BK equation and traveling wave solutions

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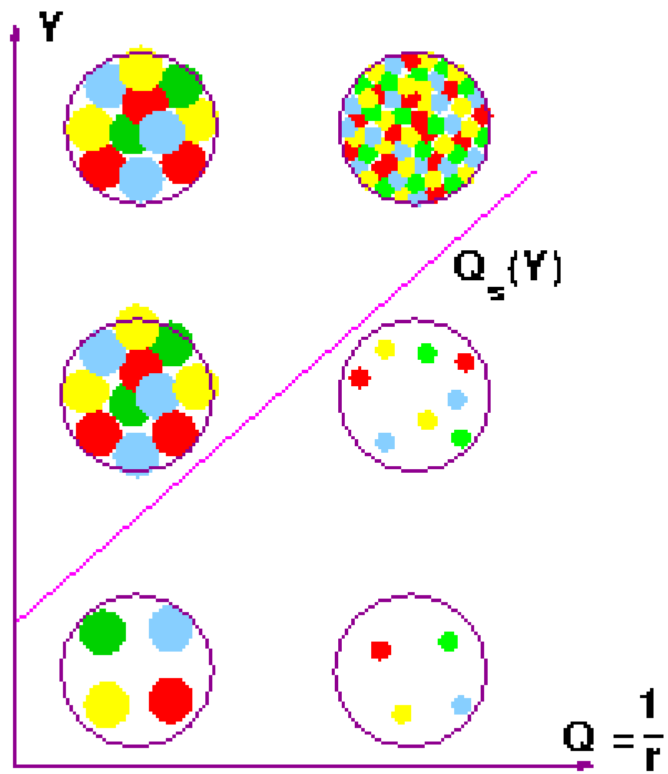
Introduction

- One of the most intriguing problems in Quantum Chromodynamics is the growth of the cross sections for hadronic interactions with energy; the increase of energy causes a fast growth of the gluon density and consequently of the cross section

- It is believed that at this regime gluon recombination might be important and it would decrease the growth of the parton density; this is called saturation



- $Q_s(Y)$ is the so called saturation scale

- The nonlinear saturarion effects are important for all $Q \lesssim Q_S(Y)$, which is known as saturation region

Introduction

- There has been a large amount of work devoted to the description and understanding of QCD in the high energy limit corresponding to the *saturation*
  - Theory: non-linear QCD equations describing the evolution of scattering amplitudes towards saturation - BK and JIMWLK equations
  - Phenomenology: discovery of geometric scaling in DIS at HERA

- The Balitsky-Kovchegov (BK) nonlinear equation describes the evolution in rapidity of the scattering amplitude of a dipole off a given target; this equation has been shown to lie in the same universality class as the Fisher-Kolmogorov-Petrovsky-Piscounov (FKPP) equation

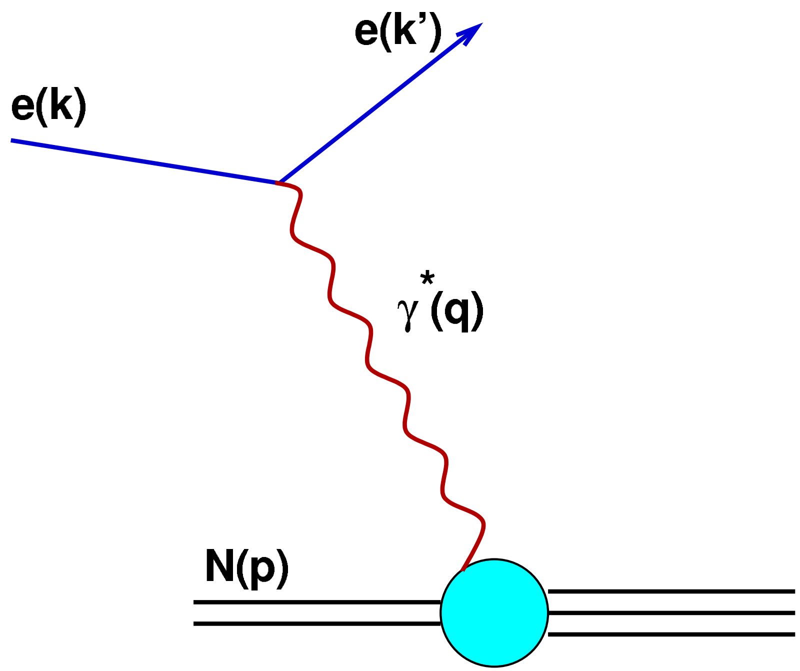
- Geometric scaling has a natural explanation in terms of the so-called traveling wave solutions of BK equation

- The evolution at intermediate energies is well understood and is described by a linear equation; the deep saturation regime can also be evaluated in some models

- However, the transition between these two regimes is still a challenge

Deep Inelastic Scattering (DIS)

Kinematics and variables



- The total energy squared of the photon-nucleon system

$$s = (p+q)^2$$

- Photon virtuality

$$q^2 = (k-k') = -Q^2 < 0$$

- The Bjorken variable

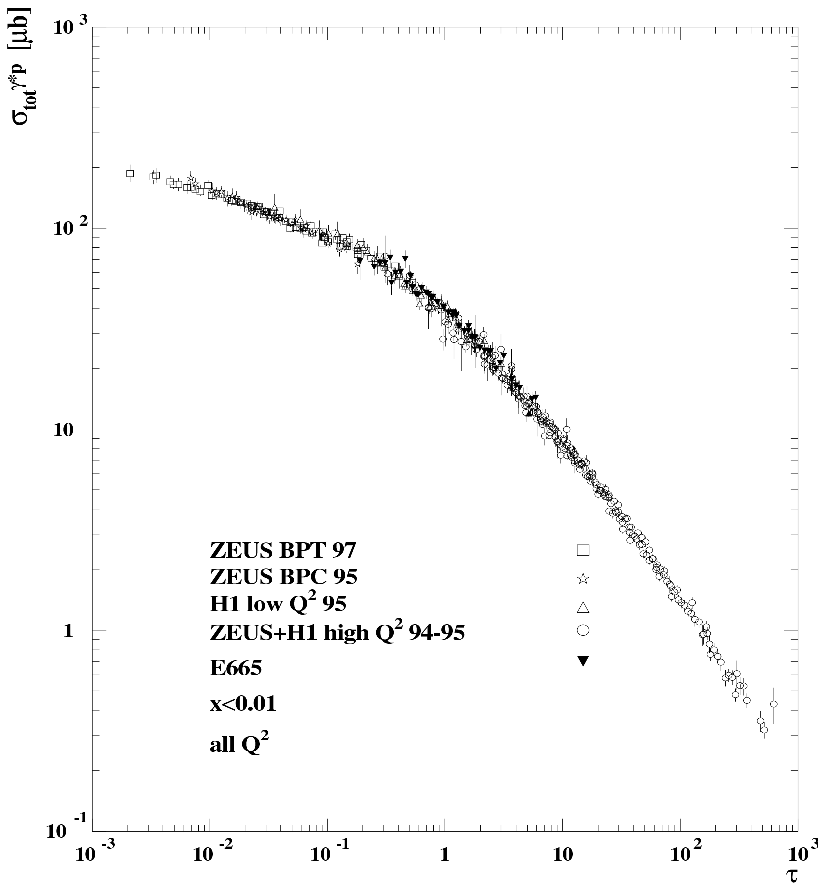
$$x \equiv x_{Bj} = \frac{Q^2}{2p\cdot q} = \frac{Q^2}{Q^2+s}$$

- The high energy limit:

$$s \to \infty, \quad x \approx \frac{Q^2}{s} \to 0$$

Geometric Scaling

- Geometric scaling is a phenomenological feature of high energy deep inelastic scattering (DIS) which has been observed in the HERA data on inclusive $\gamma^* - p$ scattering, which is expressed as a scaling property of the virtual photon-proton cross section

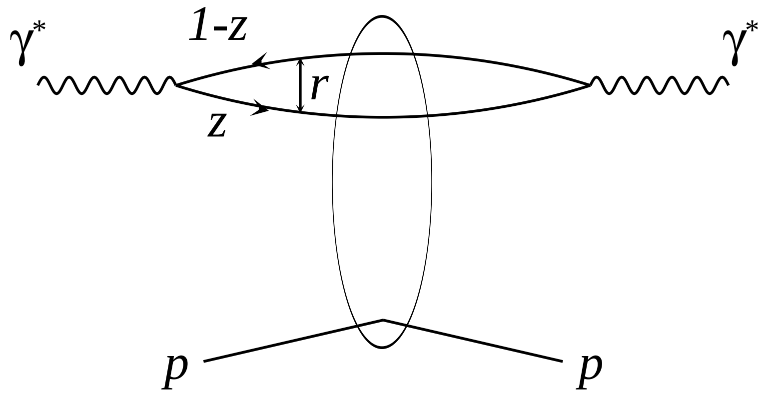


$$\sigma^{\gamma^* p}(Y, Q) = \sigma^{\gamma^* p}(\tau)\,, \quad \tau = \frac{Q^2}{Q_s^2(Y)}$$

where Q is the virtuality of the photon, $Y = \log 1/x$ is the total rapidity and $Q_s(Y)$ is an increasing function of Y called saturation scale

Dipole frame

- It is convenient to work within the QCD dipole frame of DIS



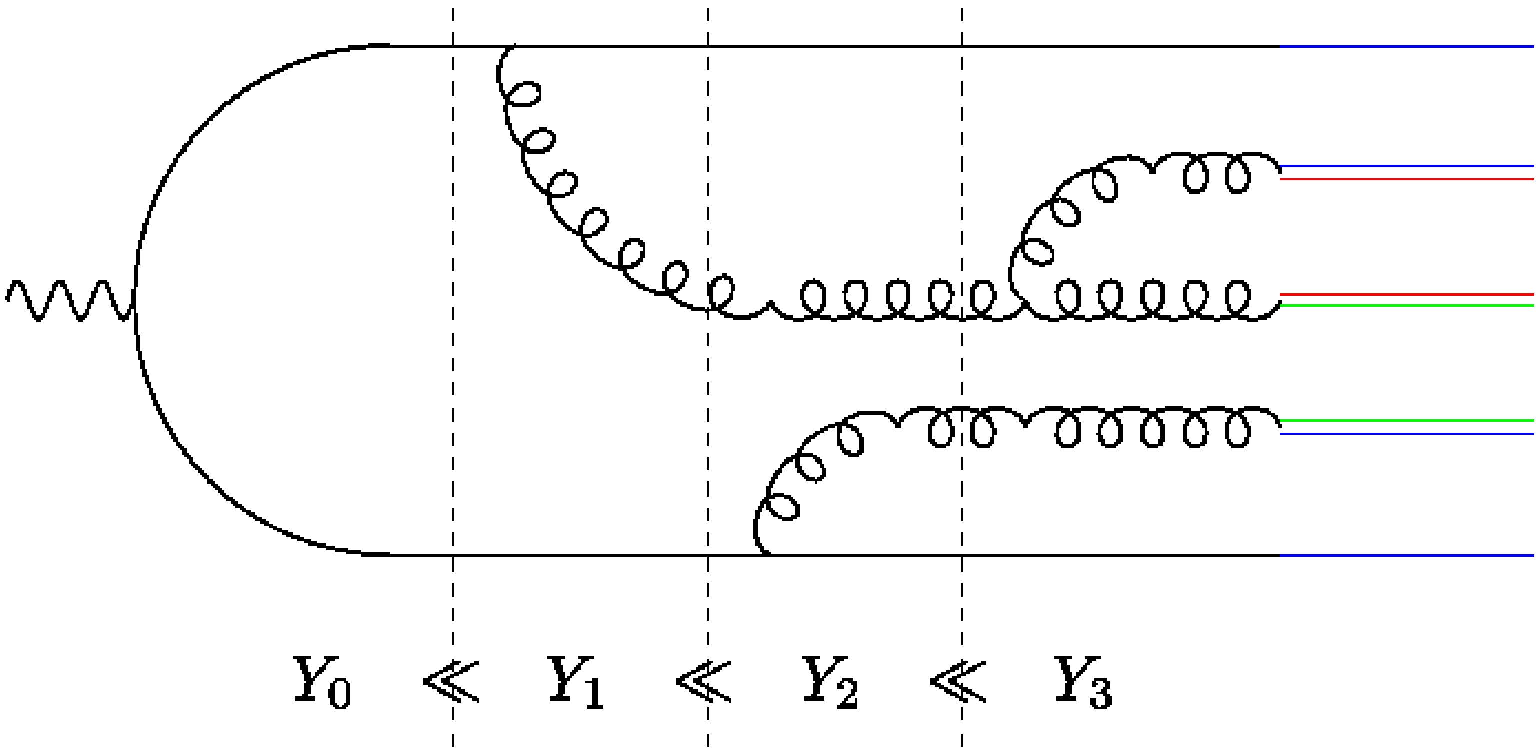
- In the LLA of perturbative QCD (pQCD), the cross section factorizes as

$$\sigma^{\gamma^* p}(Y, Q) = \int d^2\mathbf{r} \int_0^1 dz \, |\psi(z, \mathbf{r}, Q)|^2 \, \sigma_{dip}^{\gamma^* p}(Y, \mathbf{r}) \quad (1)$$

$\sigma_{dip}^{\gamma^* p}(Y, \mathbf{r})$ is the dipole-proton cross section, z is the fraction of photon's momentum carried by the quark and $\mathbf{r}$ is the transverse separation of the quark-anti-quark pair

The Balitsky-Kovchegov equation

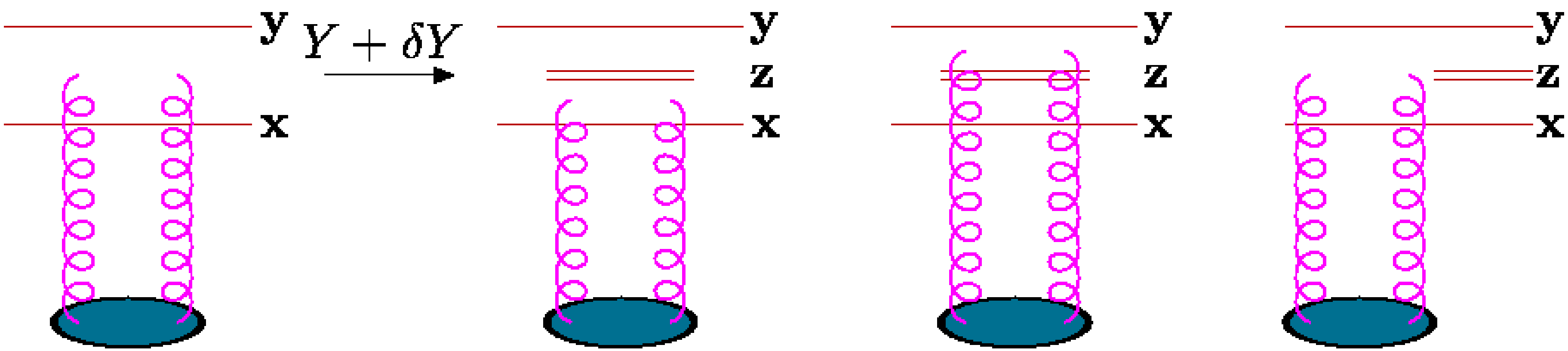
- Consider a fast-moving $q\bar{q}$



- In the large N_c limit the gluons emitted can be replaced by quark-anti-quark pairs, which interact with the target via two gluon exchanges

Balitsky-Kovchegov equation

- Consider a small increase in rapidity

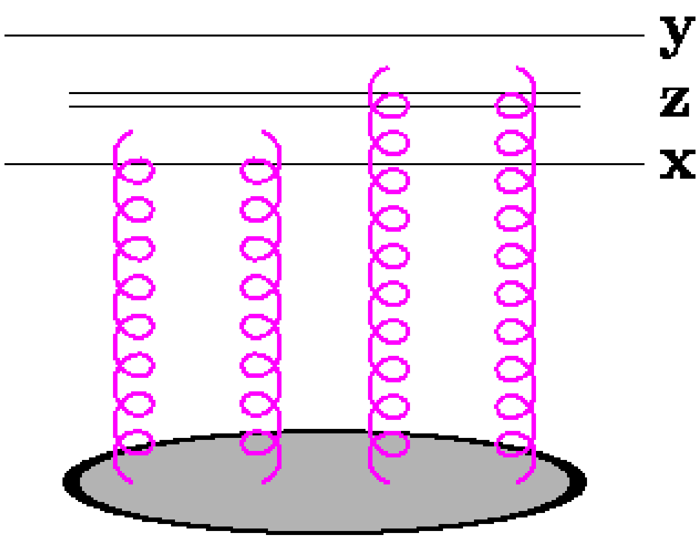


$$\partial_Y N(\mathbf{x},\mathbf{y},Y) = \bar{\alpha}\int d^2\mathbf{z}\,\frac{(\mathbf{x}-\mathbf{y})^2}{(\mathbf{x}-\mathbf{z})^2(\mathbf{z}-\mathbf{y})^2}\,[N(\mathbf{x},\mathbf{z},Y)+N(\mathbf{z},\mathbf{y},Y)-N(\mathbf{x},\mathbf{y},Y)] \tag{2}$$

where $\bar{\alpha}=\alpha_s N_c/\pi$

- This equation is equivalent to BFKL equation
  - Fast growth of the cross sections with powers of the energy
  - Problem of diffusion in the infrared: the momentum of some gluons become of the order of Λ_{QCD}

Balitsky-Kovchegov equation



- Now we consider multiple scatterings
- In the evolution, these multiple scatterings appear as a term proportional to N^2

$$\partial_Y N(\mathbf{x},\mathbf{y},Y) = \bar{\alpha}\int d^2\mathbf{z}\frac{(\mathbf{x}-\mathbf{y})^2}{(\mathbf{x}-\mathbf{z})^2(\mathbf{z}-\mathbf{y})^2}\left[N(\mathbf{x},\mathbf{z},Y)+N(\mathbf{z},\mathbf{y},Y)-N(\mathbf{x},\mathbf{y},Y)\right.$$
$$\left.-N(\mathbf{x},\mathbf{z},Y)N(\mathbf{z},\mathbf{y},Y)\right] \quad (3)$$

- Of course, the quadratic term is important when $N \approx 1$

BK equation in momentum space

- Let us consider that the amplitude N is independent of the impact parameter $\mathbf{b} = \frac{\mathbf{x}+\mathbf{y}}{2}$ and of the direction of $\mathbf{r} = \mathbf{x} - \mathbf{y}$

- Then

$$N(\mathbf{x}, \mathbf{y}) \to N(\mathbf{r}, \mathbf{b}) \to N(r) \tag{4}$$

- The dipole cross section is proportional to the forward scattering amplitude N through the relation

$$\sigma_{dip}^{\gamma^* p}(Y, r) = 2\pi R_p^2 N(r, Y) \tag{5}$$

- We define the forward scattering amplitude in momentum space $\mathcal{N}(Y,\, k)$

$$\mathcal{N}(Y,\, k) = \int_0^\infty \frac{dr}{r} J_0(kr) N(Y,\, r) \tag{6}$$

- The BK equation then reads

$$\partial_Y \mathcal{N} = \bar{\alpha}\chi(-\partial_L)\mathcal{N} - \bar{\alpha}\mathcal{N}^2, \qquad \bar{\alpha} = \frac{\alpha_s N_c}{\pi} \tag{7}$$

- In this picture geometric scaling property reads $\mathcal{N}(Y,\, k) = \mathcal{N}\left(\frac{k}{Q_s(Y)}\right)$

BK equation in momentum space

- In this equation

$$\chi(\gamma) = 2\psi(1) - \psi(\gamma) - \psi(1-\gamma) \quad (8)$$

is the characteristic function of the well known Balitsky-Fadin-Kuraev-Lipatov (BFKL) kernel, and $L = \log\left(k^2/k_0^2\right)$, where k_0 is some fixed low momentum scale

- The kernel χ is an integro-differential operator which may be defined with the help of the formal series expansion

$$\begin{aligned}\chi(-\partial_L) \quad = \quad & \chi(\gamma_0)\mathbf{1} + \chi'(\gamma_0)(-\partial_L - \gamma_0\mathbf{1}) + \frac{1}{2}\chi''(\gamma_0)(-\partial_L - \gamma_0\mathbf{1})^2 \\ & + \frac{1}{6}\chi^{(3)}(\gamma_0)(-\partial_L - \gamma_0\mathbf{1})^3 + \ldots \end{aligned} \quad (9)$$

for some γ_0 between 0 and 1

BK and FKPP equations

- The Fisher and Kolmogorov-Petrovsky-Piscounov (FKPP) equation is a famous equation in non-equilibrium statistical physics, whose dynamics is called reaction-diffusion dynamics,

$$\partial_t u(x,t) = \partial_x^2 u(x,t) + u - u^2, \tag{10}$$

where t is time and x is the coordinate.

- It has been shown that, after the change of variables

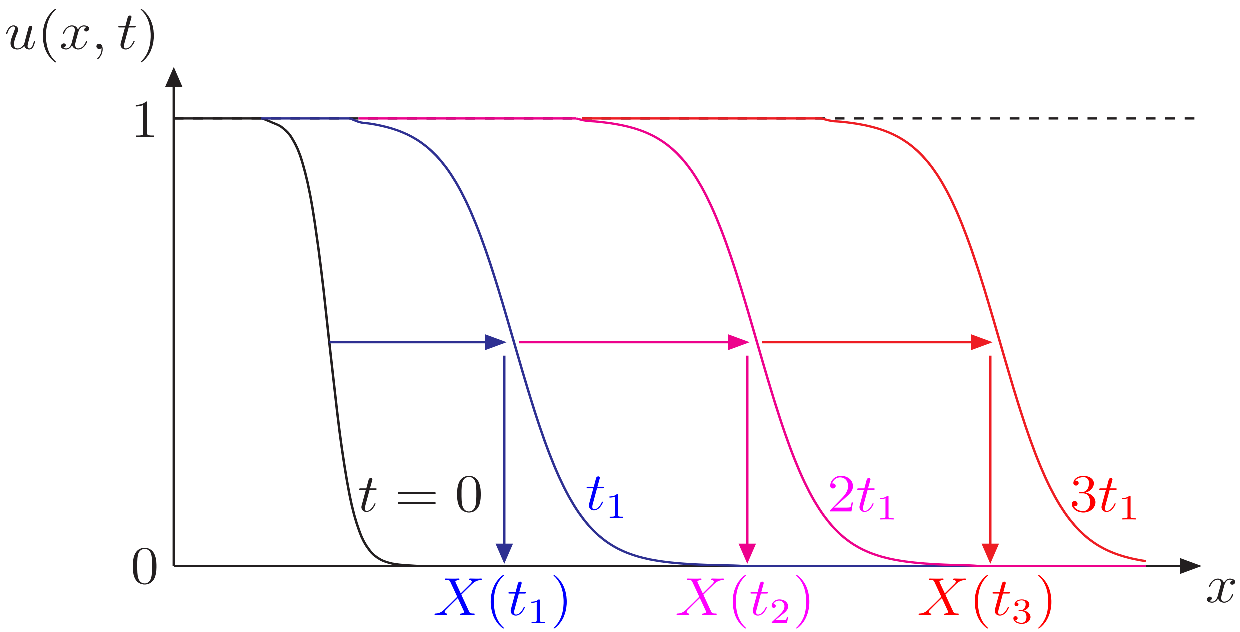
$$t \sim \bar{\alpha} Y, \quad x \sim \log(k^2/k_0^2), \quad u \sim \mathcal{N} \tag{11}$$

BK equation reduces to FKPP equation, when its kernel is approximated by the first three terms of the expansion, the so-called diffusive approximation or saddle point approximation

$$\chi(-\partial_L) = \chi(\gamma_c)\mathbf{1} + \chi'(\gamma_c)(-\partial_L - \gamma_c\mathbf{1}) + \frac{1}{2}\chi''(\gamma_c)(-\partial_L - \gamma_c\mathbf{1})^2, \tag{12}$$

Traveling wave solutions

- The FKPP evolution equation admits the so-called traveling wave solutions
  - For a traveling wave solution one can define the position of a wave front $x(t) = v(t)t$, irrespective of the details of the nonlinear effects
  - At larges times, the shape of a traveling wave is preserved during its propagation, and the solution becomes only a function of the scaling variable $x - vt$



Traveling waves and saturation

- In the language of saturation physics the position of the wave front is nothing but the saturation scale

$$x(t) \sim \ln Q_s^2(Y) \tag{13}$$

and the scaling corresponds to the geometric scaling

$$x - x(t) \sim \ln k^2/Q_s^2(Y) \tag{14}$$

- Summarizing:

$$\begin{aligned} \textit{Time } t &\quad\longrightarrow\quad Y \\ \textit{Space } x &\quad\longrightarrow\quad L \\ \textit{Wave front } u(x-vt) &\quad\longrightarrow\quad \mathcal{N}(L-vY) \\ \textit{Traveling Waves} &\quad\longrightarrow\quad \textit{Geometric Scaling} \end{aligned} \tag{15}$$

Scattering amplitude

- The linear part of the BK equation is solved by

$$\mathcal{N}(k,Y) = \int \frac{d\gamma}{2\pi i} \mathcal{N}_0(\gamma)\exp(-\gamma L + \bar{\alpha}\chi(\gamma)Y) \tag{16}$$

- The velocity of the front is given by

$$v = v_g = \min_{\gamma} \bar{\alpha}\frac{\chi(\gamma)}{\gamma} = \bar{\alpha}\frac{\chi(\gamma_c)}{\gamma_c} = \bar{\alpha}\chi'(\gamma_c) \tag{17}$$

  where γ_c is the saddle point of the exponential phase factor. This fixes, for the BFKL kernel, $\gamma_c = 0.6275...$, $v = 4.88\bar{\alpha}$

- In terms of QCD variables, the dipole forward scattering amplitude in momentum space near the wave front reads

$$\begin{aligned} \mathcal{N}(\tau, Y) &\propto \sqrt{\frac{2}{\bar{\alpha}\chi''(\gamma_c)}} \log\left(\frac{k^2}{Q_s^2(Y)}\right)\left(\frac{k^2}{Q_s^2(Y)}\right)^{-\gamma_c} \exp\left(-\frac{\log^2\left(\frac{k^2}{Q_s^2(Y)}\right)}{2\bar{\alpha}\chi''(\gamma_c)Y}\right), \\ Q_s^2(Y) &= Q_0^2 \exp\left(\bar{\alpha}\frac{\chi(\gamma_c)}{\gamma_c}Y - \frac{3}{2\gamma_c}\log Y\right). \end{aligned} \tag{18}$$

Connecting to Saturation

- We are studying the connection between the traveling wave solution and the saturation region

- These different domains can be parametrized as

$$\mathcal{N}(\tau, Y) = c - \log\left(\frac{k}{Q_s(Y)}\right)$$

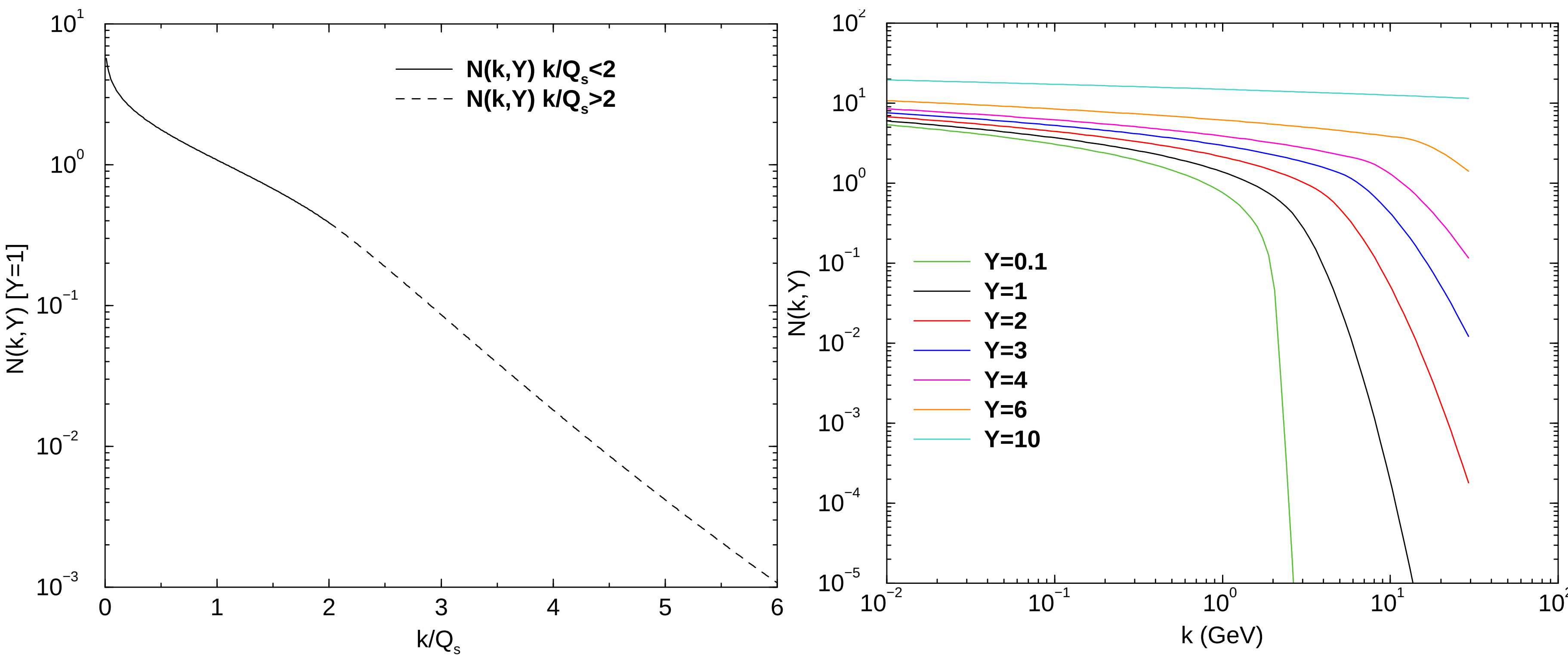
when, $k < Q_s(Y)$, and

$$\mathcal{N}(k, Y) \propto \sqrt{\frac{2}{\bar{\alpha}\chi''(\gamma_c)}} \left(\frac{k^2}{Q_s^2(Y)}\right)^{-\gamma_c} \log\left(\frac{k^2}{Q_s^2(Y)}\right) \exp\left\{-\frac{\log^2\left(\frac{k^2}{Q_s^2(Y)}\right)}{2\bar{\alpha}\chi''(\gamma_c)Y}\right\}$$

when, $k > Q_s(Y)$

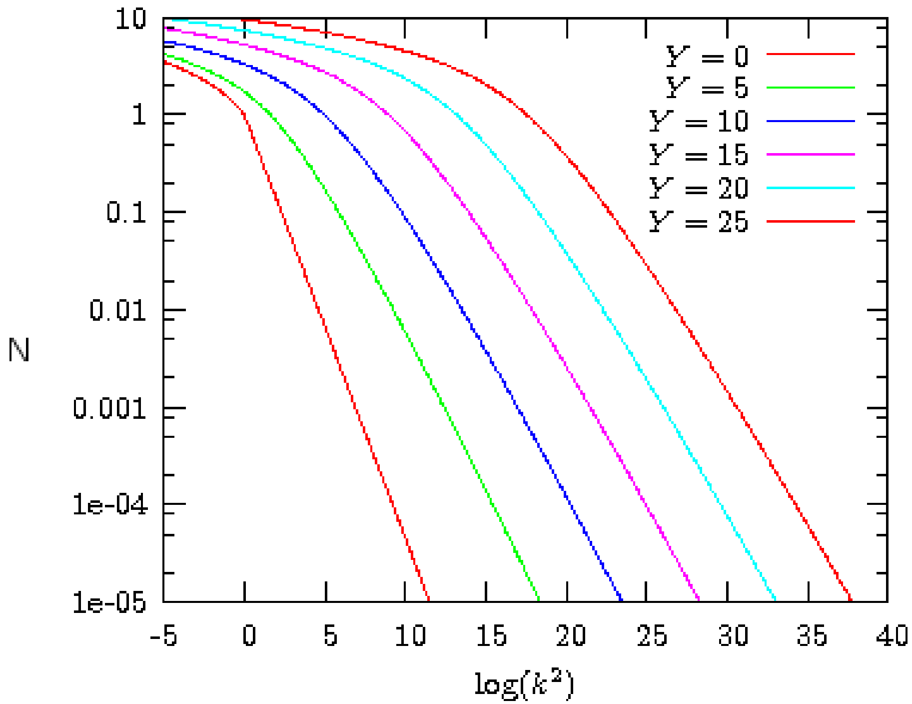
Connecting to Saturation

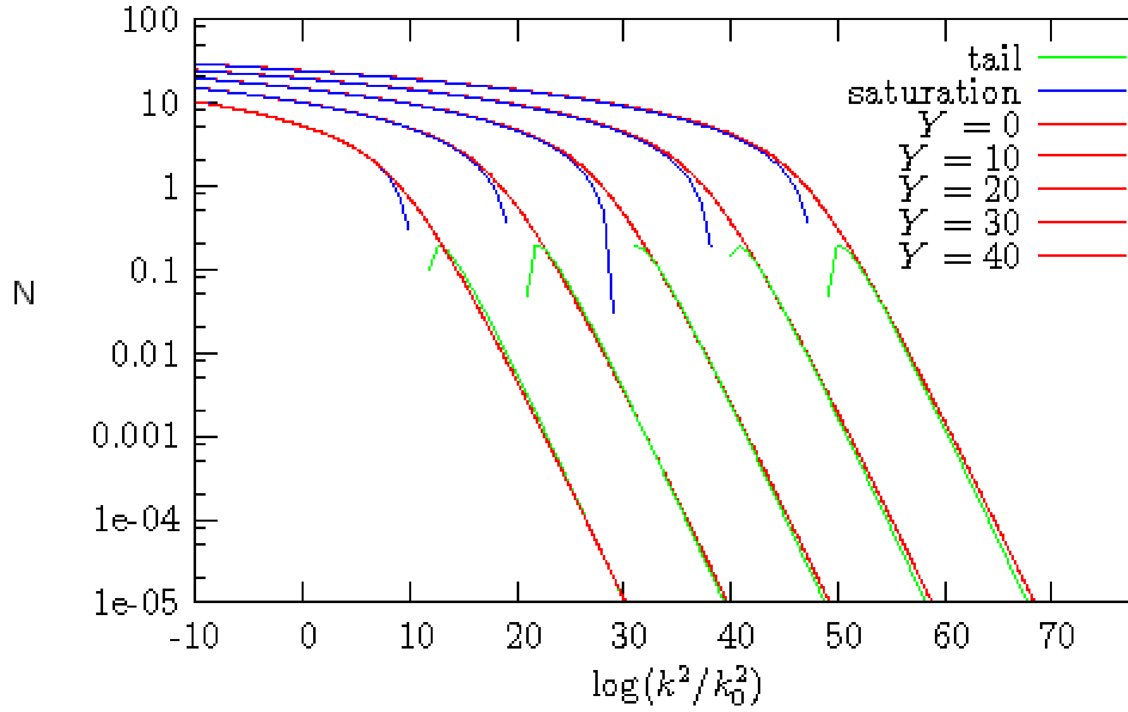
- The first attempt was to perform a **matching** between the two regions
- We imposed continuity of N and its first derivative at $k = 2Q_s(Y)$



Connecting to Saturation

- Then, a better way to obtain the connection between the two regions which satisfies this condition would be an interpolation





Next

- In order to obtain an interpolation model to connect the regions of interest we intend to build the saturation domain from the dilute one in the following way:

$$\mathcal{N} = \frac{1}{1 + \frac{1}{\mathcal{N}_{dil}}} \tag{19}$$

- This expression guarantees the correct asymptotic behaviors of the forward scattering amplitude $\mathcal{N}$. Indeed,
  - when $\mathcal{N}_{dil} \ll 1$

$$\mathcal{N} \approx \mathcal{N}_{dil} \tag{20}$$

  - when $\mathcal{N}_{dil} \gg 1$

$$\mathcal{N} \approx 1. \tag{21}$$

References

- C. Marquet, R. Peschanski, and G. Soyez, Phys.Lett.B **628**, 239 (2005).
- A. M Staśto, K. Golec-Biernat, and J. Kwiecinski, Phys. Rev. Lett. **86**, 596 (2001).
- S. Munier and R. Peschanski, Phys. Lett. **91**, 232001 (2003); Phys. Rev. D **69**, 034008 (2004).
- I. I. Balitsky, Nucl. Phys. B **463**, 99 (1996).
- Y. V. Kovchegov, Phys. Rev. D **60**, 034008 (1999); Phys. Rev. D. **61**, 074018 (2000).