Dipole scattering amplitude in momentum space

M. B. Gay Ducati

(IF – UFRGS, Porto Alegre, Brazil)

Work done with J. T. de Santana Amaral, M. A Betemps and G. Soyez

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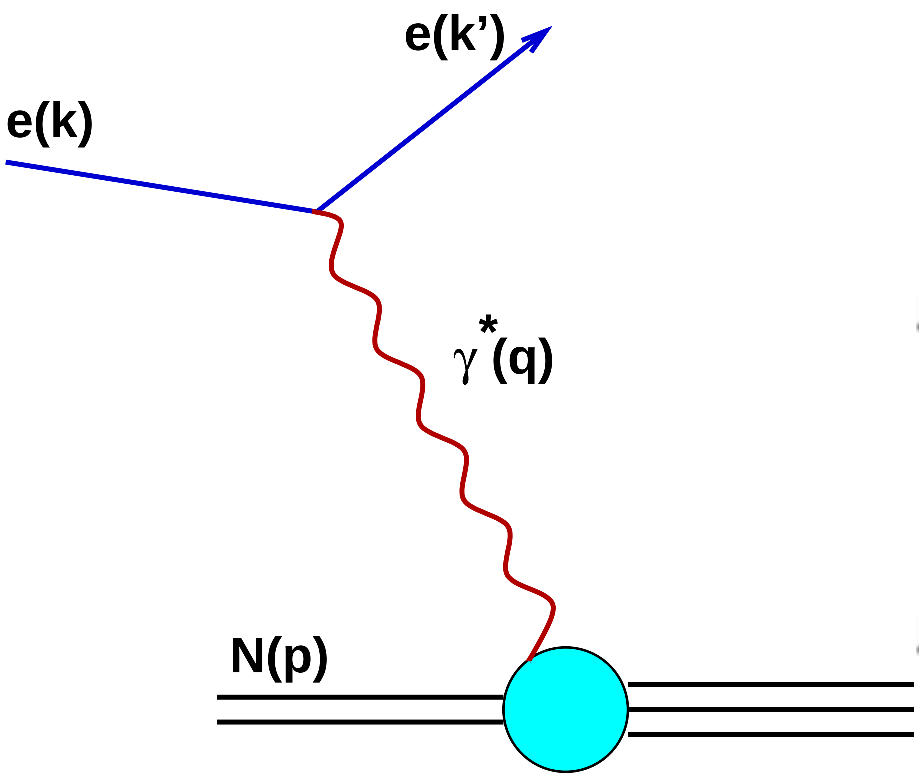
Outline

- DIS and the high energy QCD challenge
- The Dipole Model
- Scattering Amplitudes in High Energy QCD
- Description of the $\gamma^* p$ data
- Conclusions and Discussion

Deep Inelastic Scattering (DIS)

Kinematics and variables



- The total energy squared of the photon-nucleon system

$$s = (p+q)^2$$

- Photon virtuality

$$q^2 = (k-k') = -Q^2 < 0$$

- The Bjorken variable

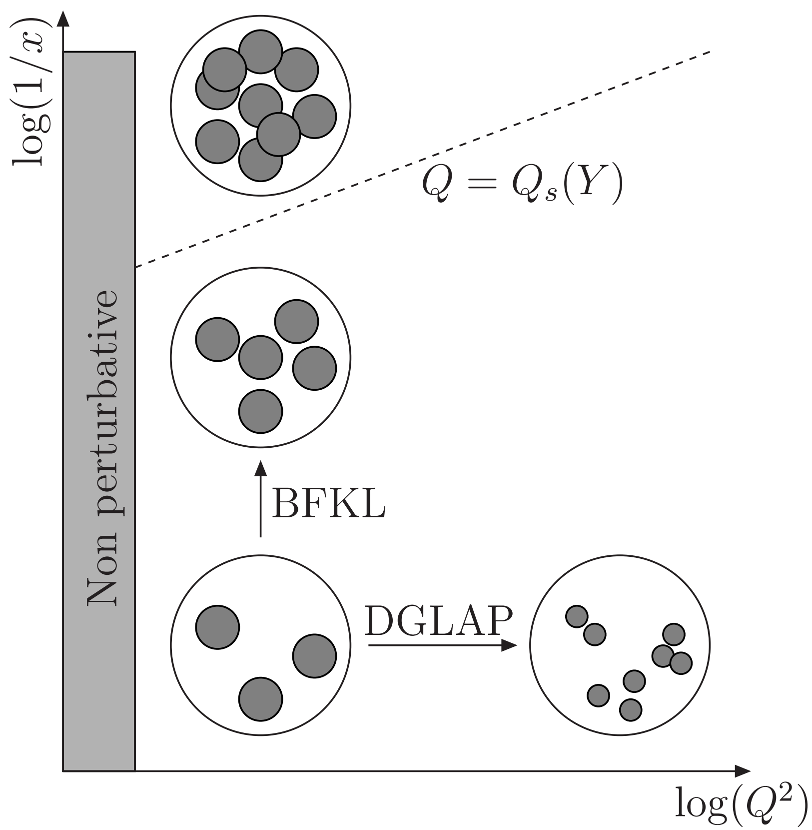
$$x \equiv x_{Bj} = \frac{Q^2}{2p \cdot q} = \frac{Q^2}{Q^2+s}$$

- The high energy limit:

$$s \to \infty, \quad x \approx \frac{Q^2}{s} \to 0$$

QCD at high energies

- As energy increases (with Q fixed) the gluon density grows fast and so does the cross sections for hadronic interactions
  - This is still a challenge in Quantum Chromodynamics
- At this regime gluon recombination and multiple scattering might be important to restore unitarity



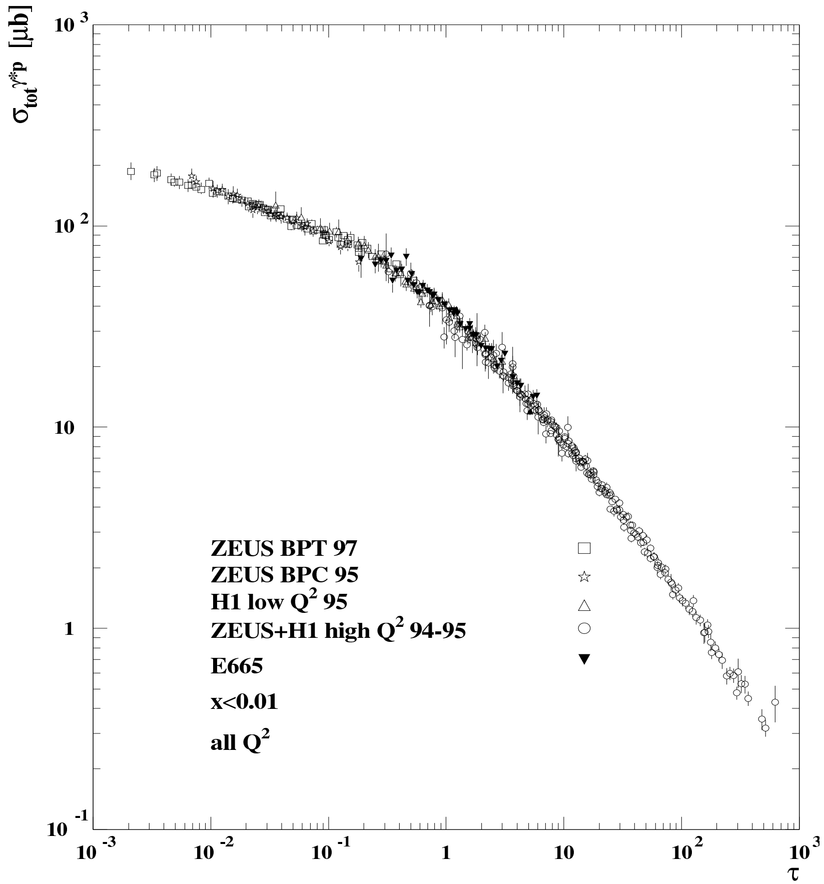
- $Q_s(Y)$ is the so called saturation scale
- The nonlinear saturation effects are important for all $Q \lesssim Q_S(Y)$, which is known as saturation region

Towards saturation

- There has been a large amount of work devoted to the description and understanding of QCD in the high energy limit corresponding to the *saturation*
  - Theory: non-linear QCD equations describing the evolution of scattering amplitudes towards saturation - AGL, BK and JIMWLK equations
  - Phenomenology: discovery of geometric scaling in DIS at HERA
- The Balitsky-Kovchegov (BK) nonlinear equation describes the evolution in rapidity of the scattering amplitude of a dipole off a given target
  - This equation has been shown to lie in the same universality class as the Fisher-Kolmogorov-Petrovsky-Piscounov (FKPP) equation
- Geometric scaling has a natural explanation in terms of the so-called traveling wave solutions of BK equation
- The evolution at intermediate energies is well understood and is described by the linear BFKL equation
- The deep saturation regime can also be evaluated in some models, but the transition between these two regimes is still a challenge

Geometric Scaling

- Geometric scaling is a phenomenological feature of high energy deep inelastic scattering (DIS) which has been observed in the HERA data on inclusive $\gamma^* - p$ scattering, which is expressed as a scaling property of the virtual photon-proton cross section



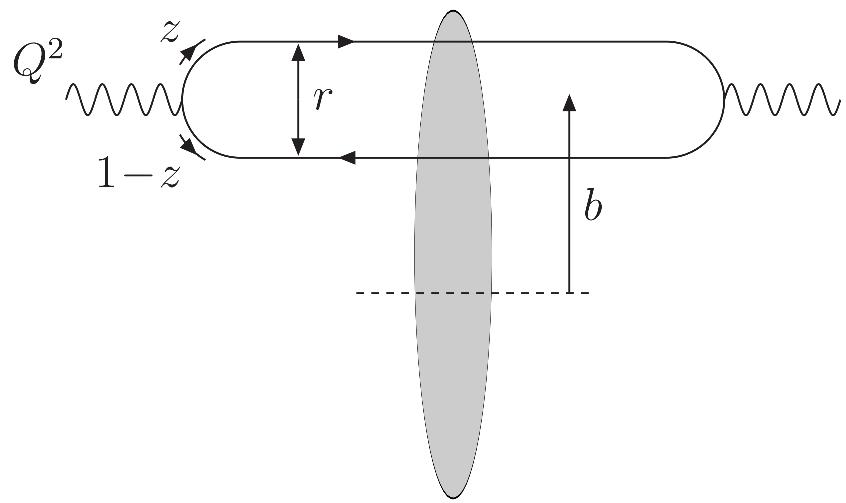
$$\sigma^{\gamma^* p}(Y, Q) = \sigma^{\gamma^* p}(\tau)\,, \quad \tau = \frac{Q^2}{Q_s^2(Y)}$$

where Q is the virtuality of the photon, $Y = \log 1/x$ is the total rapidity and $Q_s(Y)$ is an increasing function of Y called saturation scale

[Stasto, Golec Biernat and Kwiecinsky, 2001]

$\sigma^{\gamma^* p}$ cross section

- Consider the collision between a virtual photon and a proton at high energy; in a frame where the proton carries most of the total energy one can consider that the photon fluctuates into a $q\bar{q}$ pair



- The cross section

$$\sigma_{T,L}^{\gamma^* p}(Y,Q) = \int d^2r \int_0^1 dz \, \left|\Psi_{T,L}(r,z;Q^2)\right|^2 \sigma_{dip}^{\gamma^* p}(r,Y), \tag{1}$$

$\sigma_{dip}^{\gamma^* p}(Y,r)$ is the dipole-proton cross section, z is the fraction of photon's momentum carried by the quark, r is the size of the dipole and b is the impact parameter

Dipole-proton cross section

- The transverse and longitudinal photon wavefunctions

$$|\Psi_T(r,z;Q^2)|^2 = \frac{2N_c\alpha_{em}}{4\pi^2}\sum_q e_q^2 \left\{\left[z^2+(1-z)^2\right]\bar{Q}_q^2 K_1^2(\bar{Q}_q r) + m_q^2 K_0^2(\bar{Q}_q r)\right\} \quad (2)$$

and

$$|\Psi_L(r,z;Q^2)|^2 = \frac{2N_c\alpha_{em}}{4\pi^2}\sum_q e_q^2 \left\{4Q^2 z^2(1-z)^2 K_0^2(\bar{Q}_q r)\right\} \quad (3)$$

where $\bar{Q}_q = z(1-z)Q^2 + m_q^2$, m_q the light quark mass and $K_{0,1}$ are the Mc Donald functions of rank zero and one, respectively

- If one treats the proton as an homogeneous disk of radius R_p, one can write the dipole-proton cross section in terms of the dipole-proton forward scattering amplitude $T(r,Y)$

$$\sigma_{dip}^{\gamma^* p}(r,Y) = 2\pi R_p^2 T(r,Y)$$

F_2 structure function (I)

- The proton structure function F_2 can be obtained from the $\gamma^* p$ cross section through the relation

$$\begin{aligned} F_2(x,Q^2) &= \frac{Q^2}{4\pi^2\alpha_{em}}\left[\sigma_T^{\gamma^* p}(x,Q^2 + \sigma_L^{\gamma^* p}(x,Q^2)\right] \\ &= \frac{Q^2}{4\pi^2\alpha_{em}}\sigma^{\gamma^* p}(x,Q^2) \end{aligned} \quad (4)$$

- It is possible to express the $\gamma^* p$ cross section in terms of the scattering amplitude in momentum space, $\tilde{T}(k,Y)$, through the Fourier transform

$$\begin{aligned} \tilde{T}(k,Y) &= \frac{1}{2\pi}\int \frac{d^2r}{r^2}\, e^{i\mathbf{k}\cdot\mathbf{r}}\, T(r,Y) \\ &= \int_0^\infty \frac{dr}{r} J_0(kr)\, T(r,Y) \end{aligned} \quad (5)$$

F_2 structure function (II)

- After a bit of algebra one obtains

$$F_2(x,Q^2) = \frac{Q^2 R_p^2 N_c}{4\pi^2} \int_0^\infty \frac{dk}{k} \int_0^1 dz\, |\tilde{\Psi}(k,z;Q^2)|^2 \tilde{T}(k,Y) \tag{6}$$

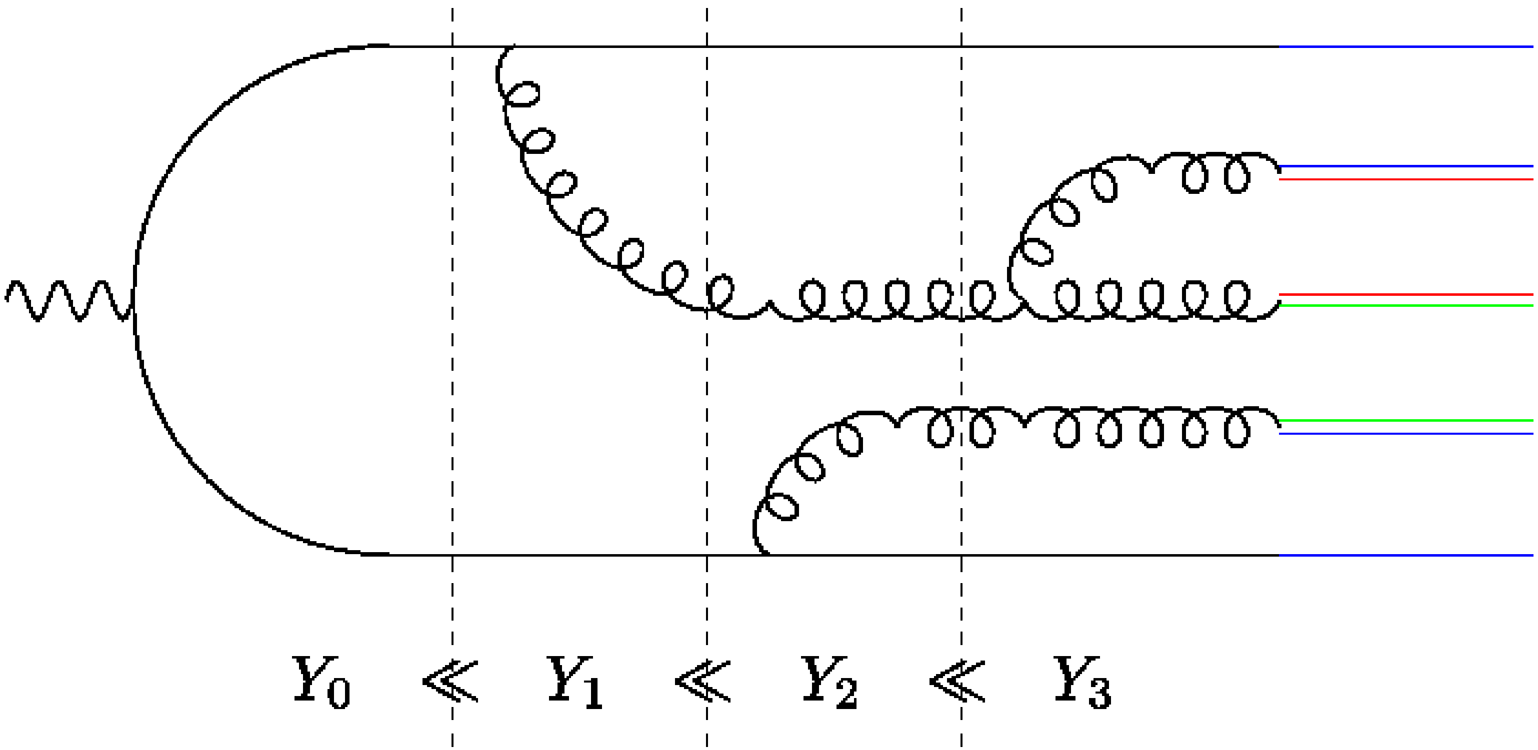
- The wavefunction is now expressed in momentum space

$$|\tilde{\Psi}(k^2,z;Q^2)|^2 = \sum_q \left(\frac{4\bar{Q}_q^2}{k^2+4\bar{Q}_q^2}\right)^2 e_q^2 \left\{ [z^2+(1-z)^2] \left[\frac{4(k^2+\bar{Q}_q^2)}{\sqrt{k^2(k^2+4\bar{Q}_q^2)}} \text{arcsinh}\left(\frac{k}{2\bar{Q}_q}\right) + \frac{k^2-2\bar{Q}_q^2}{2\bar{Q}_q^2} \right] + \frac{4Q^2z^2(1-z)^2+m_q^2}{\bar{Q}_q^2} \left[\frac{k^2+\bar{Q}_q^2}{\bar{Q}_q^2} - \frac{4\bar{Q}_q^4+2\bar{Q}_q^2k^2+k^4}{\bar{Q}_q^2\sqrt{k^2(k^2+4\bar{Q}_q^2)}} \text{arcsinh}\left(\frac{k}{2\bar{Q}_q}\right) \right] \right\}$$

- The amplitude $\tilde{T}(k,Y)$, as we shall see, obeys the Balitsky-Kovchegov equation in momentum space, where the asymptotic behaviour of its solutions is naturally expressed

Dipole evolution

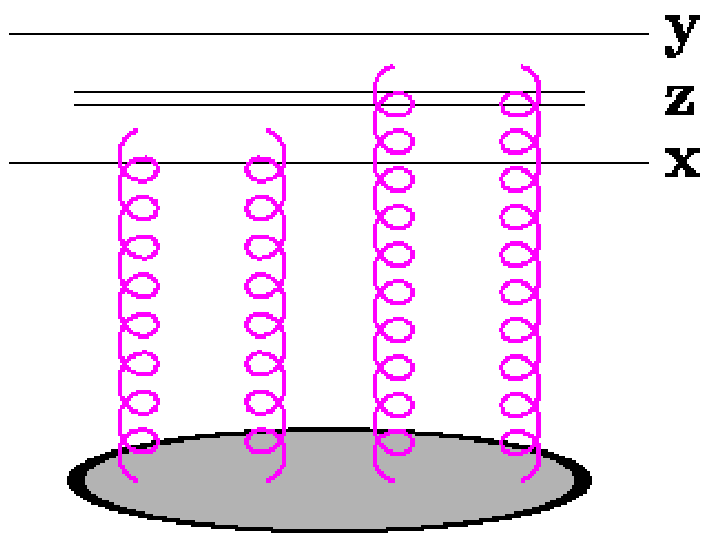
- Consider a fast-moving $q\bar{q}$



- In the large N_c limit the gluons emitted can be replaced by quark-anti-quark pairs, which interact with the target via two gluon exchanges

Balitsky-Kovchegov equation

Multiple scattering



- In the evolution of the scattering amplitude, the multiple scattering appears as a term proportional to T^2 ($\bar{\alpha} = \alpha_s N_c/\pi$)

$$\partial_Y T(\mathbf{x},\mathbf{y},Y) = \bar{\alpha} \int d^2\mathbf{z} \frac{(\mathbf{x}-\mathbf{y})^2}{(\mathbf{x}-\mathbf{z})^2(\mathbf{z}-\mathbf{y})^2} [T(\mathbf{x},\mathbf{z},Y) + T(\mathbf{z},\mathbf{y},Y) - T(\mathbf{x},\mathbf{y},Y) - T(\mathbf{x},\mathbf{z},Y)T(\mathbf{z},\mathbf{y},Y)] \quad (7)$$

- This is the Balitsky-Kovchegov (BK) equation in coordinate space

BK equation in momentum space

- If one neglects the dependence on the impact parameter $\mathbf{b} = (\mathbf{x} + \mathbf{y})/2$ and integrates out the remaining angular dependence of $\mathbf{r}$, the BK equation becomes an equation for $T(r, Y)$
- After performing the Fourier transform the equation can be expressed in momentum space

$$\partial_Y \tilde{T} = \bar{\alpha}\chi(-\partial_L)\tilde{T} - \bar{\alpha}\tilde{T}^2 \tag{8}$$

where

$$\chi(\gamma) = 2\psi(1) - \psi(\gamma) - \psi(1-\gamma) \tag{9}$$

is the characteristic function of the Balitsky-Fadin-Kuraev-Lipatov (BFKL) kernel and $L = \log(k^2/k_0^2)$, with k_0 some fixed soft scale.

- The kernel χ is an integro-differential operator which may be defined with the help of the formal series expansion

$$\begin{aligned}\chi(-\partial_L) &= \chi(\gamma_0)\mathbf{1} + \chi'(\gamma_0)(-\partial_L - \gamma_0\mathbf{1}) + \frac{1}{2}\chi''(\gamma_0)(-\partial_L - \gamma_0\mathbf{1})^2 \\ &\quad + \frac{1}{6}\chi^{(3)}(\gamma_0)(-\partial_L - \gamma_0\mathbf{1})^3 + \ldots \end{aligned} \tag{10}$$

for some γ_0 between 0 and 1

BK and FKPP equations

- It has been shown [Munier and Peshcanski, 03] that, after the change of variables

$$t \sim \bar{\alpha} Y, \quad x \sim \log(k^2/k_0^2), \quad u \sim \tilde{T} \tag{11}$$

BK equation reduces to Fisher and Kolmogorov-Petrovsky-Piscounov (FKPP) equation, when its kernel is approximated by the first three terms of the expansion, the so-called diffusive approximation

$$\chi(-\partial_L) \approx \chi(\gamma_c)\mathbf{1} + \chi'(\gamma_c)(-\partial_L - \gamma_c \mathbf{1}) + \frac{1}{2}\chi''(\gamma_c)(-\partial_L - \gamma_c \mathbf{1})^2, \tag{12}$$

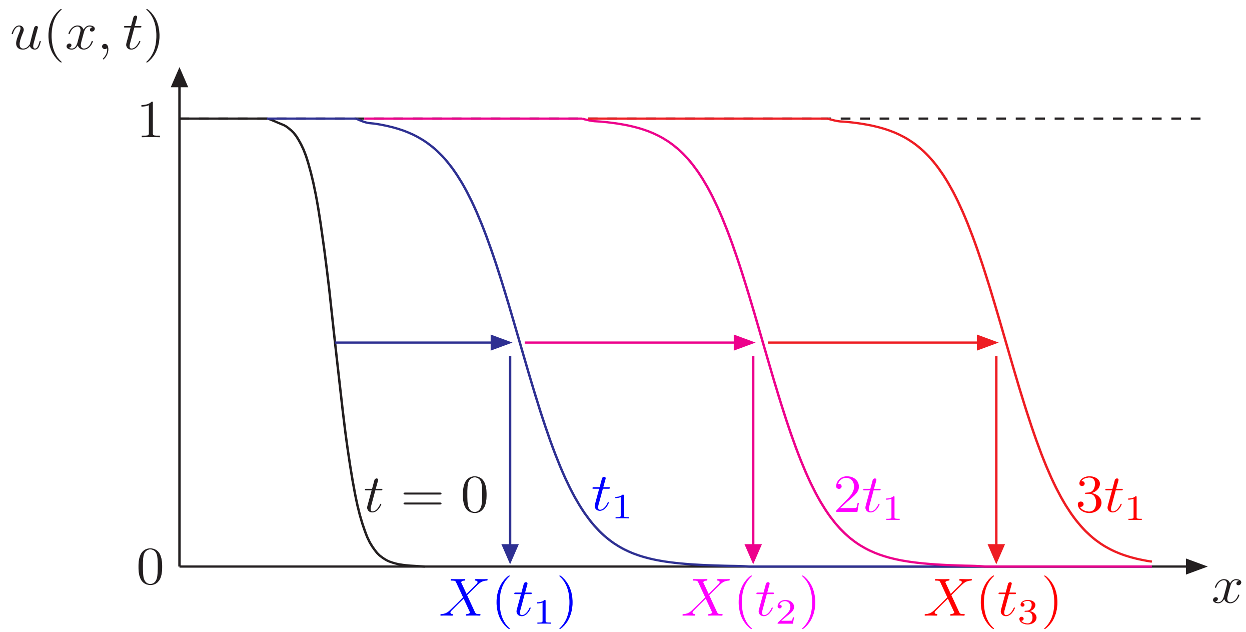
- The FKPP equation is a known equation in non-equilibrium statistical physics, whose dynamics is called reaction-diffusion dynamics,

$$\partial_t u(x,t) = \partial_x^2 u(x,t) + u - u^2, \tag{13}$$

where t is time and x is the coordinate.

Traveling wave solutions

- The FKPP evolution equation admits the so-called traveling wave solutions
  - For a traveling wave solution one can define the position of a wave front $x(t) = v_c(t)t$, irrespective of the details of the nonlinear effects
  - At large times, the shape of a traveling wave is preserved during its propagation, and the solution becomes only a function of the scaling variable $x - v_c t$



Traveling waves and saturation

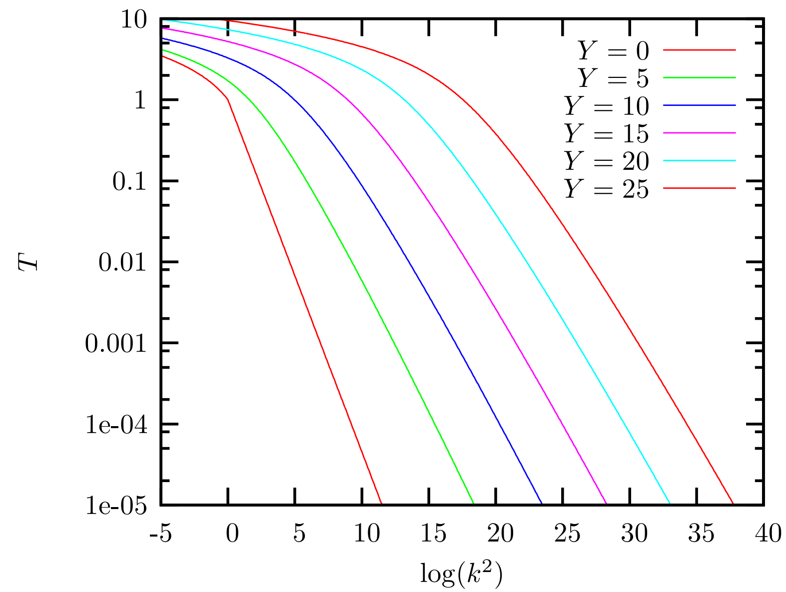
- In the language of saturation physics the position of the wave front is nothing but the saturation scale

$$x(t) \sim \ln Q_s^2(Y)$$

and the scaling corresponds to the geometric scaling

$$x - x(t) \sim \ln k^2/Q_s^2(Y)$$

Numerical solution to BK equation



- Summarizing:

$$\begin{aligned} \text{Time } t &\longrightarrow Y \\ \text{Space } x &\longrightarrow L \\ \text{Wave front } u(x - vt) &\longrightarrow \tilde{T}(L - v_c Y) \\ \text{Traveling Waves} &\longrightarrow \text{Geometric Scaling} \end{aligned} \quad (14)$$

Behaviour at $k \gg Q_s$

- This property of the FKPP equation is actually true if one considers the BK equation with the full BFKL kernel
- At asymptotic rapidities, the amplitude $\tilde{T}(k,Y)$, instead of depending separately on k and Y, depends only on the scaling variable $k^2/Q_s^2(Y)$, where we have introduced the *saturation scale* $Q_s^2(Y) = k_0^2 \exp(v_c Y)$, measuring the position of the wavefront
- A more detailed calculation allows also for the extraction of two additional subleading corrections, resulting into the following expression for the tail of the scattering amplitude

$$T(k,Y) \overset{k \gg Q_s}{\approx} \left(\frac{k^2}{Q_s^2(Y)}\right)^{-\gamma_c} \log\left(\frac{k^2}{Q_s^2(Y)}\right) \exp\left[-\frac{\log^2\left(k^2/Q_s^2(Y)\right)}{2\bar{\alpha}\chi''(\gamma_c)Y}\right] \tag{15}$$

where the saturation scale

$$Q_s^2(Y) = k_0^2 \exp\left(\bar{\alpha} v_c Y - \frac{3}{2\gamma_c}\log(Y) - \frac{3}{\gamma_c^2}\sqrt{\frac{2\pi}{\bar{\alpha}\chi''(\gamma_c)}}\frac{1}{\sqrt{Y}}\right) \tag{16}$$

Critical parameters

- The critical parameters γ_c and v_c are obtained from the knowledge of the BFKL kernel alone and correspond to the selection of the slowest possible wave:

$$v_c = \min_\gamma \bar{\alpha} \frac{\chi(\gamma)}{\gamma} = \bar{\alpha} \frac{\chi(\gamma_c)}{\gamma_c} = \bar{\alpha}\chi'(\gamma_c)$$

- For the leading-order BFKL kernel, one finds $\gamma_c = 0.6275...,$ and $v_c = 4.88\bar{\alpha}$
- The geometric scaling expresses the fact that when one moves along the saturation line, the behaviour of the scattering amplitudes remains unchanged
- The last term in the expression for the tail introduces an explicit dependence on the rapidity Y and hence violates geometric scaling. However, this term can be neglected when

$$\frac{\log^2\left(k^2/Q_s^2(Y)\right)}{2\bar{\alpha}\chi''(\gamma_c)Y} < 1$$

  This means that geometric scaling is obtained for

$$\log\left(k^2/Q_s^2(Y)\right) \lesssim \sqrt{2\chi''(\gamma_c)\bar{\alpha}Y}$$

Dipole scattering amplitude

- In order to complete the description, we also need expressions for $\tilde{T}$ around the saturation scale and at saturation
- In the infrared domain , one can show that the amplitude behaves like

$$\tilde{T}\left(\frac{k}{Q_s(Y)}, Y\right) \stackrel{k \ll Q_s}{=} c - \log\left(\frac{k}{Q_s(Y)}\right) \tag{17}$$

  where c is an unfixed constant

- We are now left with the matching around the saturation scale
  - The easiest way is to use the expression for the tail given previously for $k > Q_s$ and the above expression for $k < Q_s$ and match the constant c to obtain a continuous distribution
- The problem with this definition by parts is that it may introduce oscillations in the coordinate space amplitude $T(r, Y)$ which may even lead to negative amplitudes
- Then the best way to obtain the description of the transition to the saturation region is to perform an analytic interpolation between both asymptotic behaviours.

The model

- Our starting point to describe the transition to saturation is an expression which is monotonically decreasing with L and which reproduces (up to the logarithmic factor), the amplitude for geometric scaling (tail)

$$T_{\text{dil}} = \exp\left[-\gamma_c \log\left(\frac{k^2}{Q_s^2(Y)}\right) - \frac{L_{\text{red}}^2 - \log^2(2)}{2\bar{\alpha}\chi''(\gamma_c)Y}\right] \tag{18}$$

with

$$L_{\text{red}} = \log\left[1 + \frac{k^2}{Q_s^2(Y)}\right] \quad \text{and} \quad Q_s^2(Y) = k_0^2\, e^{\bar{\alpha} v_c Y} \tag{19}$$

- This result is unitarised *à la* Glauber-Mueller *i.e.* $T_{\text{unit}} = 1 - \exp(-T_{\text{dil}})$ and we reinsert both logarithmic behaviours in the infrared and in the ultraviolet
- We obtain that the following choice gives good results:

$$T(k,Y) = \left[\log\left(\frac{k}{Q_s} + \frac{Q_s}{k}\right) + 1\right]\left(1 - e^{-T_{\text{dil}}}\right) \tag{20}$$

- The equations above determine our model for the scattering amplitude, to be inserted into the expression for the F_2 structure function

Dataset

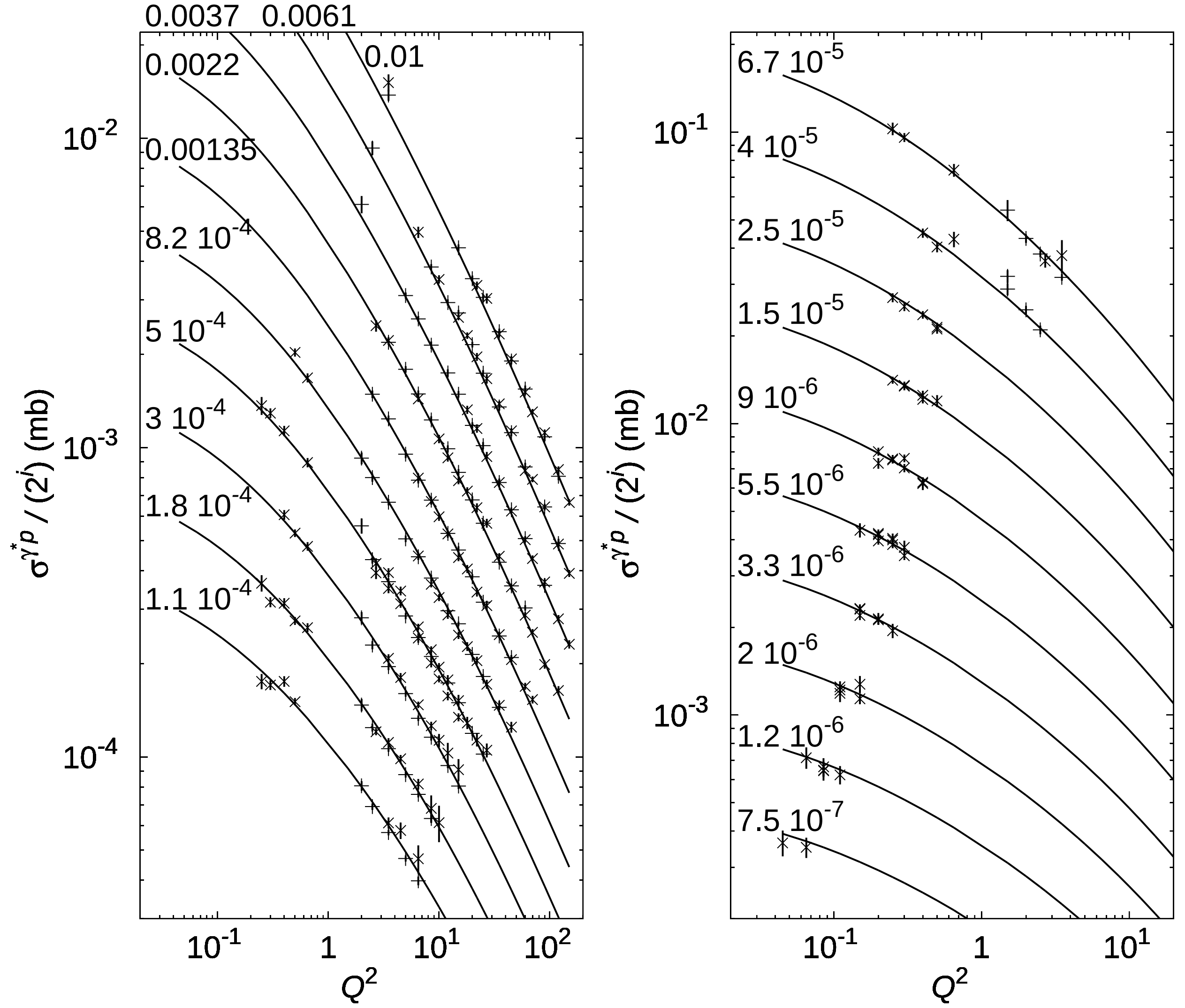
- We fit all the last HERA measurements of the proton structure function from H1 and ZEUS
- Our analysis is restricted to the following kinematic range:

$$\begin{cases} x \leq 0.01, \\ 0.045 \leq Q^2 \leq 150 \text{ GeV}^2 \end{cases}$$

  - The first limit comes from the fact that our approach is meant to describe the high-energy amplitudes *i.e.* the small x behaviour
  - The second cut prevents to reach too high values of Q^2 for which DGLAP corrections need to be included properly
- Total amount of data points: 279; we have allowed for a 5% renormalisation uncertainty on the H1 data
- We have kept $\gamma_c = 0.6275$ and $\bar{\alpha} = 0.2$ fixed assumed different situations for the quarks masses: the light-quarks mass m_q has been set to 50 or 140 MeV while we have used $m_c = m_q$ or $m_c = 1.3$ GeV for the charm mass
- v_c, χ''_c , k_0^2 and R_p are free parameters

Results (I): $\sigma^{\gamma^* p}$ cross section

$m_q = 50$ MeV and $m_c = 1.3$ GeV



H1 [EPJC 21, 2002] and ZEUS [EPJC 12, 2000; EPJC 21, 2001]

Results (II): Parameters

- The parameters obtained from the fit to the experimental data for F_2^p:

Masses	k_0^2 (10^{-3} GeV^2)	v_c	χ_c''	R_p (GeV^{-1})
$m_q = 50$ MeV, $m_c = 50$ MeV	3.782 ± 0.293	1.065 ± 0.018	4.691 ± 0.221	2.770 ± 0.045
$m_q = 50$ MeV, $m_c = 1.3$ GeV	7.155 ± 0.624	0.965 ± 0.017	2.196 ± 0.161	3.215 ± 0.065
$m_q = 140$ MeV, $m_c = 1.3$ GeV	3.917 ± 0.577	0.807 ± 0.025	2.960 ± 0.279	4.142 ± 0.167

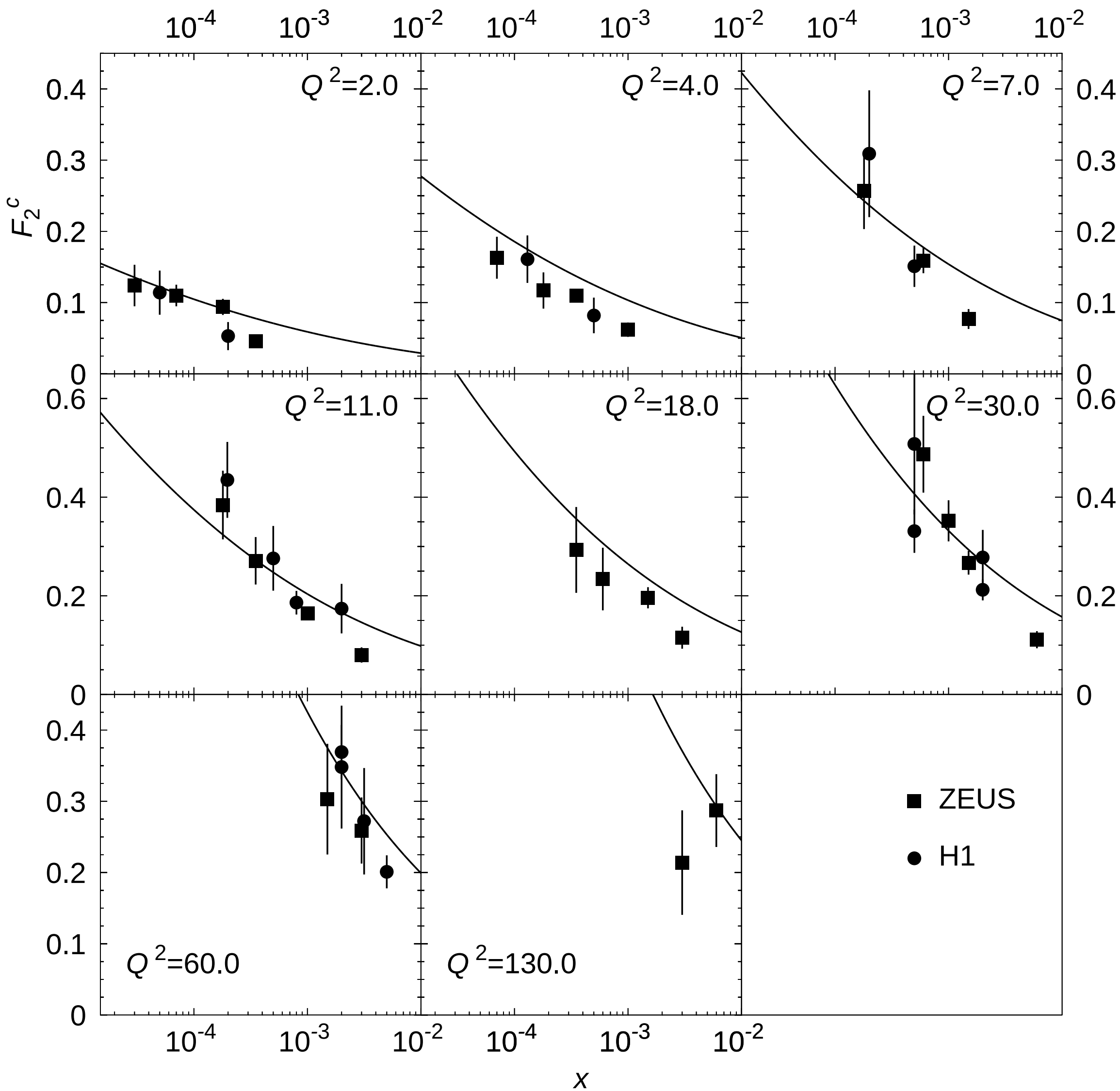
- The χ^2 per point

Masses	χ^2/nop
$m_q = 50$ MeV, $m_c = 50$ MeV	0.960
$m_q = 50$ MeV, $m_c = 1.3$ GeV	0.988
$m_q = 140$ MeV, $m_c = 1.3$ GeV	1.071

- Good agreement with the measurements of F_2^p due to the small χ^2

Results (III): prediction for F_2^c

$m_q = 50$ MeV and $m_c = 1.3$ GeV



H1 [PLB 528, 2002; EPJC 45, 2006] and ZEUS [PRD 69, 2004]

The saturation scale

- Within our parametrization, the saturation scale Q_s corresponds to the energy-dependent scale at which the dipole scattering amplitude is $T(k = Q_s(Y), Y) = [1 + \log(2)](1 - 1/e) \approx 1.07$

- For the fit corresponding to $m_q = 50$ MeV and $m_c = 1.3$ GeV, we obtain a saturation scale $Q_s = 0.206$ GeV for $x = 10^{-4}$ and $Q_s = 0.257$ GeV for $x = 10^{-5}$

- These values, although they seem rather small, correspond to large values for T

- If, instead, we extract the saturation scale by requiring that $T = 1/2$ when $k = Q_s$, we get $Q_s = 0.296$ (resp. $Q_s = 0.375$) GeV for $x = 10^{-4}$ (resp. $x = 10^{-5}$)

- These last values are still a bit smaller than the saturation scales observed in previous studies ($Q_s \approx 1$ GeV for $x \approx 10^{-5}$) and, hence, tend to confirm the tendency for the saturation scale to decrease when heavy-quark effects are taken into account

Summary

- We have investigated the traveling-wave solutions of the BK equation which describe the forward scattering amplitude at high energies and tested their phenomenological implications for the virtual photon-proton scattering
- We have proposed an expression for the amplitude in momentum space which interpolates between the behaviour of the dipole-proton amplitude at saturation and the traveling-wave, ultraviolet, amplitudes predicted by perturbative QCD from the BK equation
- This expression was used to compute the proton structure function F_2 (in the framework of the dipole model) and tested against the HERA data
- We verify a good agreement with the measurements of F_2^p due to the small χ^2 provided by the fit
- Moreover, the F_2^c predicted by the parametrization is in reasonable agreement with the experimental results, which shows the robustness of the model proposed

Analysis: previous models

- It is interesting to compare our results with those from previous approaches, namely
  - GBW model [Golec-Biernat and Wüsthoff, 1999]
  - IIM model or CGC fit [Iancu, Itakura and Munier, 2004]
  - recent developments concerning the Bartels-Golec-Biernat-Kowalski (BGK) model [Golec-Biernat and Sapeta, 2006]

Parametrization	Quark masses	nop	χ^2/nop
GBW	$m_q = 140$ MeV, $m_c = 1.5$ GeV	372	1.5
IIM	$m_q = 140$ MeV, no charm	156	0.81
BGK	$m_q = 0$ MeV, $m_c = 1.3$ GeV	288	1.06
This work	$m_q = 50$ MeV, $m_c = 1.3$ GeV	279	0.988

- These three models were developed in coordinate space and not in momentum space
- Our choice is directly motivated by the analysis of the BK equation in momentum space leading to universal asymptotic results on which we heavily rely

Analysis: conclusions

- Our model can be differentiated from the previous ones at two levels:
  - The analysis is based on the BK equation to account for unitarity effects. Thus, we expect it to be more precise, especially in the small-x and low Q^2 domain under study
  - We improved the IIM model by including massive charm
- Moreover, in the case of GBW model, one obtains a Fourier transform of the dipole cross section which presents an unrealistic perturbative behaviour, in the case of IIM it presents non-positivity values [Betemps and Gay Ducati, 2004]
- These problems are tamed in our model, where the inverse Fourier transform (scattering amplitude in coordinate space) remains between 0 and 1
- We then conclude that the dipole scattering amplitude proposed in this work should be a good parametrization to investigate the properties of the observables at RHIC and LHC energies, considering the dipole approach