BK equation and traveling wave solutions

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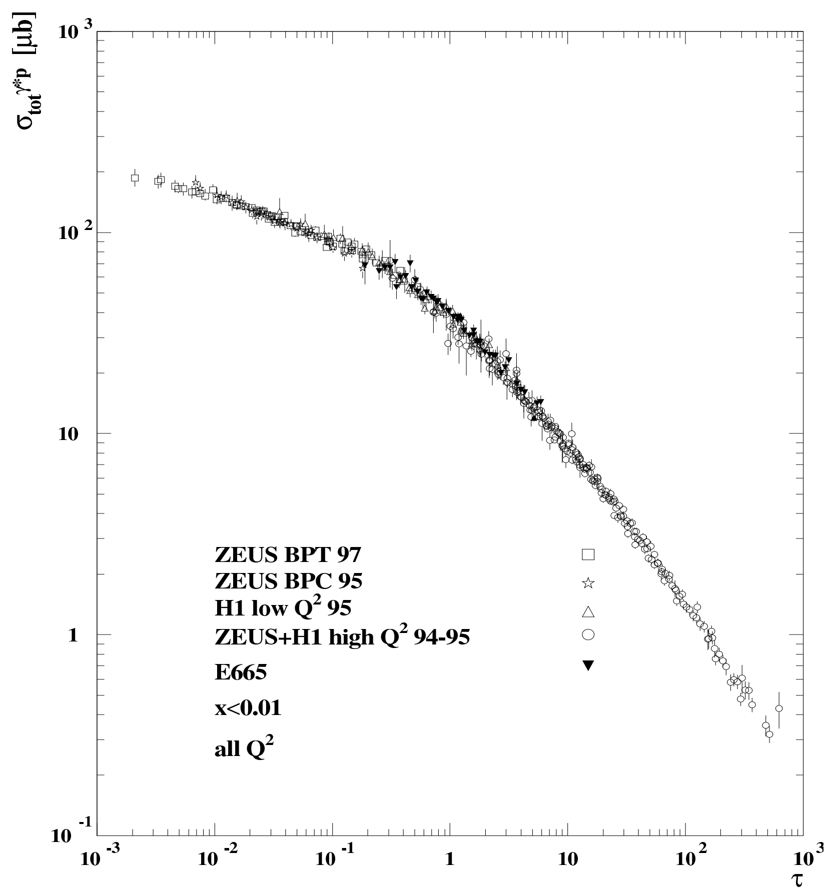
Introduction

- There has been a large amount of work devoted to the description and understanding of **quantum chromodynamics** (QCD) in the high energy limit corresponding to **saturation**
  - Theory: non-linear QCD equations describing the evolution of scattering amplitudes towards saturation
  - Phenomenology: discovery of **geometric scaling** in DIS at HERA
- The Balitsky-Kovchegov (BK) nonlinear equation describes the evolution in rapidity of the scattering amplitude of a dipole off a given target; assuming an independence on the impact parameter, the BK equation has been shown to lie in the same universality class as the **Fisher-Kolmogorov-Petrovsky-Piscounov** (FKPP) equation
- Geometric scaling has a natural explanation in terms of the so-called **traveling wave solutions** of BK equation
- The evolution at low energy is well understood and is described by a linear equation; the deep saturation regime can also be evaluated in some models
- However, the **transition** between these two regimes is still a challenge

Geometric Scaling and BK equation

- **Geometric scaling** is a phenomenological feature of high energy deep inelastic scattering (DIS) which has been observed in the HERA data on inclusive $\gamma^* - p$ scattering, which is expressed as a scaling property of the virtual photon-proton cross section

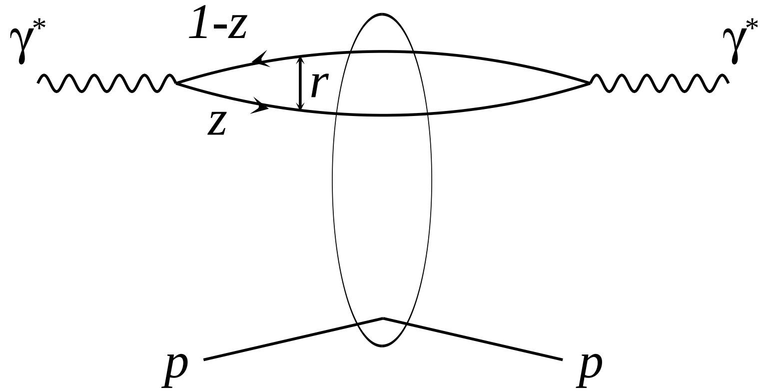


$$\sigma^{\gamma^* p}(Y, Q) = \sigma^{\gamma^* p}(\tau), \quad \tau = \frac{Q^2}{Q_s^2(Y)}$$

where Q is the virtuality of the photon, $Y = \log 1/x$ is the total rapidity and $Q_s(Y)$ is an increasing function of Y called **saturation scale**

Dipole frame

- It is convenient to work within the QCD dipole frame of DIS



- In the LLA of perturbative QCD (pQCD), the cross section factorizes as

$$\sigma^{\gamma^* p}(Y, Q) = \int_0^\infty r\,dr \int_0^1 dz\ |\psi(z,\, r,\, Q)|^2\, \sigma_{dip}^{\gamma^* p}(Y,\, r) \tag{1}$$

$\sigma_{dip}^{\gamma^* p}(Y, r)$ is the dipole-proton cross section, taken to be proportional to the forward scattering amplitude $N(Y, r)$ through the relation

$$\sigma_{dip}^{\gamma^* p}(Y, r) = 2\pi R_p^2 N(r, Y) \tag{2}$$

BK equation in momentum space

- We define the forward scattering amplitude in momentum space $\mathcal{N}(Y,\, k)$

$$\mathcal{N}(Y,\, k) = \int_0^\infty \frac{dr}{r} J_0(kr) N(Y,\, r) \tag{3}$$

and in this picture geometric scaling property reads

$$\mathcal{N}(Y,\, k) = \mathcal{N}\left(\frac{k}{Q_s(Y)}\right) \tag{4}$$

- $\mathcal{N}$ obeys the Balitsky-Kovchegov (BK) equation

$$\partial_Y \mathcal{N} = \bar{\alpha}\chi(-\partial_L)\mathcal{N} - \bar{\alpha}\mathcal{N}^2, \qquad \bar{\alpha} = \frac{\alpha_s N_c}{\pi} \tag{5}$$

BK equation in momentum space

- In this equation

$$\chi(\gamma) = 2\psi(1) - \psi(\gamma) - \psi(1-\gamma) \tag{6}$$

is the characteristic function of the well known Balitsky-Fadin-Kuraev-Lipatov (BFKL) kernel, and $L = \log\left(k^2/k_0^2\right)$, where k_0 is some fixed low momentum scale

- The kernel χ is an integro-differential operator which may be defined with the help of the formal series expansion

$$\begin{aligned}\chi(-\partial_L) \quad &= \quad \chi(\gamma_0)\mathbf{1} + \chi'(\gamma_0)(-\partial_L - \gamma_0\mathbf{1}) + \frac{1}{2}\chi''(\gamma_0)(-\partial_L - \gamma_0\mathbf{1})^2 \\ &\quad + \frac{1}{6}\chi^{(3)}(\gamma_0)(-\partial_L - \gamma_0\mathbf{1})^3 + \ldots \end{aligned} \tag{7}$$

for some γ_0 between 0 and 1, *i.e.* for the principal branch of the function χ

BK and FKPP equations

- The **Fisher and Kolmogorov-Petrovsky-Piscounov** (FKPP) equation is a famous equation in non-equilibrium statistical physics, whose dynamics is called **reaction-diffusion dynamics**,

$$\partial_t u(x,t) = \partial_x^2 u(x,t) + u - u^2, \tag{8}$$

  where t is time and x is the coordinate.

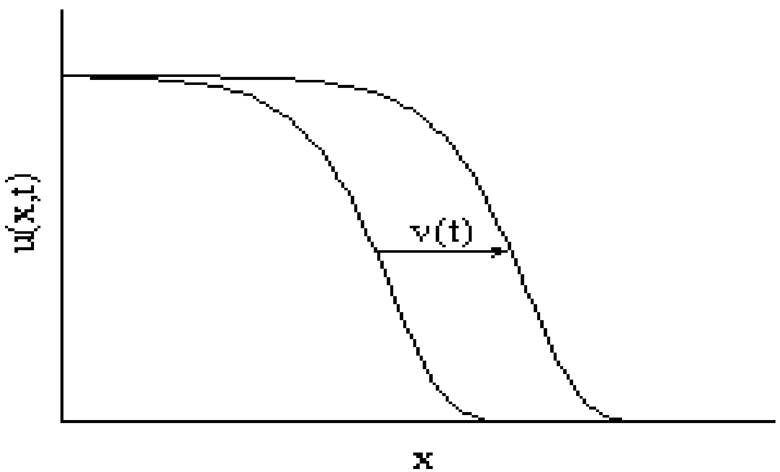
- It has been shown that, after the change of variables

$$t \sim \bar{\alpha} Y, \quad x \sim \log(k^2/k_0^2), \quad u \sim \mathcal{N} \tag{9}$$

  BK equation reduces to FKPP equation, when its kernel is approximated by the first three terms of the expansion, the so-called **diffusive approximation** or **saddle point approximation**

Traveling wave solutions

- The FKPP evolution equation admits the so-called **traveling wave solutions**
  - For a traveling wave solution one can define the position of a wave front $x(t) = v(t)$, irrespective of the details of the nonlinear effects
  - At larges times, the shape of a traveling wave is preserved during its propagation, and the solution becomes only a function of the scaling variable $x - vt$



- In the language of saturation physics the position of the wave front is nothing but the saturation scale

$$x(t) \sim \ln Q_s^2(Y)$$

and the scaling corresponds to the **geometric scaling**

$$x - vt \sim \ln k^2/Q_s^2(Y)$$

Traveling wave solutions

- The linear part of the BK equation is solved by

$$\mathcal{N}(k,Y) = \int \frac{d\gamma}{2\pi i} \mathcal{N}_0(\gamma)\exp(-\gamma L + \bar{\alpha}\chi(\gamma)Y) \tag{10}$$

- The velocity of the front is given by

$$v = v_g = \min_{\gamma} \bar{\alpha}\frac{\chi(\gamma)}{\gamma} = \bar{\alpha}\frac{\chi(\gamma_c)}{\gamma_c} = \bar{\alpha}\chi'(\gamma_c) \tag{11}$$

where γ_c is the saddle point of the exponential phase factor

- This fixes, for the BFKL kernel,

$$\gamma_c = 0.6275..., \quad v = 4.88\bar{\alpha} \tag{12}$$

Scattering amplitude

- In terms of QCD variables, the dipole forward scattering amplitude in momentum space near the wave front reads

$$\mathcal{N}(\tau, Y) \propto \sqrt{\frac{2}{\bar{\alpha}\chi''(\gamma_c)}} \log\left(\frac{k^2}{Q_s^2(Y)}\right)\left(\frac{k^2}{Q_s^2(Y)}\right)^{-\gamma_c} \exp\left(-\frac{\log^2\left(\frac{k^2}{Q_s^2(Y)}\right)}{2\bar{\alpha}\chi''(\gamma_c)Y}\right) \quad (13)$$

- The saturation scale is defined as

$$Q_s^2(Y) = Q_0^2 \exp\left(\bar{\alpha}\frac{\chi(\gamma_c)}{\gamma_c}Y - \frac{3}{2\gamma_c}\log Y\right). \quad (14)$$

Q_0 absorbs undetermined constants but remains of order k_0.

Connecting to Saturation

- We are studying the connection between of the traveling wave solution with the saturation region

- These different domains can be parametrized as

$$\mathcal{N}(\tau, Y) = c - \log\left(\frac{k}{Q_s(Y)}\right)$$

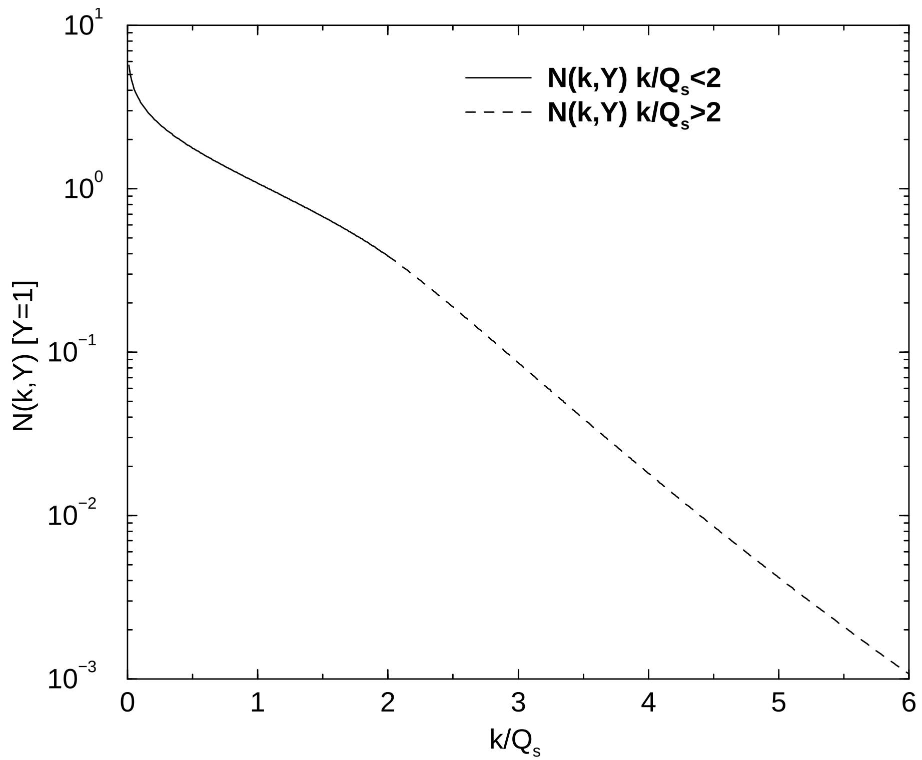
when, $k < 2Q_s(Y)$, and

$$\mathcal{N}(k, Y) \propto \sqrt{\frac{2}{\bar{\alpha}\chi''(\gamma_c)}}\left(\frac{k^2}{Q_s^2(Y)}\right)^{-\gamma_c} \log\left(\frac{k^2}{Q_s^2(Y)}\right) \exp\left\{-\frac{\log^2\left(\frac{k^2}{Q_s^2(Y)}\right)}{2\bar{\alpha}\chi''(\gamma_c)Y}\right\}$$

when, $k > 2Q_s(Y)$

Connecting to Saturation

- The first attempt was to perform a **matching** between the two regions



- However, a matching procedure does not necessarily imply a positive Fourier transform of the scattering amplitude. Then, a better way to obtain the connection between the two regions which satisfies this condition would be an **interpolation**

Next

- In order to obtain an interpolation model to connect the regions of interest we intend to build the saturation domain from the dilute one in the following way:

$$\mathcal{N} = \frac{1}{1 + \frac{1}{\mathcal{N}_{dil}}} \tag{15}$$

- This expression guarantees the correct asymptotic behaviors of the forward scattering amplitude $\mathcal{N}$. Indeed,
  - when $\mathcal{N}_{dil} \ll 1$

$$\mathcal{N} \approx \mathcal{N}_{dil} \tag{16}$$

  - when $\mathcal{N}_{dil} \gg 1$

$$\mathcal{N} \approx 1. \tag{17}$$

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