Numerical Solutions of the QCD Evolution Equations

An overview

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Overview

- *Quick* introduction on evolution equations (see J.T. Amaral seminars): *DGLAP, BFKL, GRV, AGL, JIMWLK, BK*
- Mathematical Background:
  - Numerical Methods on ODE's and integral equations
  - Orthogonal Chebyshev polynomials
- Physical application in evolution equations: several solutions in recent years
- Summary and plans for future work

Quick Introduction on Evolution Equations

- Rise of cross sections with energy $\Leftrightarrow$ gluon density $\Leftrightarrow$ dense color system
- Violation of Froissart-Martin limit (unitarity of the scattering amplitude)
- To avoid the increase $\Rightarrow$ parton recombination & partonic saturation
- Introduction of a **non-linear term** to reach the saturation
- QCD evolutions time line: *linear to non-linear*

DGLAP, BFKL, GRV, AGL, CCFM, BK, JIMWLK

- Recent huge interest in effective theories: Color Glass Condensate (CGC), JIMWLK and BK equations
- Numerical studies to further application in phenomenology

Linear to Non-linear

- BFKL equations violates the Froissart bound (among other problems: singularity structure in the Regge plane, IR diffusion,...)
- Physical image: in large densities there is *shadowing*: superposition of the partons
- modification of the linear equation $\Rightarrow$ addiction of a non-linear term (summation of *fan diagram*) $\Rightarrow$ GLR & AGL equations.
- Dipole representation $\Rightarrow$ *onium* emitting a color dipole
- dipole generator functional $\Rightarrow$ equation for dipole-target scattering amplitude $\Rightarrow$ *dipole BFKL*

Balitskiĭ-Kovchegov Equation

- More recent development: ***Balitskiĭ-Kovchegov equation***
- DIS in very high energy: multiple color dipole scattering with nucleus.
- Interpretation: one dipole splits in two with the non-linear term encodes the recombination of the dipoles.
- incoherent scattering: each dipole scattering independently with the target (correlations are recent field of investigation)
- non-linear equation for high density parton systems with: geometric scaling, saturation scale, IR diffusion suppression.
- evolution eq. in rapidity with suitable initial conditions, valid in the LLA (α_s fixed)
- Diffusion equation with non-linear term $\Rightarrow$ solution as a propagation wave $\Rightarrow$ eq. Fisher-Kolmogorov-Petrovsky-Piscounov.

Mathematical Background

Differential Equations

- Initial value problem $\Rightarrow$ existence & uniqueness of solution

(1) $$y^{(n)} = f(x, y, \ldots, y^{(n+1)}) \qquad y(a) = y_a,\ y(b) = y_b$$

- Grid:

$$x \in [a, b] \to x_j = x_0, x_1, \ldots, x_N \Rightarrow y_0, y_1, \ldots, y_N$$

- Numerical methods: Euler, Runge-Kutta, Prediction-correlation
- The idea: giving x_i, obtains y_i, goes to x_{i+1} and so on

Mathematical Background

Integral Equations

- Fredholm equations: 1^{st} & 2^{nd} kinds

(2) $$g(t) = \int_a^b K(t,s)\, f(s)\, ds$$

(3) $$f(t) = \lambda \int_a^b K(t,s)\, f(s)\, ds + g(t)$$

- Volterra equations: 1^{st} & 2^{nd} kinds

(4) $$g(t) = \int_a^t K(t,s)\, f(s)\, ds$$

(5) $$f(t) = \int_a^t K(t,s)\, f(s)\, ds + g(t)$$

where $g(t)$ is the independent term, $K(t,s)$ is the kernel and λ is the eigenvalue.

- Relation with the linear system of equations

(6) $$\mathbb{K} \cdot \mathbf{f} = \mathbf{g} \qquad (\mathbb{K} - \lambda \mathbb{1}) \cdot \mathbf{f} = \mathbf{g}$$

Mathematical Background

Numerical Solution of Integral Equations

- Quadrature rules

(7)
$$f(t) = \lambda \int_a^b K(t,s)\, f(s)\, ds + g(t) \Rightarrow$$
$$f(t) \approx \lambda \sum_{j=1}^{N} w_j\, K(t, s_j)\, f(s_j) + g(t) \Rightarrow$$

(8)
$$f(t_i) \approx \lambda \sum_{j=1}^{N} w_j\, K(t_i, s_j)\, f(s_j) + g(t_i)$$

where t_i are the quadrature points.

- The unknowns are the $f(t_i) \Rightarrow$ linear system of equations.

Mathematical Background

(T)Chebyshev Polynomial Expansion

(9)
$$f(x) \approx \sum_{j=1}^{N} V_j(x)\, f(x_j)$$

$$V_j(x) = \frac{2}{N} \sum_{k=1}^{N} v_k\, T_{k-1}(x_j) T_{k-1}(x), \quad v_k = \begin{cases} 1/2\ , k = 1 \\ 1 \quad\ , k \neq 1 \end{cases}$$

Orthogonal set of polynomials:

$$T_n(x) = \cos(n \arccos\ x), \quad T_0(x) = 1,\ T_1(x) = x$$

$$T_{n+1}(x) = 2x T_n(x) - T_{n-1}(x),\ n \geq 1$$

$$\int_{-1}^{1} \frac{T_i(x) T_j(x)}{\sqrt{1-x}} dx = \begin{cases} 0 \quad\ , i \neq j \\ \pi/2\ , i = j \neq 0 \\ \pi \quad\ , i = j = 0 \end{cases}, \quad \sum_{k=1}^{m} T_i(x_k) T_j(x_k) = \begin{cases} 0 \quad\ , i \neq j \\ m/2\ , i = j \neq 0 \\ m \quad\ , i = j = 0 \end{cases}$$

Zeros & extrema of $T_n(x),\ k = 0, \ldots, n$:

$$x_k = \cos\left(\frac{\pi(k-1/2)}{n}\right), \quad \tilde{x}_k = \cos\left(\frac{\pi k}{n}\right)$$

Physical Applications

DGLAP by Laguerre polynomials $L_n(x)$

$$Q^2 \frac{\partial}{\partial Q^2} \tilde{q}^{\pm}_{\mathrm{NS}}(x, Q^2) = \frac{\alpha_s(Q^2)}{2\pi} \int_x^1 \frac{dz}{z} \tilde{q}^{\pm}_{\mathrm{NS}}(x/z, Q^2) \tilde{P}^{\pm}_{\mathrm{NS}}(z)$$

Change of variables:

$$x, z \to \exp(-x', z') \Rightarrow \tilde{q}^{\pm}_{\mathrm{NS}}(e^{-x'}, Q^2) = \sum_{n=0}^{\infty} \tilde{q}^{\pm}_{\mathrm{NS}}(n, Q^2) L_n(x')$$

Using orthogonal proprieties of Laguerre polynomials:

$$Q^2 \frac{\partial}{\partial Q^2} \tilde{q}^{\pm}_{\mathrm{NS}}(n, Q^2) = \frac{\alpha_s(Q^2)}{2\pi} \sum_{m=0}^{n} \left[\tilde{q}^{\pm}_{\mathrm{NS}}(m, Q^2) \left(\tilde{P}^{\pm}_{\mathrm{NS}}(n-m) - \tilde{P}^{\pm}_{\mathrm{NS}}(n-m-1) \right) \right]$$

- System of ODE's in Q^2. Solution:

$$\tilde{q}^{\pm}_{\mathrm{NS}}(x, Q^2) \approx \sum_{n=0}^{30} \tilde{q}^{\pm}_{\mathrm{NS}}(n, Q^2) L_n(\ln 1/x)$$

Physical Applications

CCFM (Ciafaloni,Catani,Fiorani,Marchesini) numerical solution

- Summation in x and Q^2

$$F(x, Q_T, Q) = F^0(x, Q_T, Q; \mu^2) + \int_x^1 \frac{dz}{z} \int \frac{d^2q}{\pi q^2} \bar{\alpha}_s \Theta(Q - zq)\Theta(q^2 - \mu^2) \times \tilde{A}_{\rm NS}(z, q, Q_T; \mu^2) F(x/z, Q'_T, q)$$

- change of variables to a compact interval: $Q^2, Q_T^2 \to \tau, \tau_F \in [-1, 1]$ and $f = Q_T^2 F$

$$f(x, Q_T, Q) = \sum_{i,j=1}^{N} V_i(\tau_T) V_j(\tau) f(x, Q_i, Q_{Fj}) \Rightarrow$$

$$f_{ij}(x) = f_{ij}^0(x) + \int_x^1 \frac{dz}{z} \sum_{k,l=1}^{N} A_{ij,kl}(z) f_{kl}(x/z)$$

- The integro-differential equation turns out in an integral equation.

BFKL LO & NLO: Kwieciński/Motyka

- Calculation of cross-section of $\gamma\gamma \to J/\psi J/\psi$ in the BFKL Pomeron framework

$$\Im m\, A(W^2, t) = \int \frac{d^2k}{\pi} \frac{\Phi_0(k^2, Q^2)\Phi(x, k, Q)}{\left[\left(k + \frac{Q}{2}\right)^2 + s_0\right]\left[\left(k - \frac{Q}{2}\right)^2 + s_0\right]}$$

$\Phi_0(k^2, Q^2)$ is the impact factor of the transition $\gamma \to J/\psi$; s_0 is a IR regulator.

$$\begin{aligned}\Phi(x, k, Q) &= \Phi_0(k^2, Q^2) + \frac{3\alpha_s(\mu^2)}{2\pi^2} \int_x^1 \frac{dx'}{x'} \int \frac{d^2k'}{(k' - k)^2 + s_0} \left\{\left[\frac{k_1^2}{k_1'^2 + s_0} + \frac{k_2^2}{k_2'^2 + s_0}\right.\right. \\ &- \left. Q^2 \frac{(k' - k)^2 + s_0}{(k_1'^2 + s_0)(k_2'^2 + s_0)}\right](*)\Phi(x', k', Q) - \left[\frac{k_1^2}{k_1'^2 + (k' - k)^2 + 2s_0}\right. \\ &+ \left.\left. \frac{k_2^2}{k_2'^2 + (k' - k)^2 + 2s_0}\right]\Phi(x', k, Q)\right\}\end{aligned}$$

$(*) = \Theta\left[\left(k^2 + Q^2/4\right)x'/x - k'^2\right]$ is the kinematic veto to include NLO corrections.

Physical Applications

Solution's *modus operandi*

1. Fourier series expansion (angular dependence: $\phi = \angle(k, Q)$)

$$\Phi(x,k,Q) = \sum_{m=0}^{\infty} \tilde{\Phi}(x,k^2,Q^2)\ \cos(2m\phi) \overset{m=0}{\Rightarrow}$$

$$\begin{aligned}\widetilde{\Phi}_0(x,k^2,Q^2) &= \Phi_0(k^2,Q^2) + \frac{3\alpha_s(\mu^2)}{\pi}\int_x^1 \frac{dx'}{x'}\int_0^\infty dk'^2 R(k^2,k'^2,Q^2)\widetilde{\Phi}_0(x',k'^2,Q^2) \\ &- S(k^2,Q^2)\widetilde{\Phi}_0(x',k^2,Q^2)\end{aligned}$$

2. Change of variables: $k^2, Q^2 \in [0,\infty] \to \kappa^2, \xi^2 \in [0,1]$
3. Chebyshev polynomial expansion:

$$\widetilde{\Phi}_0(x,k^2,Q^2) = \sum_{i,j=1}^{N} V_i(\kappa^2)V_j(\xi^2)\,\underbrace{\widetilde{\Phi}_0(x,\kappa_i^2,\xi_j^2)}_{\equiv\widetilde{\Phi}_{ij}(x)}$$

4. Integral equation

$$\widetilde{\Phi}_{ij}(x) = \Phi_{ij}^0 + \int_x^1 \frac{dz}{z}\sum_{l=1}^{N}\int_0^1 d\kappa' V_l(\kappa')R_{ijl}\widetilde{\Phi}_{il}(z) + S_{ijl}V_l(\kappa')\widetilde{\Phi}_{ij}(z)$$

Physical Application: BK equation

Tel-Aviv Group

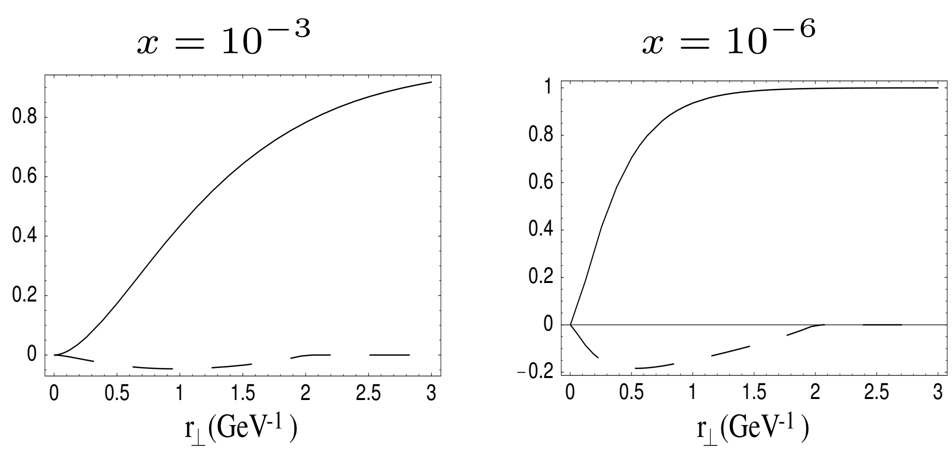
BK equation:

$$\frac{dN(x_{01},y)}{dy} = \frac{C_F \alpha_s}{\pi^2} \int d^2 x_2 \frac{x_{01}^2}{x_{02}^2 x_{12}^2} \left[2N(x_{02},y) - N(x_{01},y) - N(x_{02},y)N(x_{12},y)\right] \quad (10)$$

$$N(r,x,b) = 1 - \exp[-\kappa(x,r)S(b)], \quad \kappa(x,r) = -\ln[1 - N(r,x)]$$

where N is the scattering amplitude of color dipole and S is a profile function. Features:

- Inclusion of DGLAP evolution corrections
- Without numerical procedure details
- 50-500 hours(!?) in a PC-cluster
- Result (avaliable in WWW as a grid for interpolation):



Physical Application: BK Equation

Golec-Biernat & Staśto

$$\frac{\partial N_{\mathbf{xy}}}{\partial Y} = \underbrace{\bar{\alpha}_s \int \frac{d^2\mathbf{z}}{2\pi} \frac{(\mathbf{x}-\mathbf{y})^2}{(\mathbf{x}-\mathbf{z})^2(\mathbf{y}-\mathbf{z})^2}}_{\equiv \int_{\mathbf{z}}} [N_{\mathbf{xz}} + N_{\mathbf{yz}} - N_{\mathbf{xy}} - N_{\mathbf{xz}} N_{\mathbf{yz}}] \tag{11}$$

Dipole vector, $\mathbf{r} = \mathbf{x} - \mathbf{y}$; impact parameter, $\mathbf{b} = (\mathbf{x} - \mathbf{y})/2$; rapidity, $Y = \ln(1/x)$.

- Question: how the $\mathbf{b}$-dependence propagates?
- Invariance under translations ($\mathbf{x} \to \mathbf{x} + \mathbf{b}$) $\Rightarrow$ without $\mathbf{b}$-dependence $\Rightarrow$ infinite and uniform target
- Cylindric symmetric target $\Rightarrow$ 3 freedom degrees: r, θ, b and Y
- Numerical method (based in the Salam work in dipole BFKL): grid in the variables + initial condition as Glauber-Muller one:

$$N^{(0)}(r,b) = 1 - \exp[-r^2 S(b)], \quad S(b) = 10\exp[-b^2/2]$$

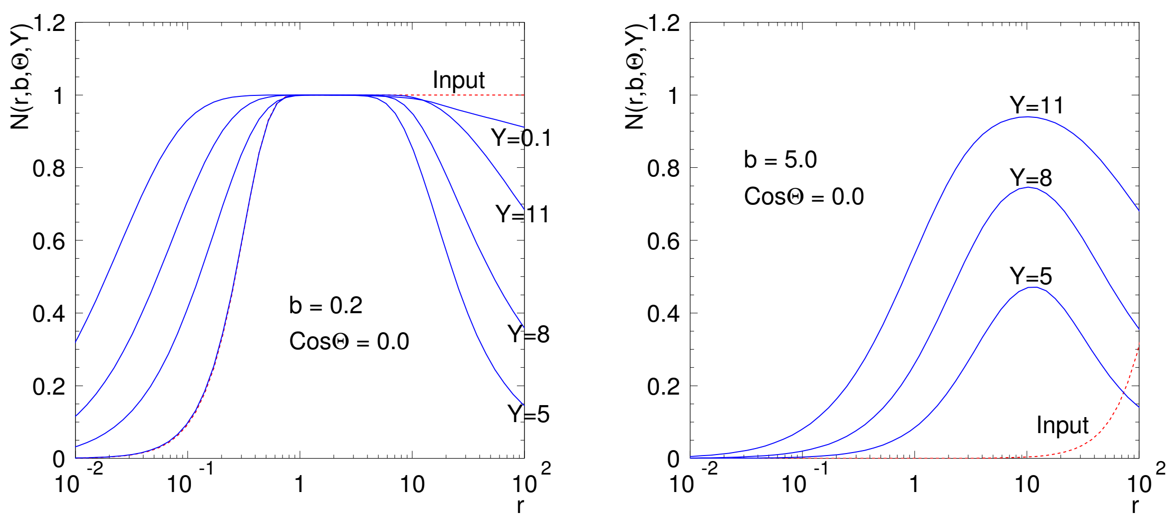
- Solution by a Rugge-Kutta-like method:

$$N_{xy}^{(1)} = N_{xy}^{(0)} + \Delta Y \int_{\mathbf{z}} N_{xz}^{(0)} + N_{yz}^{(0)} - N_{xy}^{(0)} - N_{xz}^{(0)} N_{yz}^{(0)}$$

Physical Application: BK Equation

$$N_{xy}^{(2)} = N_{xy}^{(0)} + \frac{\Delta Y}{2}\int_{\mathbf{z}} N_{xz}^{(0)} + N_{yz}^{(0)} - N_{xy}^{(0)} - N_{xz}^{(0)} N_{yz}^{(0)}$$
$$+ \frac{\Delta Y}{2}\int_{\mathbf{z}} N_{xz}^{(1)} + N_{yz}^{(1)} + N_{xz}^{(1)} - N_{xz}^{(1)} N_{yz}^{(1)}$$
$$+ \frac{\Delta Y}{6}\int_{\mathbf{z}} [N_{xz}^{(1)} - N_{xz}^{(0)}][N_{yz}^{(1)} - N_{yz}^{(0)}] \quad (12)$$

- 5 interactions in a grid $r \times b \times \theta = (100, 100, 20)$
- 300 min. (!) in a 2.5 GHz PC with 2.0 GB RAM
- Results:

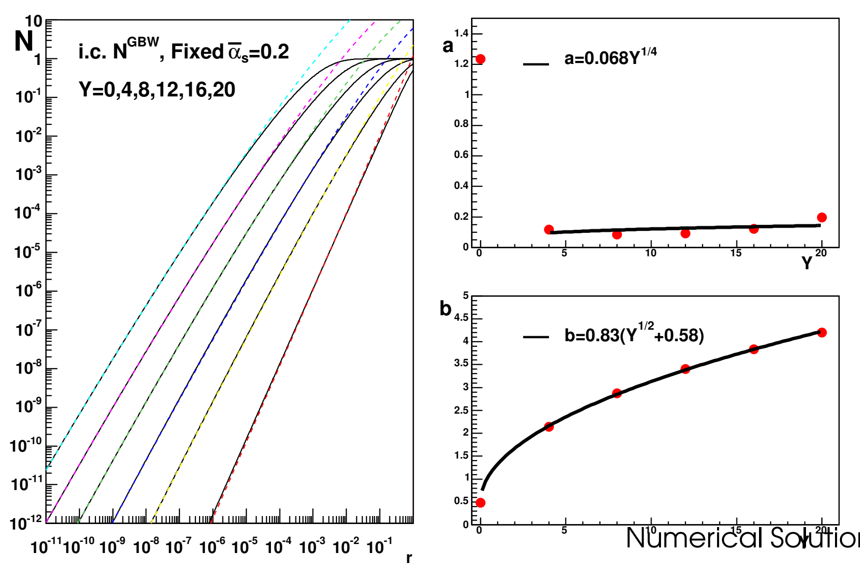


Physical Application: BK Equation

Spanish studies: Albacete *et al.*

Numerical solution by

- Second order Runge-Kutta method with step $\Delta Y = 0.1$
- Grid in $r \in [10^{-12}, 10^{2}]$ with 1200 points logarithm spaced
- Integrals made with Simpson rule
- Linear interpolation inside the grid, power-like extrapolation when $r < r_{\min}$ and constant when $r > r_{\min} \to$ saturation
- Initial conditions:
  - Golec-Biernat/Wüsthoff: $N^{\rm GBW}(r) = 1 - \exp(-r^2 Q_s'/4)$
  - McLerran-Venogupalan: $N^{MV}(r) = 1 - \exp[-(r^2 Q_s'/4)\ln(1/(r^2\Lambda^2_{\rm QCD}) + e)]$
  - (Albacete,Arnesto?)-Salgado: $N^{AS}(r) = 1 - \exp[-(rQ_s')^c]$
- Results:



Physical Application: BK Equation

BKSolver: Enberg/Golec-Biernat

(13) $$\frac{\partial\phi(k)}{\partial Y} = \bar{\alpha}_s \int_0^\infty \frac{dl^2}{l^2}\left[\frac{l^2\phi(l) - k^2\phi(k)}{|k^2 - l^2|} + \frac{k^2\phi(k)}{\sqrt{4l^4 + k^4}}\right] - \bar{\alpha}_s\phi^2(k)$$

Numerical method:

- Change of variables
  1. $L = \ln k^2,\ L' = \ln l^2$
  2. $\begin{bmatrix} L \\ L' \end{bmatrix} = (M_1 + M_2)\begin{bmatrix} u \\ v \end{bmatrix} - M_1$ with $u, v \in [0, 1]$ and M_i lower/upper bounds

     in $\ln k^2 \Longrightarrow$

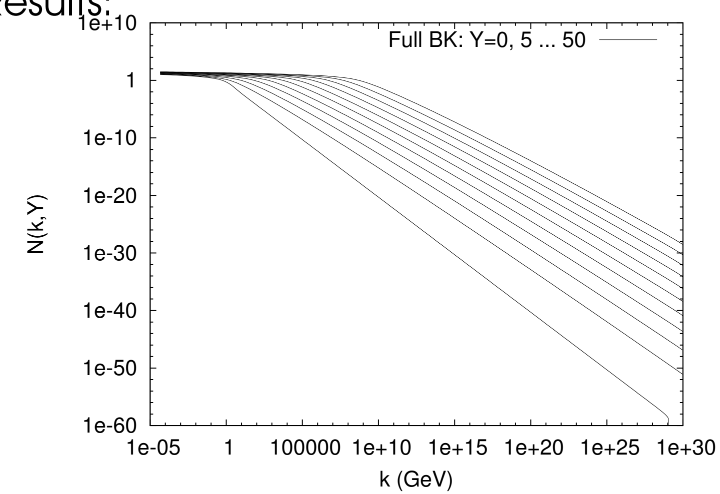
$$\frac{\partial\widehat{\phi}(u)}{\partial Y} = (M_1 + M_2)\int_0^1 dv \left[\frac{\widehat{\phi}(v) - e^{(M_1+M_2)(u-v)}\widehat{\phi}(v)}{|e^{(M_1+M_2)(u-v)} - 1|} + \frac{\widehat{\phi}(u)}{\sqrt{1 + 4e^{-2(M_1+M_2)(u-v)}}}\right] - \bar{\alpha}_s\widehat{\phi}^2(u)$$

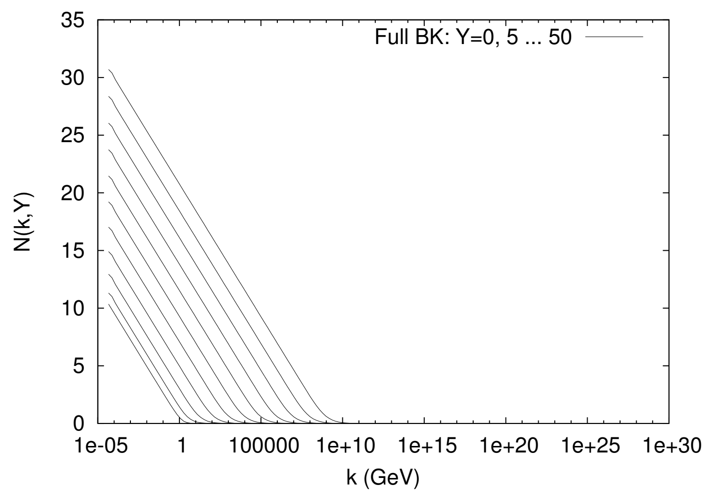
  3. Expansion in terms of Chebyshev polynomials

$$\int_a^b f(t)dt \approx \sum_{j,k=0}^{N}{}'' \frac{2}{N} f(\tilde{x}_k)T_j(\tilde{x}_k)\underbrace{\int_a^b T_j(t)dt}$$

Physical Application: BK Equation

- Resolution method:
  1. Grid in u and v in the extrema of Chebyshev polynomials;
  2. The integral is calculated by the approximation formula;
  3. McLerran-Venugopalan initial condition;
  4. System of ODE's for each point of the grid by embedded Runge-Kutte-Fehlberg 4,5 method;
  5. Summation over large N done by Fast Fourier Transform (FFT) $\longrightarrow$ less numerical errors.
- Advantages:
  1. small grid in momentum $\longrightarrow$ "small" ODE systems
  2. fast: 256 points in k grid, $Y < 50 \Longrightarrow$ 20-25 s in a Laptop Pentium IV 2 GHz.
  3. The program also gives the saturation scale $Q_s(Y)$ and $\frac{\partial \log Q_s^2(Y)}{\partial Y}$.
- Results:





Conclusions

- Several methods to solve the evolution equations ⇒ numerical conceivability
- *BKSolver* is free, modificable, well documented and fast.
- The others: only indications and/or tables of results and are slow
- Comparison between the different methods
- Applications in phenomenology in near future

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