Saturation Physics: A Brief Introduction

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Outline

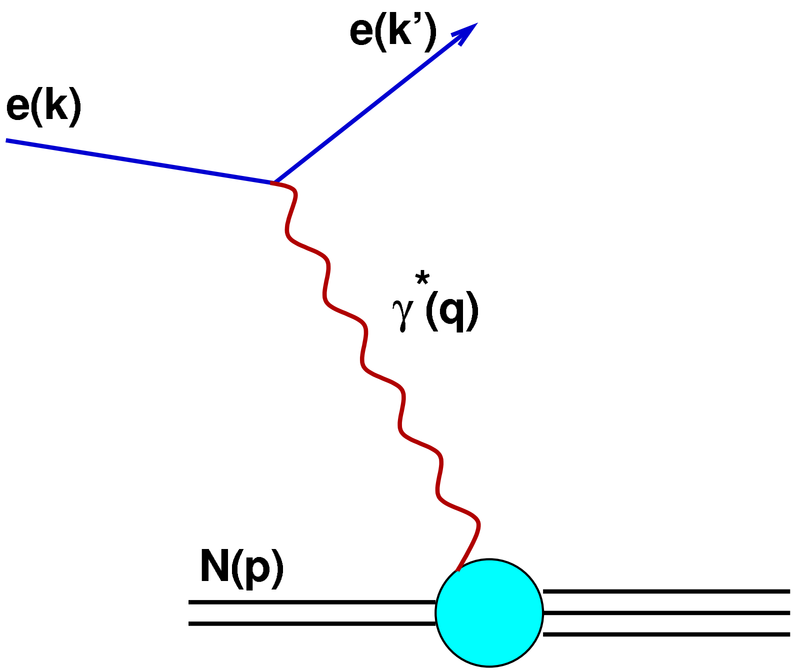
- Introduction
- BFKL equation
- GLR-MQ equation
- Glauber-Mueller Model
- Next Seminar

Introduction

- One of the most intriguing problems in Quantum Chromodynamics is the growth of the cross sections for hadronic interactions with energy
- Since gluons carry color charge and couple to each other, the increase of energy will cause a fast growth of the gluon density and consequently of the cross section
- It is believed that at this regime gluon recombination might be important and it would decrease the growth of the parton density; this is called perturbative parton saturation
- The dense colored medium formed at these high energies can be obtained through many processes; as an example, deep inelastic lepton-nucleon scattering (DIS)

Introduction

DIS kinematics and variables



- The total energy squared of the photon-nucleon system

$$s = (p+q)^2$$

- Photon virtuality

$$q^2 = (k-k') = -Q^2 < 0$$

- The Bjorken variable

$$x \equiv x_{Bj} = \frac{Q^2}{2p\cdot q} = \frac{Q^2}{Q^2+s}$$

- **The high energy limit:**

$$s \to \infty, \quad x \approx \frac{Q^2}{s} \to 0, \quad Y = \ln 1/x \to \infty$$

Introduction

Light cone (LC) coordinates

- Let z be the longitudinal axis of the collision. For an arbitrary four-vector $v^\mu = (v^0, v^1, v^2, v^3)$, we define

$$v^+ \equiv \frac{1}{\sqrt{2}}(v^0 + v^3), \quad v^- \equiv \frac{1}{\sqrt{2}}(v^0 - v^3), \quad v_\perp \equiv (v^1, v^2)$$

- In particular, $x^+ = (t+z)/\sqrt{2}$ is the LC "time" and $x^- = (t-z)/\sqrt{2}$ is the LC "coordinate"
- The invariant scalar product of two four-vectors:

$$p \cdot x = p^- x^+ + p^+ x^- - p_\perp \cdot x_\perp$$

- The light cone dispersion relation takes the form

$$p^- = \frac{1}{2} \frac{p_\perp^2 + m^2}{p^+}$$

BFKL Equation

- Describes the (high energy) behavior of scattering amplitudes and gluon distribution functions resumming powers of radiative corrections of order $(\alpha_s \ln s/Q^2)^n$

Physical Picture

- Consider an ultrarelativistic gluon scattering on a target at rest
- In the infinite momentum frame: the gluon has a very large p^+
- The probability of emission of another gluon with $k_1^+ \ll p^+$:

$$dP_1 = \frac{\alpha_s N_c}{\pi} \frac{d^2 k_1}{\underline{k}_1^2} \frac{dk_1^+}{k_1^+}$$

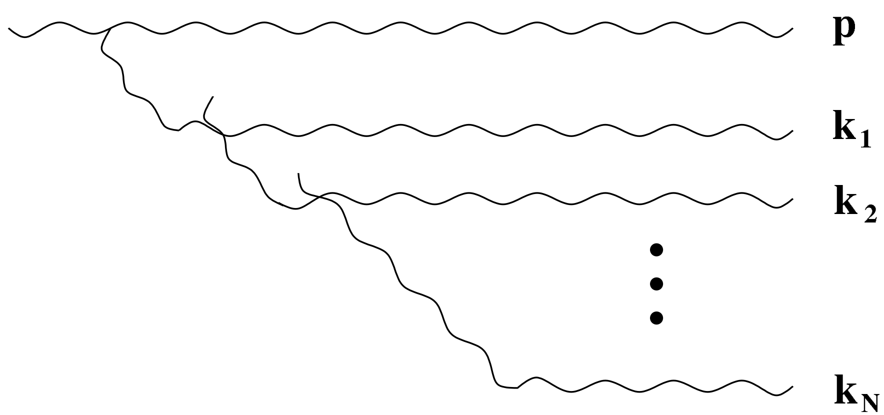
- Defining the *rapidity* variable $y_1 = \ln \frac{p^+}{k_1^+}$

$$dP_1 = c\frac{\alpha_s N_c}{\pi} dy_1$$

- Now, let us consider that the original gluon emits a cascade of gluons with longitudinal momenta progressively smaller

$$p^+ \gg k_1^+ \gg k_2^+ \gg \ldots \gg k_N^+$$

BFKL Equation



- The probability of ith gluon emission is

$$dP_i \propto \frac{\alpha_s N_c}{\pi} dy_i$$

- The number of gluons emitted in rapidity interval Y is

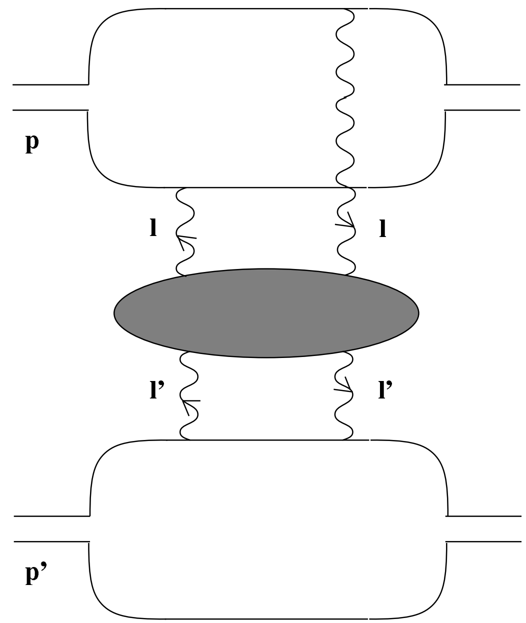
$$N(Y) = e^{c \frac{\alpha_s N_c}{\pi} Y},$$

that is, each power of α_s gets enhanced by a power of rapidity Y (equivalent to the logarithm of center of mass energy s) such that the equation above resums powers of $\alpha_s Y$

BFKL Equation

The Equation

- Let us consider a scattering of a bound heavy quark-antiquark (quarkonium or onium) state on another quarkonium



- The cross section:

$$\sigma_{tot} = 2\int d^2x \int_0^1 dz \int d^2x' \int_0^1 dz' \Phi_{q\bar{q}}(\underline{x}, z)\Phi_{q\bar{q}}(\underline{x}', z')F(\underline{x}, \underline{x}', Y)$$

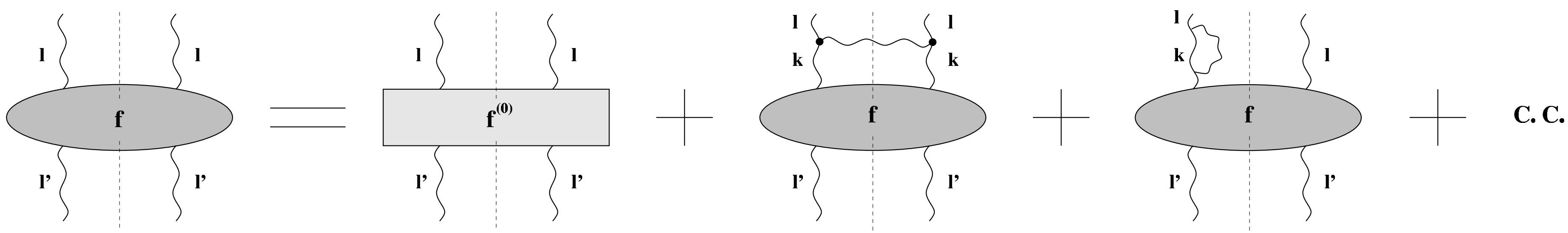
BFKL Equation

- In the center of mass frame, each of the onia carries a large light cone momenutm, p^+ and p^-
- $Y = \ln(s/4M^2)$ is the rapidity variable and

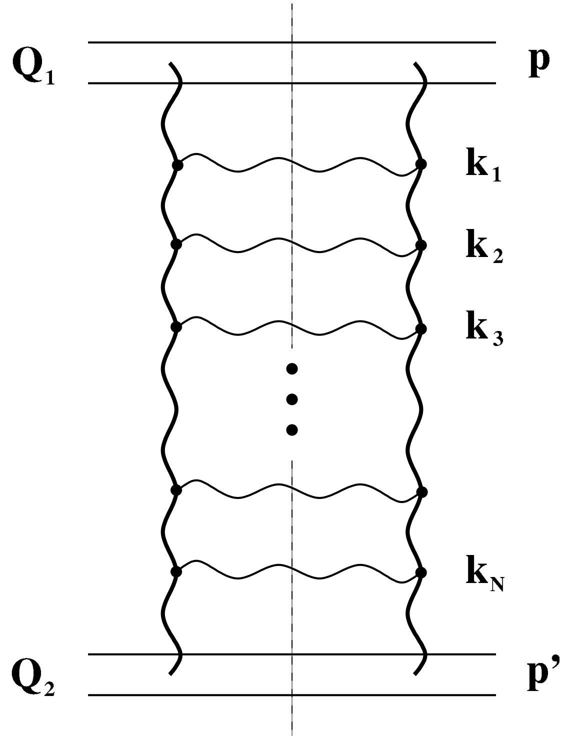
$$
\begin{aligned}
F(\underline{x}, \underline{x}', Y) &= \frac{\alpha_s^2 C_F}{N_c} \int \frac{d^2 l d^2 l'}{\underline{l}^2} \underline{l}'^2 (2 - e^{-i\underline{l}\cdot\underline{x}} - e^{i\underline{l}\cdot\underline{x}})(2 - e^{-i\underline{l}'\cdot\underline{x}'} - e^{i\underline{l}'\cdot\underline{x}'}) \\
&\times \quad f(\underline{l}, \underline{l}', Y)
\end{aligned}
$$

- The BFKL equation for the amplitude $f(\underline{l}, \underline{l}', Y)$ reads

$$
\frac{\partial f(\underline{l}, \underline{l}', Y)}{\partial Y} = \frac{\alpha_s N_c}{\pi} \int \frac{d^2 k}{(\underline{k} - \underline{l})^2} \left[f(\underline{k}, \underline{l}', Y) - \frac{\underline{l}^2 f(\underline{l}, \underline{l}', Y)}{\underline{k}^2 + (\underline{k} - \underline{l})^2} \right]
$$



BFKL Equation



- The solution should contain all iteractions of the kernel
- Iteration leads to the ladder diagrams

← *interaction characterized by typical transverse momentum scales Q_1 and Q_2 via a BFKL-evolved amplitude*

- **The solution:**

$$f(\underline{l},\underline{l}',Y) \approx \frac{1}{2\pi^2 l l'}\sqrt{\frac{\pi}{14\zeta(3)\bar{\alpha}_s Y}}\exp\left[(\alpha_P-1)Y-\frac{\ln^2 l/l'}{14\zeta(3)\bar{\alpha}_s Y}\right]$$

where

$$\alpha_P - 1 = \frac{4\alpha_s N_c}{\pi}\ln 2, \quad \bar{\alpha}_s \equiv \frac{\alpha_s N_c}{\pi}$$

BFKL Equation

- **Problems of the BFKL Evolution:**
  - Cross sections mediated by the BFKL exchange grow as a power of energy

$$\sigma \sim e^{(\alpha_P - 1)Y} \sim s^{\alpha_P - 1}$$

    and this behavior violates the Froissart bound

$$\sigma \leq \frac{cosnt}{m_\pi^2} \ln^2 s$$

  - Diffusion term $\Rightarrow$ the distribution of gluons' transverse momentum in the ladder may have significant fluctuations towards high and low momenta
  - At certain very high energy the momentum of some gluons in the middle of the ladder would become of the order of Λ_{QCD}; this invalidates the application of perturbative QCD and, consequently, of the BFKL evolution

GLR-MQ Equation

- Let us define the unintegrated gluon distribution of an onium

$$\phi(x_{Bj}, \underline{k}^2) = \frac{\alpha_s C_F}{\pi} \int \frac{d^l}{\underline{l}^2} (2 - e^{-i\underline{l}\cdot\underline{x}} - e^{i\underline{l}\cdot\underline{x}}) f(\underline{l}, \underline{k}, Y = \ln 1/x_{Bj}) d^2x \, dz \, \Phi_{q\bar{q}}(\underline{x}, z)$$

  which gives the number of gluons in the onium wave function having transverse momentum $\underline{k}$ and carrying the fraction x_{Bj} of the onium "$+$" component of the momentum

- Using the BFKL solution one obtains

$$\phi(x_{Bj}, \underline{k}^2) \sim \left(\frac{1}{x_{Bj}}\right)^{\alpha_P - 1}$$

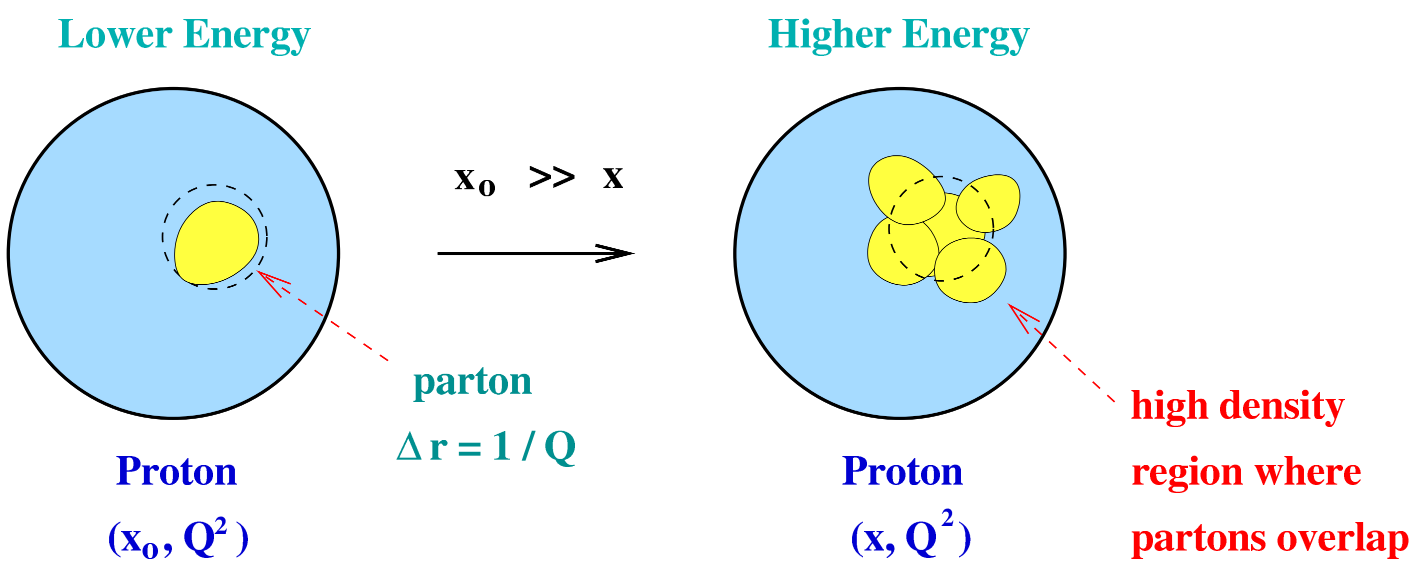
- This feature is illustrated by the gluon cascade representation of the BFKL evolution; there, the gluons are produced with comparable transverse momenta, which means that

$$r_1^{\perp} \sim r_2^{\perp} \sim \ldots r_N^{\perp}$$

- As energy increases, more and more gluons are produced in the cascade; the gluons overlap in the transverse plane, creating areas of high gluon density

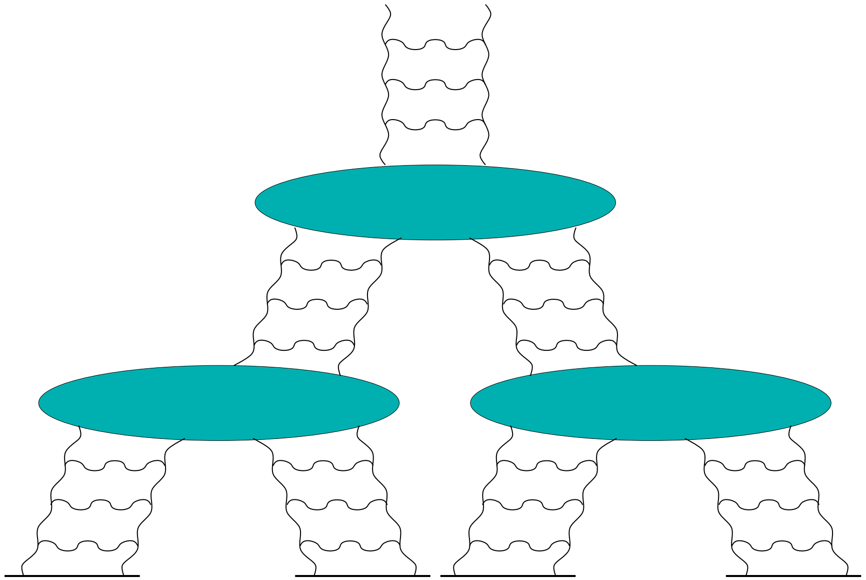
GLR-MQ Equation

- Not only the *number* of gluons, gut their *density* in the transverse plane increase with energy



- In QCD the gluon fields can not be stronger than $A_\mu \sim 1/g$ for small coupling g. Therefore, some new possibily non-linear effects should become important slowing down the density growth
- Gribov, Levin and Ryskin (GLR) considered distribution functions of a "dense" proton or nucleus. For such systems multiple ladder exchanges may become important; these ones should come in as the "fan" diagrams

GLR-MQ Equation



- Multiple ladders start from different quarks and gluons in the proton/nucleus
- Due to high density, the ladders can not stay independent forever
- As the energy increases, so does the gluon density: *recombination* of the ladders

- Suggestion: there could be an intermediate energy region where the physics of gluon didtribution is dominated by $2 \to 1$ ladder recombination only
- This recombination leads to a quadratic correction to the linear BFKL equation $\to$ *GLR evolution equation*

$$\frac{\partial \phi(x, \underline{k}^2)}{\partial \ln(1/x)} = \frac{\alpha_s N_c}{\pi^2} \int \frac{d^2 l}{(\underline{k} - \underline{l})^2} \left[\phi(x, \underline{k}^2) - \frac{\underline{k}^2 \phi(x, \underline{k}^2)}{\underline{l}^2 + (\underline{k} - \underline{l})^2} \right] - \frac{\alpha_s^2 \pi}{S_\perp} \left[\phi(x, \underline{k}^2) \right]^2$$

GLR-MQ Equation

- The quadratic term introduces damping, slowing down the gluon distribution with energy: *saturation of parton distributions*
- The ansatz of GLR was proven by Mueller and Qiu in the *double logarithmic approximation* (DLA) for the gluon distribution functions

$$xG(x,Q^2) = \int^{Q^2} dk^2 \phi(x,k^2)$$

- The DLA is a resummation of the powers of the parameter

$$\alpha_s \ln(1/x) \ln(Q^2/\Lambda^2)$$

- The BFKL equation was derived in the *leading logarithmic approximation* (LLA), corresponding to resummation of the parameter $\alpha_s \ln(1/x)$, or equivalently, $\alpha_s Y$
- Leading powers of $ln(Q^2/\Lambda^2)$ are resummed by the DGLAP equation
- Employing the DLA, Mueller and Qiu obtained

$$\frac{\partial^2 xG(x,Q^2)}{\partial \ln(1/x) \partial \ln(Q^2/\Lambda^2)} = \frac{\alpha_s N_c}{\pi} xG(x,Q^2) - \frac{\alpha_s^2 \pi}{S_\perp} \frac{1}{Q^2} \left[xG(x,Q^2)\right]^2$$

GLR-MQ Equation

which is known as GLR-MQ equation

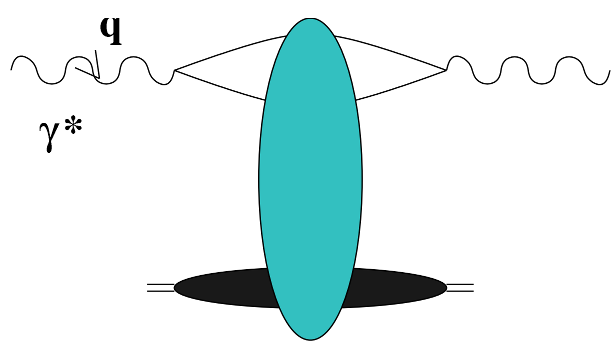
- The above equation allows one to estimate at which Q^2 the nonlinear saturation effects become important; the corresponding value is called *saturation scale*

$$Q_s^2 = \frac{\alpha_s \pi^2}{S_\perp N_c} x G(x, Q_s^2)$$

- The nonlinear saturation effects are important for all $Q \lesssim Q_s$ which is known as *saturation region*
- The quadratic damping is believed to be important only near the border of the saturation region, $Q \sim Q_s$, where non-linear effects only start to become important
- It is expected that higher order non-linear corrections would show up as one goes deeper into the saturation region towards $Q < Q_s$: quasi-classical approximation

Glauber-Mueller Model

- Let us consider deep inelastic scattering (DIS) on a large dilute nucleus
- In the rest frame of the nucleus, the interaction can be thought of as virtual photon splitting into a quark-antiquark pair, which then interacts with the nucleus



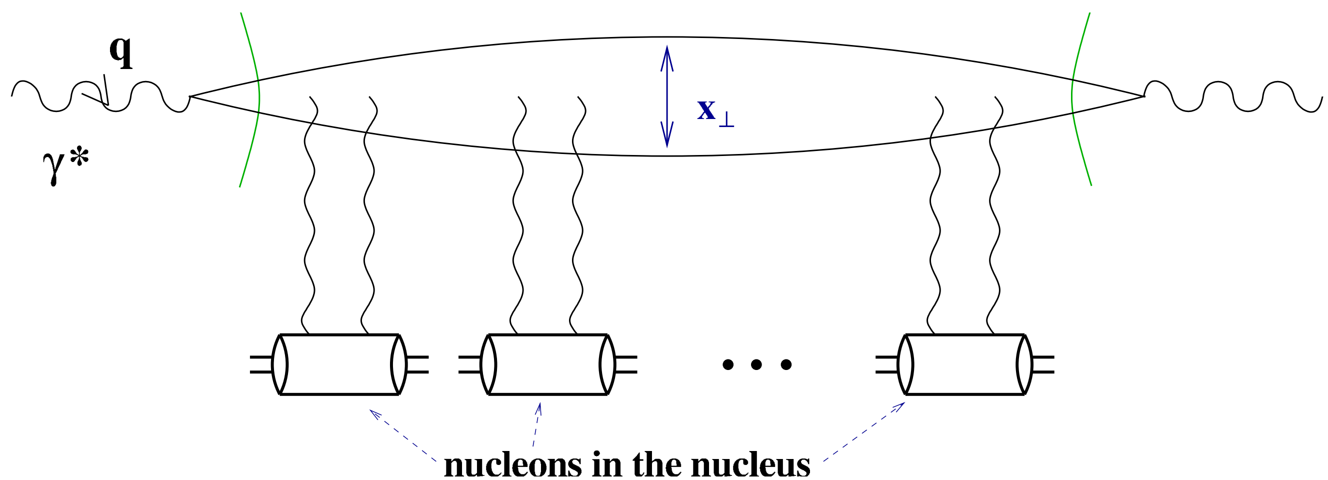
- The total cross section for the virtual photon-nucleus scattering can be written as a convolution of the virtual photon's light cone wave function with the scattering amplitude of a $q\bar{q}$ interacting with the nucleus

$$\sigma_{tot}^{\gamma^* A}(Q^2, x_{Bj}) = \int \frac{d^2x dz}{2\pi}[\Phi_T(\underline{x}, z) + \Phi_L(\underline{x}, z)] d^2b N(\underline{x}, \underline{b}, Y)$$

- The incoming photon with virtuality Q splits into a quark-antiquark pair with transverse separation $\underline{x}$ and impact parameter $\underline{b}$
- The rapidity $Y = \ln(s x_T^2) \approx \ln(1/x_{Bj})$

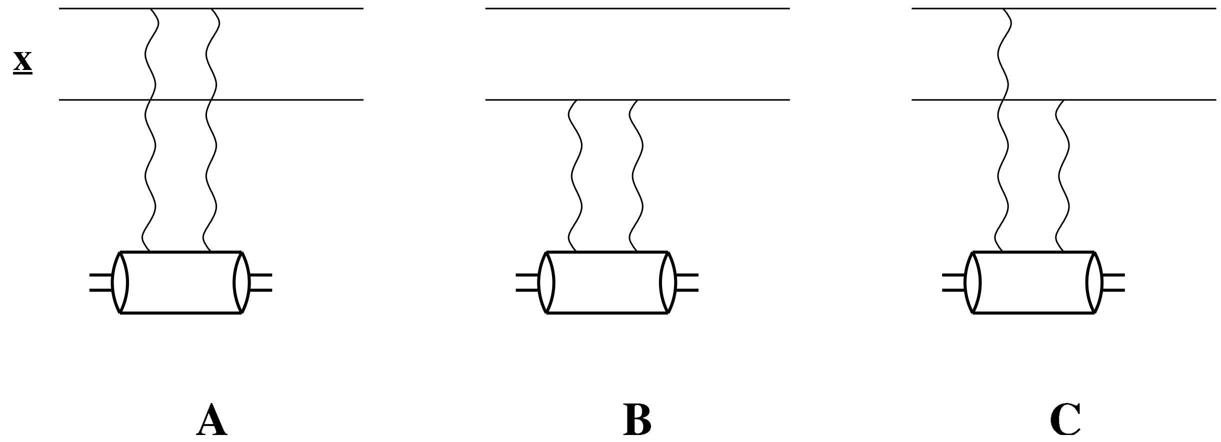
Glauber-Mueller Model

- The goal is to calculate $N(\underline{x}, \underline{b}, Y)$ including all multiple rescatterings of the dipole on the nucleons of the nucleus; then it is necessary to construct a model of the target nucleus
- One assumes the nucleons to be dilutely distributed in the nucleus
- In the covariant gauge ($\partial_\mu A^\mu = 0$) on can represent the dipole-nucleus interaction as a sequence of successive dipole-nucleon interactions



- The lowest order dipole-nucleon interaction in the amplitude is a two-gluon exchange. It can be shown that the diagrams that contribute to the scattering are the following

Glauber-Mueller Model



- One obtains for the scattering cross section of a quark dipole on a single nucleon

$$\frac{\sigma^{q\bar{q}N}}{2} = \frac{\alpha_s \pi^2}{2N_c} x_\perp^2 x_{Bj} G_N(x_{Bj}, 1/x_\perp^2)$$

- The scattering amplitude can be constructed and the result is

$$N(\underline{x}, \underline{b}, Y-0) = 1 - \exp\left[-\frac{\alpha_s \pi^2}{2N_c} \rho T(\underline{b}) x_\perp^2 x_{Bj} G_N(x_{Bj}, 1/x_\perp^2)\right]$$

which is known as *Glauber-Mueller formula*

Glauber-Mueller Model

- Defining the *saturation scale*

$$Q_s^2(\underline{b}) \equiv \frac{4\pi\alpha_s^2 C_F}{N_c}\rho T(\underline{b})$$

we can write

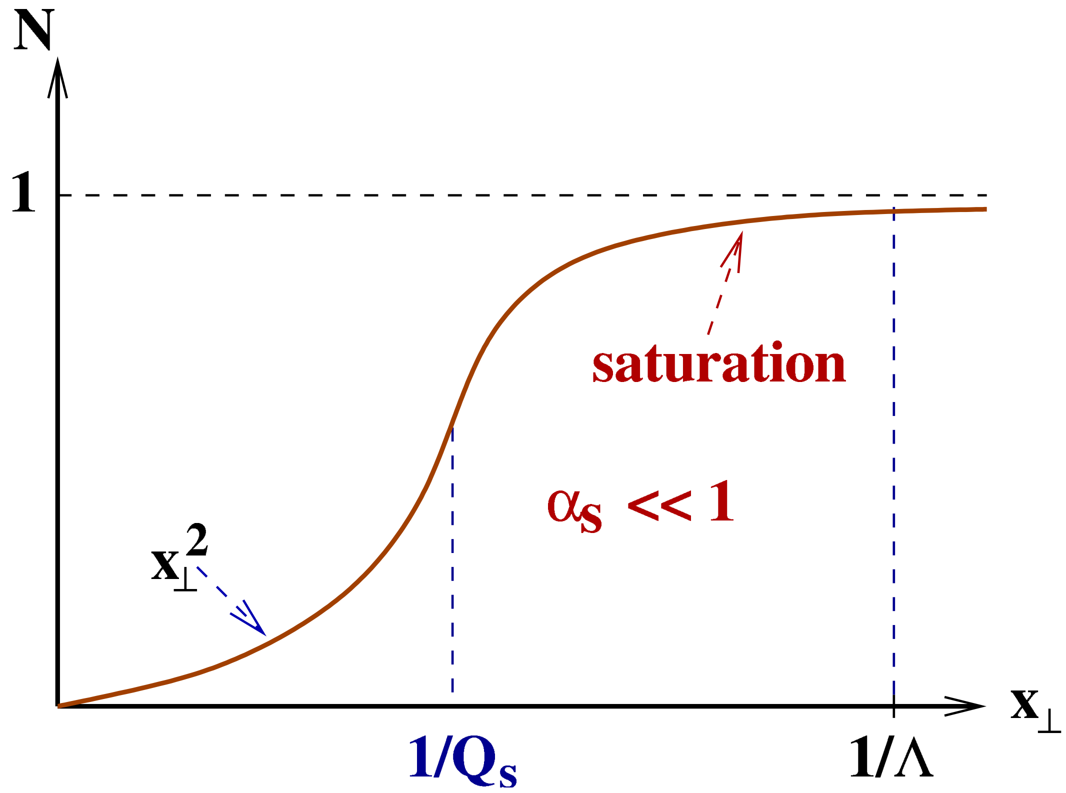
$$N(\underline{x}, \underline{b}, Y=0) = 1 - \exp\left[-\frac{x_\perp^2 Q_s^2(\underline{b}) \ln(1/x_\perp \Lambda)}{4}\right]$$

- At small $x_\perp$, $x_\perp \ll 1/Q_s$, we have $N \sim x_\perp^2$ and the amplitude is a rising function of $x_\perp$
- At large dipole sizes $x_\perp \gtrsim 1/Q_s$, the growth stops and the amplitude *saturates* at $N = 1$: black disk limit for the dipole-nucleus scattering
- Since $T(\underline{b}) \sim A^{1/3}$, the saturation scale above scales with the nuclear atomic number

$$Q_s^2 \sim A^{1/3}$$

- Then, for a very large nucleus saturation scale would become very large, $Q_s \gg \Lambda_{QCD}$, where physics is perturbative and gluons are correct degrees of freedom

Glauber-Mueller Model



Next Seminar

- AGL Equation
- The Balistky-Kovchegov Equation
- The McLerran-Venugopalan Model
- The JIMWLK Equation
- The "State of Art"