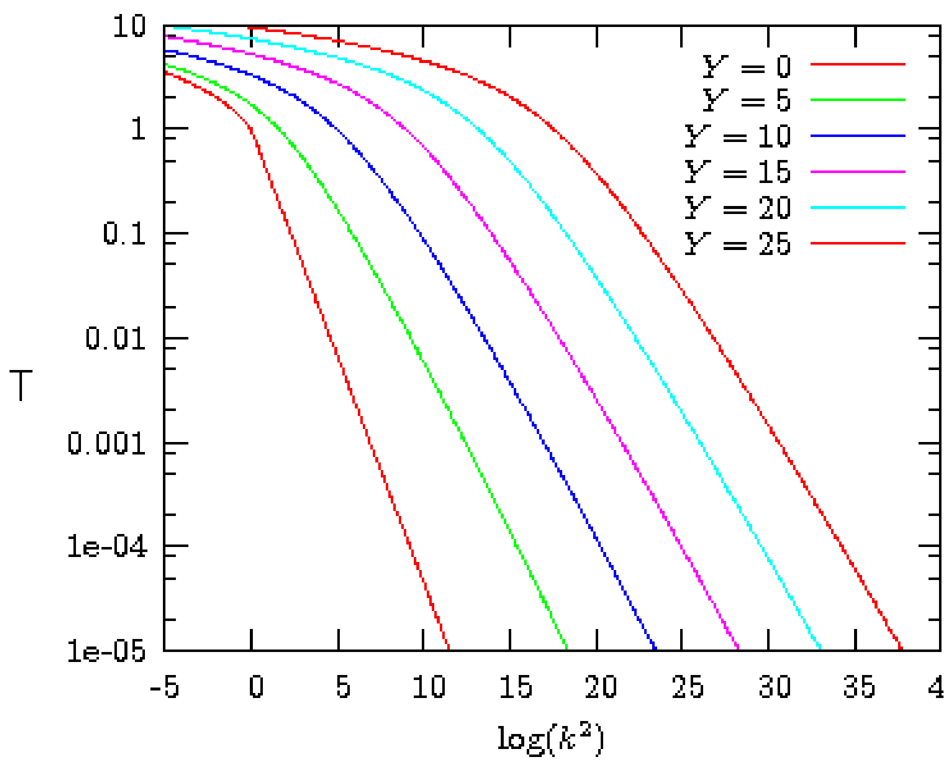
when, $k \leq Q_s(Y)$, and

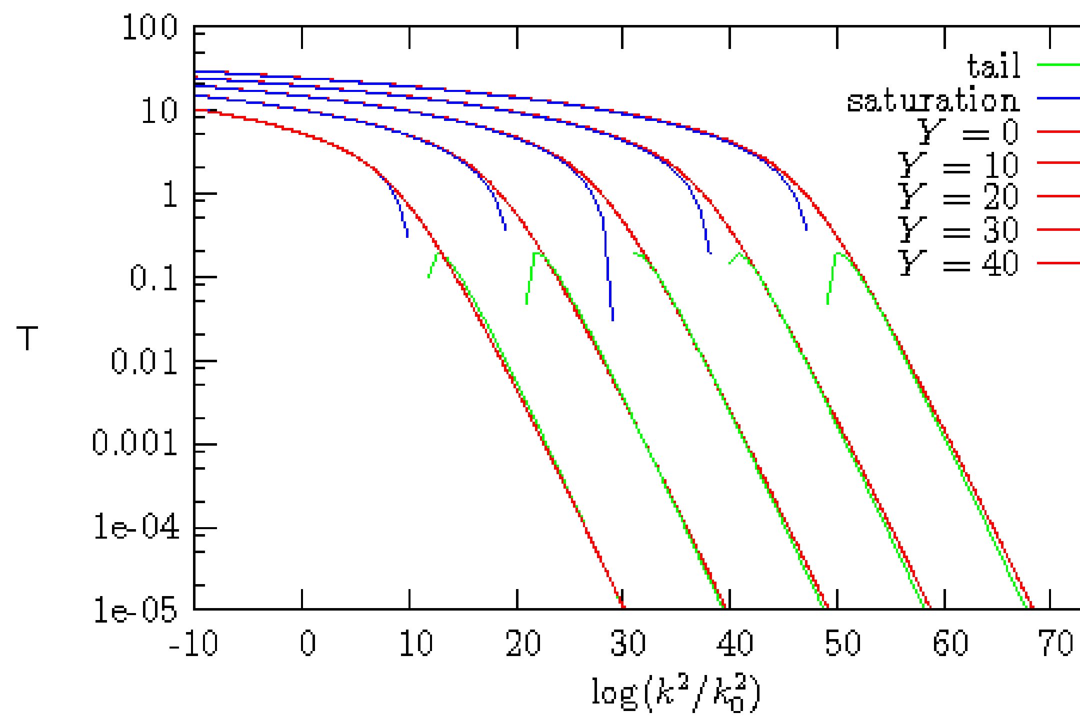
$$T(k, Y) \propto \sqrt{\frac{2}{\bar{\alpha}\chi''(\gamma_c)}} \left(\frac{k^2}{Q_s^2(Y)}\right)^{-\gamma_c} \log\left(\frac{k^2}{Q_s^2(Y)}\right) \exp\left\{-\frac{\log^2\left(\frac{k^2}{Q_s^2(Y)}\right)}{2\bar{\alpha}\chi''(\gamma_c)Y}\right\}$$

when, $k > Q_s(Y)$

Connecting to Saturation

- The first attempt was to perform a matching between the two regions by imposing continuity of T and its first derivative at $k/Q_s(Y) = 1$
- However, this procedure does not necessarily imply a positive Fourier transform. Then, a better way to obtain the connection between the two regions is an interpolation through one expression only





The model

- The idea is to build the saturarion domain from the dilute one, in such a way that the scattering amplitude satisfies the correct asymptotic behaviour, that is,

$$T_{dil} \ll 1 \Rightarrow T \sim T_{dil}, \qquad T_{dil} \gg 1 \Rightarrow T \sim 1 \tag{22}$$

- Based on the requirements above, and interpolation model was suggested by the authors and this reads

$$T(k, Y) = \frac{L_F}{1 + \frac{1}{T_{dil}}} \tag{23}$$

where

$$L_F = \sqrt{K^2 + \frac{1}{4}\log^2 \frac{k^2}{Q_s^2(Y)}} \tag{24}$$

$$T_{dil} = A\left(\frac{k^2}{Q_s^2(Y)}\right)^{-\gamma_c} \exp\left\{-\frac{\log^2\left(1 + \frac{k^2}{Q_s^2(Y)}\right)}{\bar{\alpha}\chi''(\gamma_c)Y}\right\} \tag{25}$$

Fitting procedure

- We fit the HERA data in the kinematic range

$$0.045 \leq Q^2 \leq 45\text{GeV}^2, \quad x \leq 0.01$$

and the fixed parameters are $\gamma_c = 0.6275$, $\chi''(\gamma_c) = 48.518$, $\bar{\alpha} = 0.2$, $K = 2.0$ and $A = 2.0$

- The values of light quark masses used were $m_q = 0, 10, 50, 140$ MeV , the same values used by IIM (PLB **590**, 199 (2004)) and GBW (PRD **59**, 014017 (1999)),

$$T(rQ_s, Y) = T_0 \left(\frac{rQ_s}{2}\right)^{2\left(\gamma_c + \frac{\log(2/rQ_s)}{\kappa\lambda Y}\right)} \qquad \text{for} \qquad rQ_s \leq 2$$

$$T(rQ_s, Y) = 1 - e^{-a \log^2(brQ_s)} \qquad \text{for} \qquad rQ_s > 2$$

and

$$\sigma_{\text{dipole}} = \sigma_0 \left(1 - e^{-\mathbf{r}^2 Q_s^2(x)/4}\right)$$

respectively. Note that these two parametrizations are expressed in coordinate space.

Results

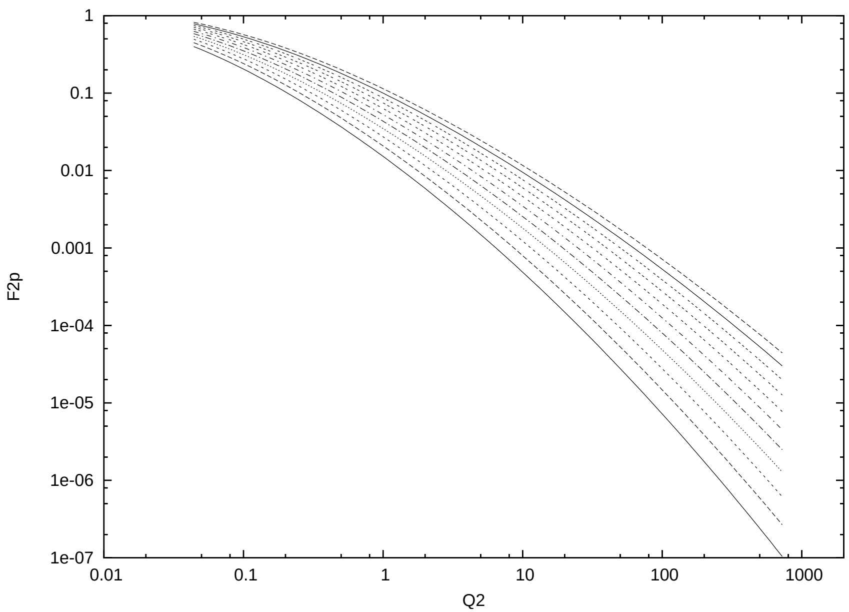
- The parametrization (which we will call Parametrization 1):

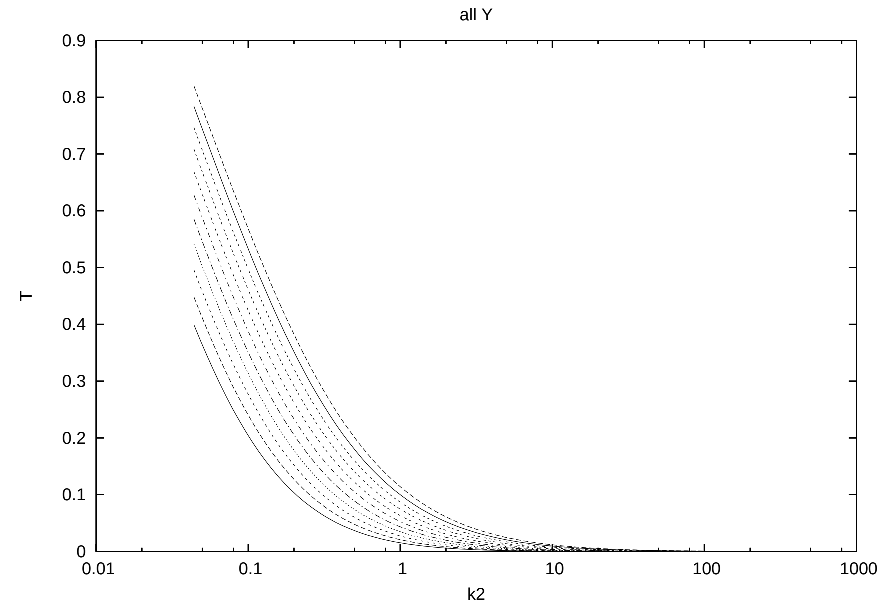
$$T(k,Y) = \frac{L_F}{1 + \frac{1}{T_{dil}}}$$

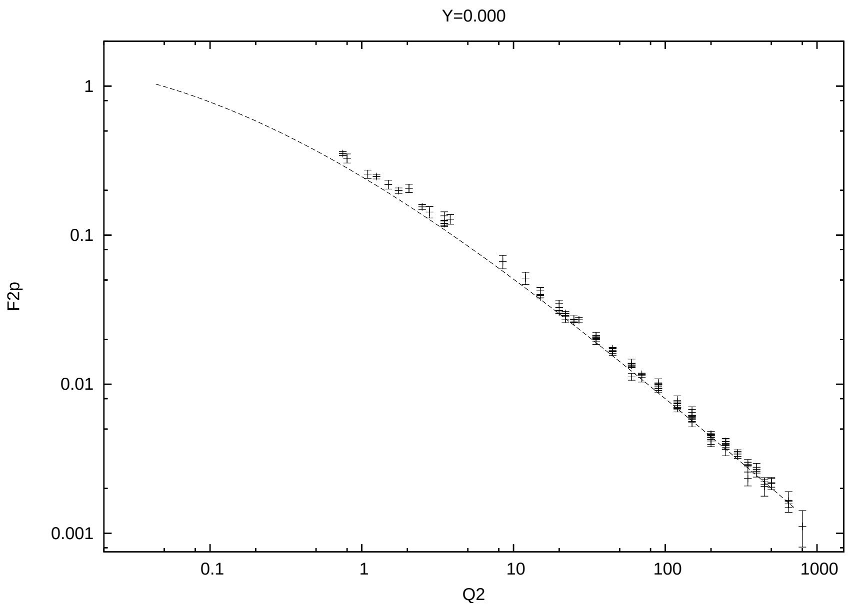
m_q	$\chi^2/d.o.f.$	
0	0.996	
10MeV	0.997	
50MeV	0.997	
140MeV	1.045	

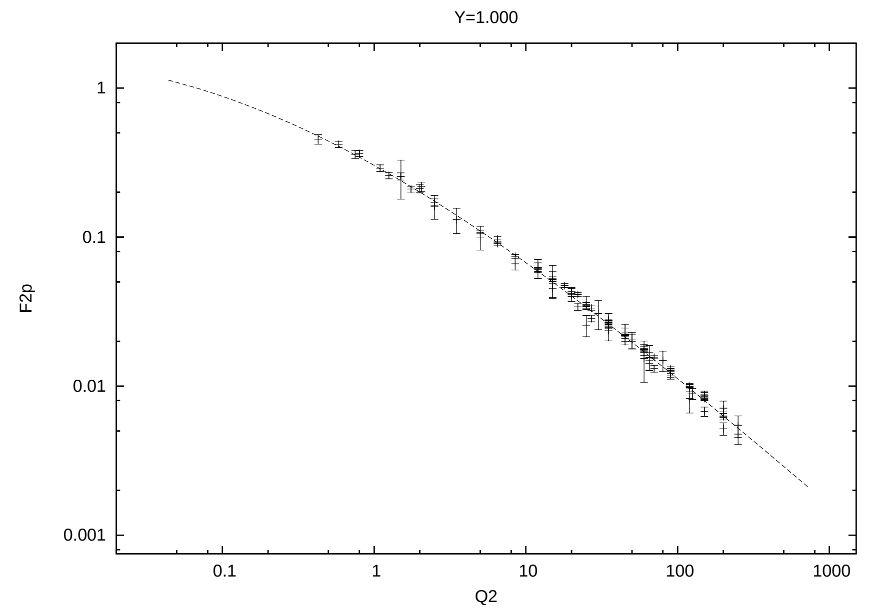
$$\chi^2(\alpha) = \sum_{i=1}^{n} \frac{(f(x_i,\alpha) - \epsilon_i)^2}{\sigma_i^2}$$

Case $m_q = 0$ (I)

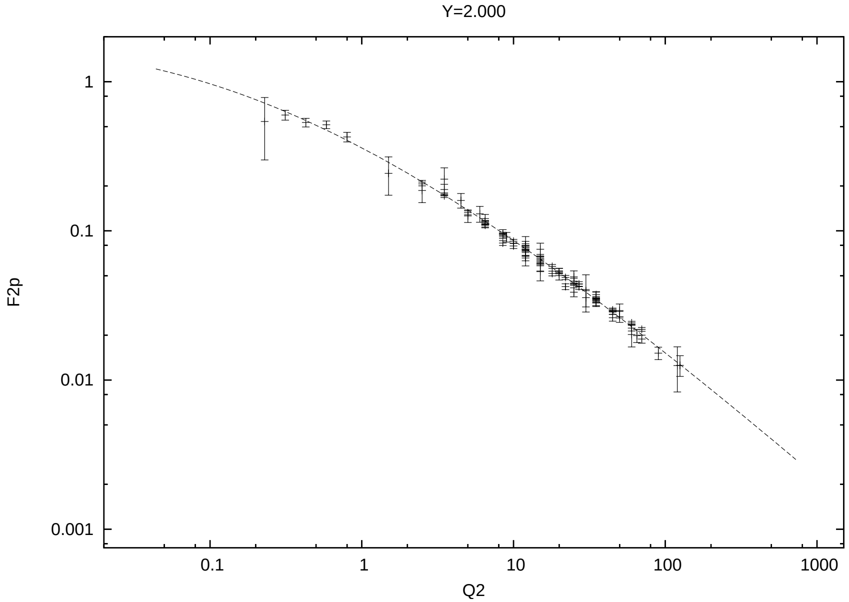

F2p
Q2
1
0.1
0.01
0.001
1e-04
1e-05
1e-06
1e-07
0.01
0.1
1
10
100
1000


all Y
T
k2
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
0.01
0.1
1
10
100
1000

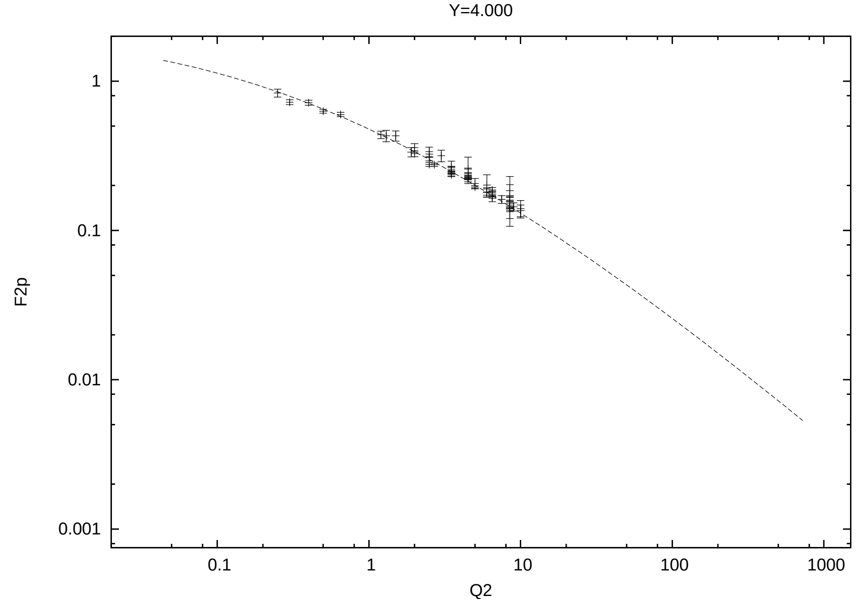

Y=0.000
F2p
Q2
1
0.1
0.01
0.001
0.1
1
10
100
1000


Y=1.000
F2p
Q2
1
0.1
0.01
0.001
0.1
1
10
100
1000

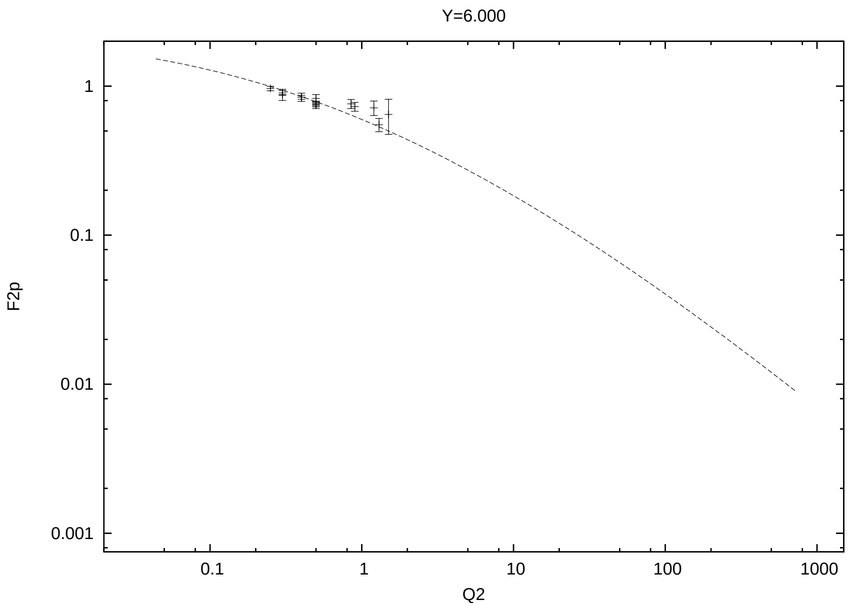
Case $m_q = 0$ (II)



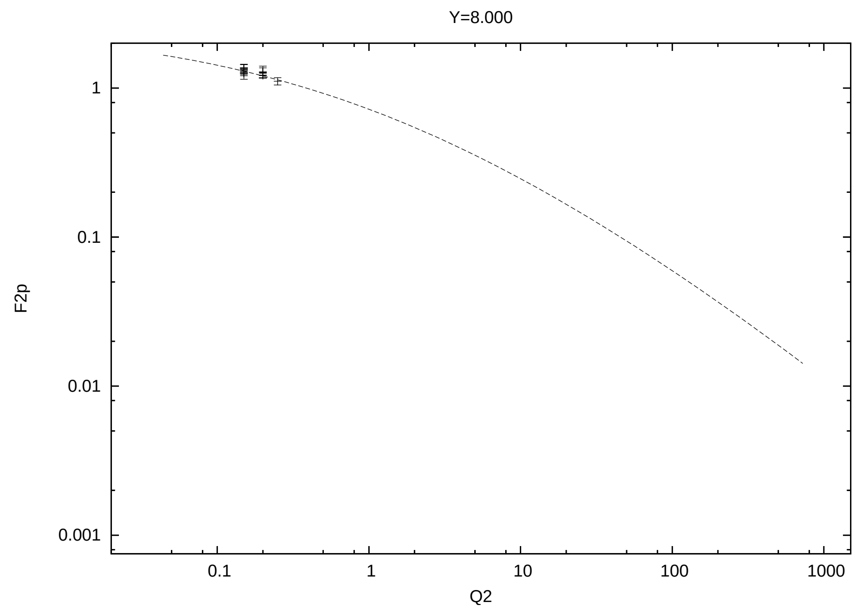
Y=2.000
F2p
Q2
1
0.1
0.01
0.001
0.1
1
10
100
1000



Y=4.000
F2p
Q2
1
0.1
0.01
0.001
0.1
1
10
100
1000

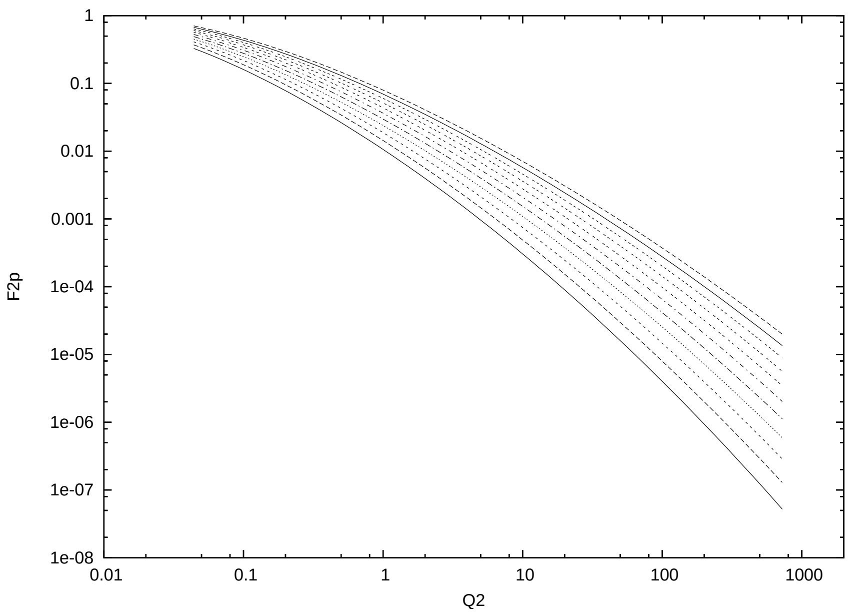


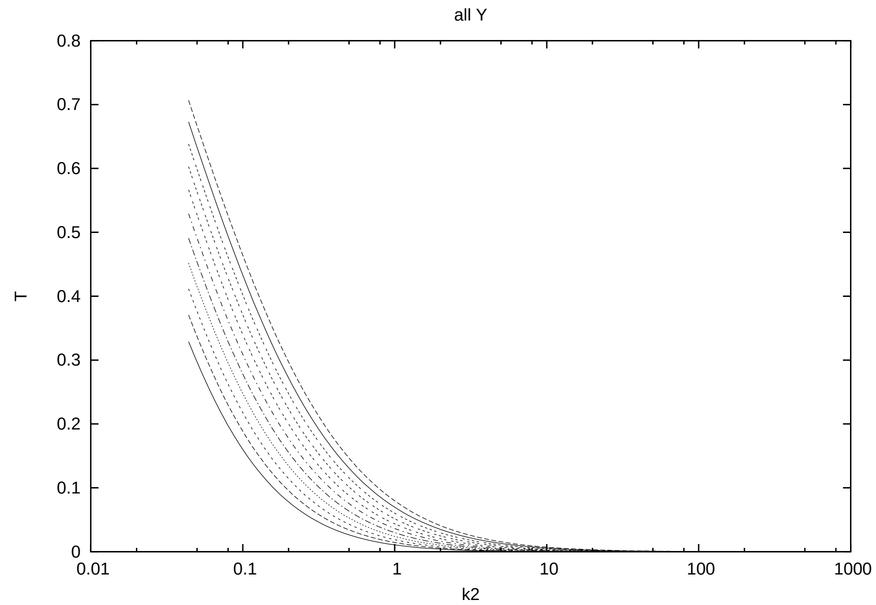
Y=6.000
F2p
Q2
1
0.1
0.01
0.001
0.1
1
10
100
1000

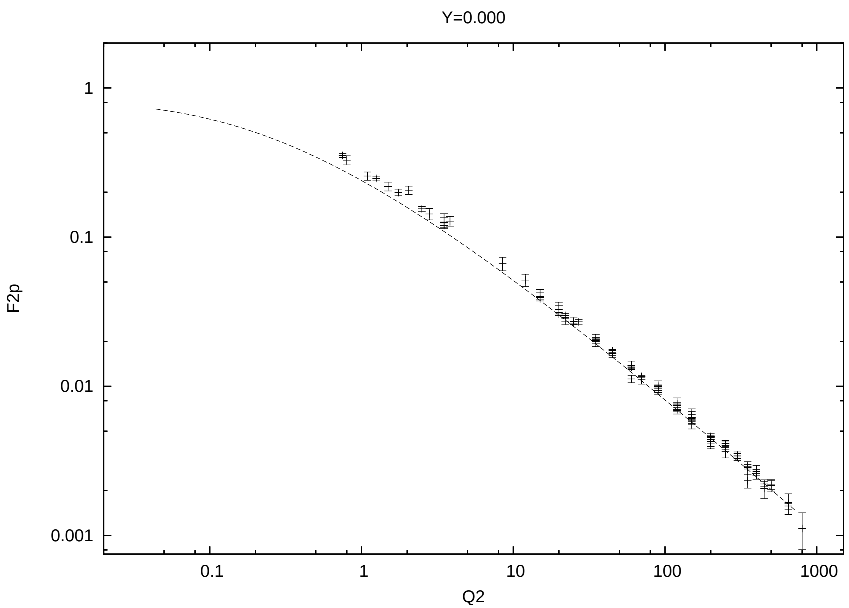


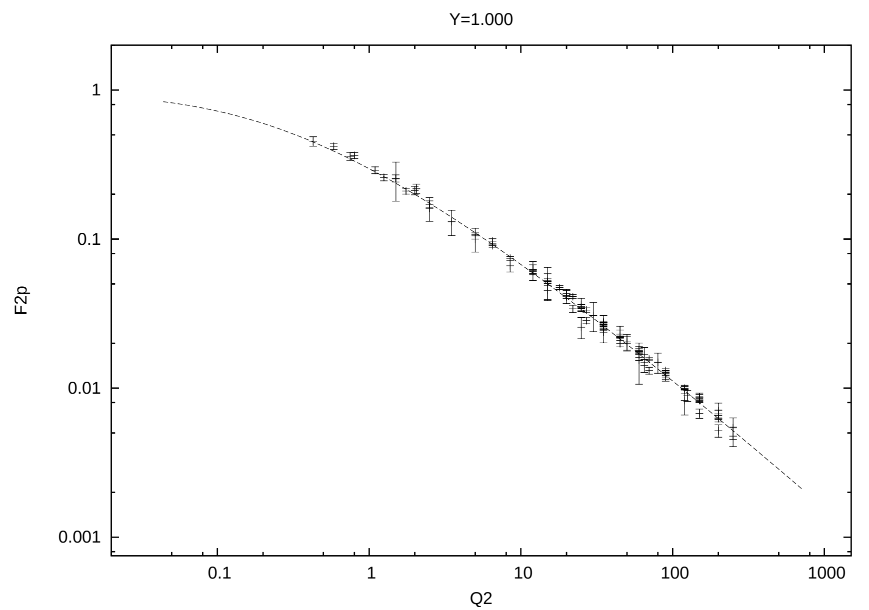
Y=8.000
F2p
Q2
1
0.1
0.01
0.001
0.1
1
10
100
1000

Case $m_q = 140$ MeV (I)

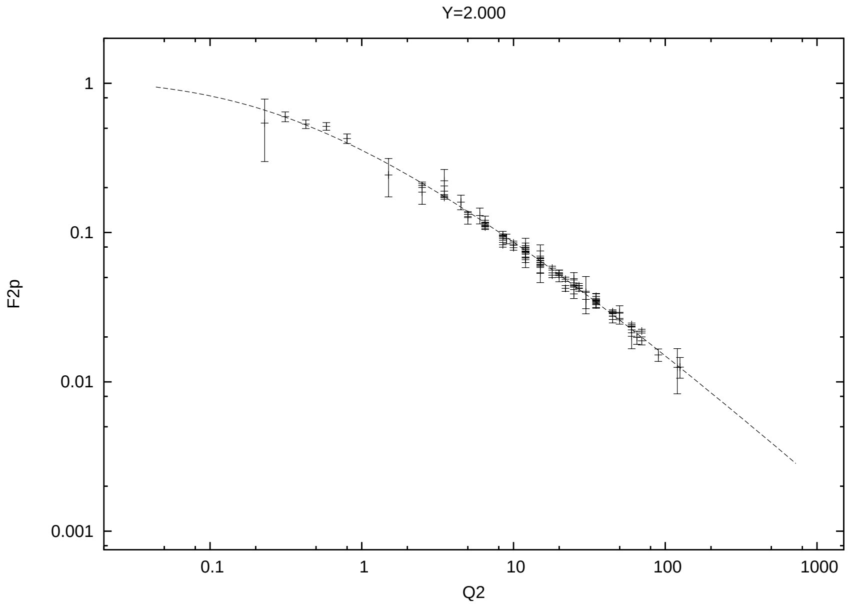


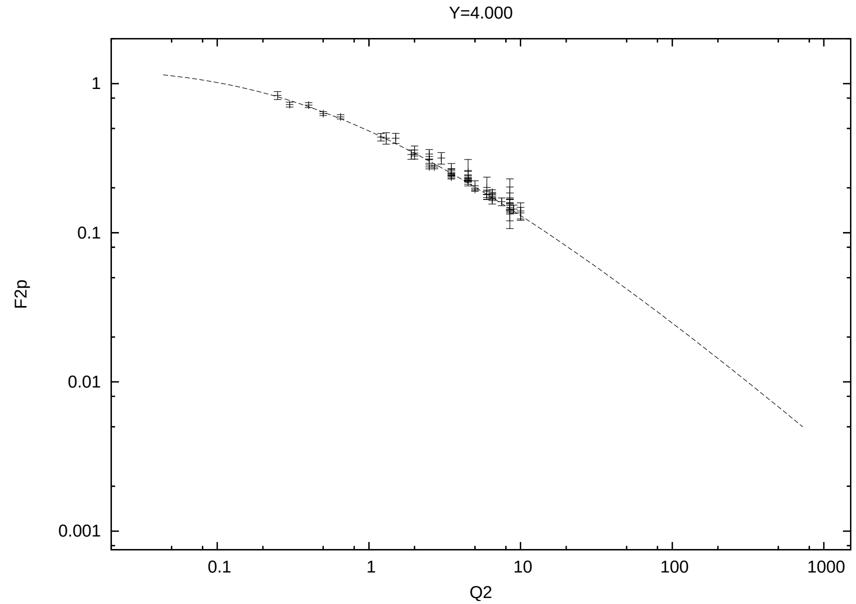


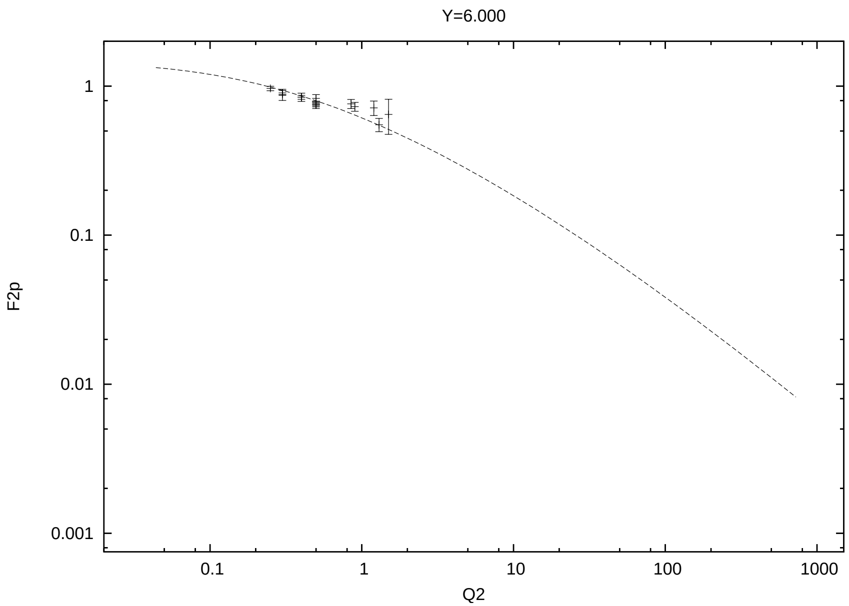


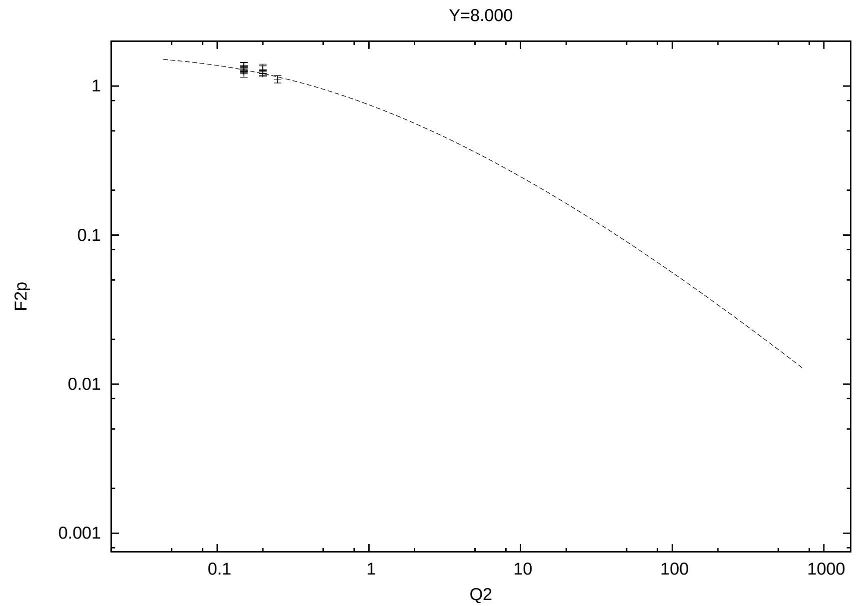


Case $m_q = 140$ MeV (II)









Results

- We tried other parametrizations, which are

Parametrization 2: $$T = \frac{L_F(1+T_{dil}^2)}{T_{dil}^2 + \frac{1}{T_{dil}}}$$

Parametrization 3: $$T = \frac{L_F(1+T_{dil})^2}{T_{dil}^2 + \frac{1}{T_{dil}}}$$

$$\chi^2/d.o.f.$$

m_q	Param[1]	Param[2]	Param[3]
0	0.996	1.072	1.53
10MeV	0.997	1.071	1.527
50MeV	0.997	1.079	1.473
140MeV	1.045	1.063	1.238

Parameters

Parametrization 1

Parameters	$m_q = 0$MeV	$m_q = 10$MeV	$m_q = 50$MeV	$m_q = 140$MeV
R_p	2.9031	2.9006	2.9368	3.3238
γ_c	0.6275	0.6275	0.6275	0.6275
v_c	1.2222	1.2222	1.2083	1.1027
χ''	48.518	48.518	48.518	48.518
$\bar{\alpha}$	0.2	0.2	0.2	0.2
k_0^2	7.9532	7.9851	8.0601	8.0717
y_0	-2.9590	-2.9534	-3.0264	-3.6975
K	2	2	2	2
A	2	2	2	2

Conclusions

- We proposed an interpolation model which describes the behaviour of the BK scattering amplitude, in momentum space, in two regions of interest - the traveling wave solution and saturarion - as well as the transition between them
- Inserting this amplitude into the expression for the F_2 proton structure function, we obtained satisfatory results for the fitting procedure to the HERA data
- Future applications to other observables
- Some important remarks:
  - Back to Evolution Equations
  - χ^2 analysis

References

- L. V. Gribov, E. M. Levin and M. G. Ryskin, Phys. Rep. **100**, 1 (1983).
- A. L. Ayala, M. B. Gay Ducati and E. M. Levin, Nucl. Phys. **B493**, 305 (1997); Nucl. Phys. **B551**, 335 (1998).
- A. M Staśto, K. Golec-Biernat, and J. Kwiecinski, Phys. Rev. Lett. **86**, 596 (2001).
- I. I. Balitsky, Nucl. Phys. B **463**, 99 (1996).
- Y. V. Kovchegov, Phys. Rev. D **60**, 034008 (1999); Phys. Rev. D. **61**, 074018 (2000).
- S. Munier and R. Peschanski, Phys. Lett. **91**, 232001 (2003); Phys. Rev. D **69**, 034008 (2004); Phys. Rev. D **70**, 077503 (2004).
- E.Iancu, K.Itakura and S.Munier, Phys. Lett. B **590**, 199 (2004).
- C. Marquet, R. Peschanski, and G. Soyez, Phys.Lett.B **628**, 239 (2005).