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Simulation of a multineuron interaction model (*)

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Abstract. — We simulate the dynamics of a multineuron model (RS model) with an energy function given by the product between the squared distances in phase space between the state of the net and the stored patterns. We obtain the relative frequency $f(m_0)$ that an arbitrary pattern is retrieved from an initial overlap m_0 and estimate the size of the basins of attraction for different activities a. Two limit cases are taken into account: when patterns and antipatterns are stored (p.a.s.) and when only the patterns are stored (o.p.s.). For the $a = 0.5$, p.a.s. nets a limit for the load parameter was not found, but for the other cases ($a \neq 0.5$ or o.p.s. configuration) the relative size of the basins of attraction may become too small.

Most common models for neural networks consist of systems of a great number of interacting spins (neurons) [1, 2] where each configuration is characterized by an N-dimensional vector **S**, given by

$$\mathbf{S} = (S_1, . \; , S_N)\,; \quad S_i = \pm 1\,. \tag{1}$$

The Hamiltonian function, and hence the configurations that minimize the energy of the system, is determined by the spin interactions considered by a given model. For the attractor neural network, when any set of P chosen states of the net (patterns), say $\boldsymbol{\xi}^\mu$, $\mu = 1, ..., P$, are implemented as minima of the Hamiltonian through convenient, *a priori* prescribed tuning of the interactions strengths, the net is said to have learnt these patterns. In this case, a given pattern $\boldsymbol{\xi}^\mu$ is retrieved when, starting from an initial, unstable configuration that is sufficiently similar to the pattern, the net relaxes towards and stabilizes at the (minimum energy) state $\boldsymbol{\xi}^\mu$.

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The most extensively studied example of such idealized systems is the Hopfield model, which in its original version considers the following Hamiltonian [2] :

$$E = - \sum_{i,j=1} J_{ij} S_i S_j \tag{2}$$

and the synaptic matrix J_{ij}, associated to the strengths of spin interactions, is given by Hebb learning rule [2, 3] :

$$J_{ij} = \frac{1}{N} \sum_{\mu=1}^{P} \xi_i^{\mu} \xi_j^{\mu} \tag{3}$$

with $\boldsymbol{\xi}^{\mu}$, $\mu = 1, . . , P$, being P vectors in the phase space representing the stored patterns. This synaptic matrix J_{ij} guarantees that these patterns are strongly correlated to minima of the Hamiltonian equation (2), if they are uncorrelated among themselves and the load parameter α, defined as

$$\alpha = \frac{P}{N} \tag{4}$$

is less than a critical value $\alpha_c \approx 0.14$ [4]. In this limit case the Hopfield model describes a content-addressable memory. The uncorrelated patterns are chosen by assigning equal probability $a = 0.5$ of each spin ξ_i^{μ} to be $+1$ or -1 at random. The value a defines the activity of the net and lower values of a (favoring spin $+1$, for instance) introduce correlation between the stored memories. Several attempts have been made to model a neural network capable to handle satisfactorily with correlated patterns and with a higher critical load capacity, here measured by α_c, both by introducing modifications in the Hopfield Hamiltonian or by proposing new models (see, for example, Refs. [4-11]).

Recently, de Almeida and Iglesias [12] proposed a new Hamiltonian function that is related to the distances in phase space between the state of the net $\mathbf{S}$ and the P patterns $\boldsymbol{\xi}^{\mu}$. This model, named RS after the brazilian state where the model was created, includes multispin interactions and can be written as

$$E = N \prod_{\mu=1}^{P} \left[\frac{1}{2N} \sum_{i=1}^{N} (S_i - \xi_i^{\mu})^2 \right] . \tag{5}$$

The expression in the product is the squared Euclidean distance between $\mathbf{S}$ and $\boldsymbol{\xi}^{\mu}$, i.e., E is a non-negative function and $E(\mathbf{S}) = 0$ if $\mathbf{S} = \boldsymbol{\xi}^{\mu}$, for any $\mu = 1, \ldots, P$. Consequently the patterns are always minima of the energy function. The capacity of the network, as well as its ability to handle with correlated patterns are greatly and qualitatively enhanced in comparison with previous models. Also, if $\boldsymbol{\xi}^{\mu}$ is a minimum of equation (5), it is not a direct consequence that its antipode $-\boldsymbol{\xi}^{\mu}$ (antipattern) should also minimize the Hamiltonian, as it happens in the Hopfield Model. Nevertheless, one can always explicitly store the antipatterns, if one wishes so. This feature of the model yields to two different limiting cases for uncorrelated patterns : when only the patterns are stored (o.p.s.) or when both patterns and antipatterns are stored (p.a.s.).

The above considerations were directly inferred from equation (5) and further conclusions require deeper analysis of the dynamics implied by the model. As the system under consideration takes into account a great number of interacting neurons, there are two different approaches to the problem : mean field calculations through statistical mechanics techniques, which preliminary results have been presented in reference [12], and numerical simulations. In this work we present some computer simulation results for the two limiting cases (o.p.s. and p.a.s. configurations) where we consider different load parameters and activities for the random patterns. In what follows we briefly explain the procedures to obtain the relative fraction $f(m_0)$ of times that an arbitrary pattern is retrieved when starting from an initial state with an overlap m_0, following the prescription by Forrest [13]. We analyze the results, obtain an estimate of the size of the basins of attraction and show that the load capacity of the net has been greatly improved.

Consider a network of N neurons and a zero-temperature dynamics such that a spin is flipped whenever the energy of the net is lowered. Equation (5) gives a lot of information about the energy landscape in phase space, but it is not in an adequate form for numerical purposes. It can be rewritten as

$$E = N \prod_{\mu=1}^{P} (1 - m_\mu) \tag{6}$$

in the o.p.s. case, or

$$E = N \prod_{\mu=1}^{P} (1 - m_\mu^2) \tag{7}$$

in the p.a.s. configuration [12]. In equations (6) and (7) m_μ stands for the overlap of the state of the net $\mathbf{S}$ with the μ-th pattern :

$$m_\mu = \frac{1}{N} \sum_{i=1}^{N} \xi_i^\mu S_i . \tag{8}$$

The numerical simulation is performed as follows :

i) initialization : construct P random patterns with activity a, and an initial state that has a given overlap m_0 with one of the stored patterns, chosen at random. Calculate the overlaps m_μ with the other ones through equation (8) and then the initial energy through (6) or (7) ;

ii) consider a virtual flip in the i-th spin and calculate virtual overlaps m_μ^* by

$$m_\mu^* = m_\mu - \frac{2}{N} S_i \xi_i^\mu ; \tag{9}$$

iii) obtain the virtual energy E^* through equations (6) or (7) using the overlaps m_μ^* ;

iv) whenever $E^* \leq E$ flip the spin and update the energy and overlaps ;

v) the updating is performed sequentially until the system reaches a stable state ;

The steps (i) to (v) are repeated several times and the averaged frequency $f(m_0)$ that the randomly chosen pattern is retrieved from an initial overlap m_0 is obtained. We used the multispin coding approach, as prescribed by Penna and Oliveira [14, 15]. Although the number of arithmetic operations per updating are of the same order as in the Hopfield model, more computing time is required because here the energy function is calculated through the product of the overlaps instead of the sum as in the Hopfield-Hebb algorithm. For more computing details see reference [17].

Figure 1 shows the relative frequency $f(m_0)$ for the p.a.s. configuration and 0.5 activity, uncorrelated sets of memories. The nets considered have $N = 128$, 256 and 512 and simulations were performed for $\alpha = 0.1$, 0.5 and 1. The average values $f(m_0)$ were obtained considering 200 relaxations (50 for the $N = 512$ net) for each of five different sets of memories for every point of the figure and the estimated errors are less than 10 %. As the patterns are always minima of the energy function, when a pattern is retrieved the final

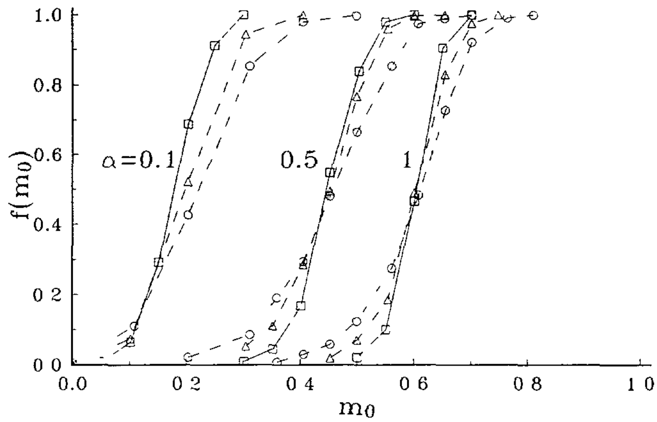


Fig. 1. — Fraction $f(m_0)$ of recalled patterns with an initial overlap m_0 for $N = 128$ (○), 256 (△) and 512 (□) and $\alpha = 0.1$, 0.5 and 1 in the p.a.s. case, and uncorrelated patterns.

overlap is always 1, i.e., there is no errors in the retrieved states. Considering the p.a.s. case, similarly to the Hopfield model, curves corresponding to nets with the same value of the load parameter α but different number of neurons N (and hence different number $P = N\alpha$ of stored patterns) superpose [13], and the rate at which $f(m_0)$ increases form 0 to 1 becomes more pronounced as the size of the system N increases for all values of α in the figure. As suggested by Forrest [13] this behavior approaches a discontinuity at $m_c(\alpha)$ as $N \to \infty$ implying that even for $\alpha = 1$ the net keeps its retrieving abilities. The value $\lim_{N \to \infty} m_c(\alpha)$ gives a measure of the size of the basins of attraction of the memories. Rough estimates, taken directly from figure 1, are $m_c(\alpha = 0.1) \approx 0.12$, $m_c(\alpha = 0.5) \approx 0.43$ and $m_c(\alpha = 1) \approx 0.61$. Comparisons with the Hopfield model value $m_c^H(\alpha = 0.1) \approx 0.37$ shows that, even at very low α the size of the basins of attraction are greatly enhanced in this model. Also, the dependence of m_c with the load parameter α is much smoother than in Hopfield Model (the basins of attraction are not drastically reduced), because here there is no critical value for the load parameter, at least in the region $\alpha \leq 1$. This effect could originate from the elimination of spurious states by the higher order synaptic connections considered by the Hamiltonian equation (2). (At $N < \infty$ a few very shallow spurious states with $E \gg 0$ may appear in numerical simulations. At $N \to \infty$ these states merge in an absolute (unstable) maximum at a point in phase space that is equidistant from every (uncorrelated) stored pattern, in agreement with mean field ($N \to \infty$) calculations [12, 16]). Higher values of α do not present any qualitative difference for the analyzed case but for a expected decrease in the size of the basins of attraction.

In figure 2, we present $f(m_0)$ *versus* m_0 for 0.5 activity, uncorrelated patterns in the o.p.s. configuration for nets with $N = 128$, 256 and 512 neurons again, the average is taken over 200 relaxations for each of five different sets of memories (50 for the $N = 512$). Here the different N curves with same number P of stored patterns (and not α) superpose. This can be explained as follows. The antipatterns are located in a region in phase space that is the specular reflexion of the corresponding region where the patterns are. When antimemories are also stored (p.a.s. case) these two regions have similar energy landscapes. But when antimemories are not considered (o.p.s. case), the antipatterns region comprises very large energy states, while its specular image contains the lower energy ones, forming an analogue of a multi-hole bowl in the closed space defined by the (hyper) surface of the (hyper) cube in

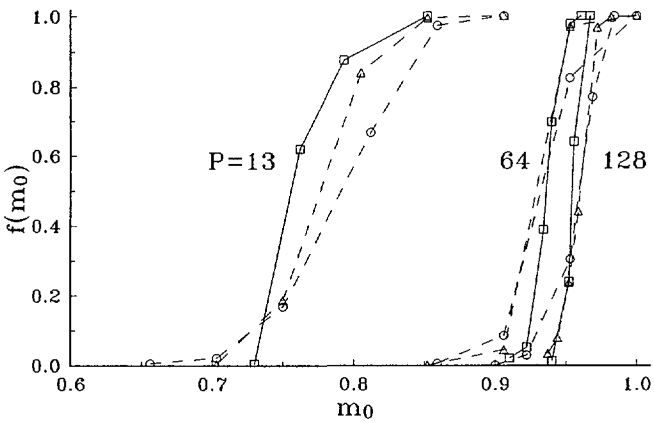


Fig. 2. — $f(m_0)$ *versus* m_0 for $N = 128$ (○), 256 (△) and 512 (□) and $P = 13$, 64 and 128 in the o.p.s. case, and uncorrelated patterns. Note the different scale on the horizontal axis.

phase space. The greater the number of patterns P is, the bigger the difference between these two regions and the deeper is the bowl. The patterns are stored inside this big basin whose center lays in the direction **C** determined by the sum of all stored uncorrelated patterns. As P increases, the sizes of the basins of attraction of each memory decrease, i.e. m_c decreases, enlarging the probability that the net stabilizes in a state which is strongly correlated to the center of the big bowl. Figure 3 presents a pictorial sketch of the energy landscape in the phase space for this case. In the limit $P \to \infty$, the energy of the state in the center of the bowl goes to zero, the basins of attraction of each stored pattern merge, and there is only one big basin with a flat, zero energy bottom in the region in phase space which is delimited by a ring obtained by linking the points associated to the patterns and contains the direction **C**. Outside this ring the energy increases monotonically from zero to infinity, reaching its maximum value at the point $-$ **C**. Hence it is not surprising that in the o.p.s. case the relative size of the basins of attraction, measured by m_c, is determined by the absolute number P while for the p.a.s. configuration the relative quantity $\alpha = P/N$, that gives a measure of the density of stored patterns, is the relevant number in defining the size of the basins of attraction. This effect is in agreement with mean field calculations [16]. Again, the rate at which $f(m_0)$ increases from 0 to 1 becomes more pronounced as the size of the system increases and we can see that the net still retrieves the stored patterns when P is of the order of N, at least for finite nets.

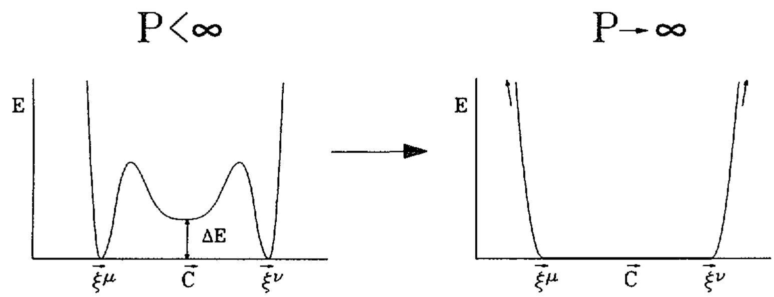
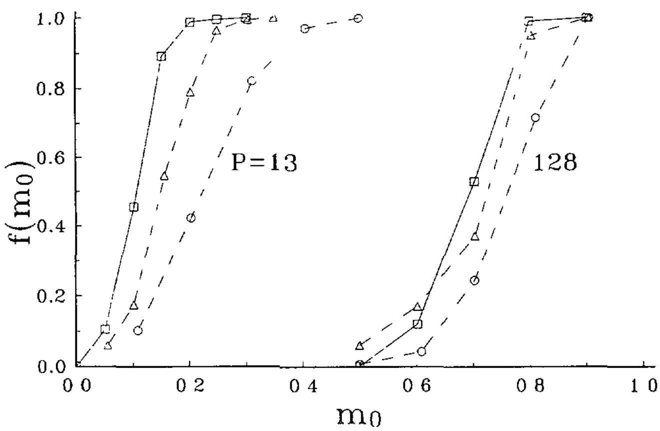
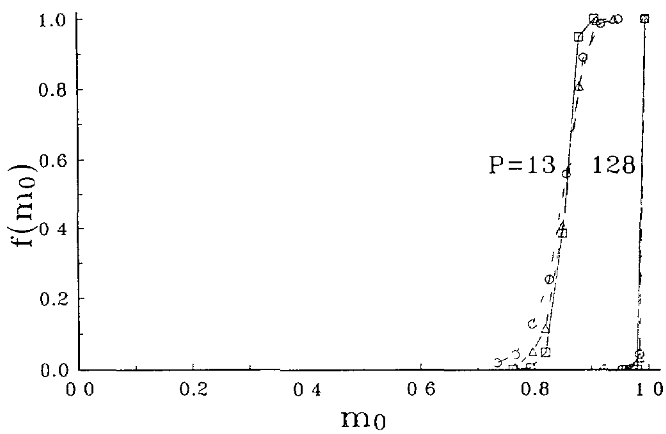


Fig. 3. — Pictorial representation of the energy landscape in the phase space for the o.p.s., $a = 0.5$ case. For $P < \infty$ the direction **C**, determined by the sum of all stored patterns, is a local minimum of the Hamiltonian, with $E > 0$. In the $P \to \infty$ the center of the bowl and all the stored patterns merge in a unique wide minimum.

Figure 4 shows the plot of $f(m_0)$ *versus* m_0 for different activities ($a = 0.2$ and 0.4) for the p.a.s. case. Here too, as in the o.p.s. configuration, the curves with same number P of stored patterns superpose and again the effect of increasing the size N of the net is to pronounce the rate at which $f(m_0)$ increases from 0 to 1. It also implies that the size of the basins of attraction decreases with α. The explanation is similar to the o.p.s., $a = 0.5$ configuration : there is a high energy region in the phase space, comprising now the $a \approx 0.5$ states, where there are not minima of energy. This region separates two multi-hole-bowl-like regions, each of them containing all the patterns or all the antipatterns and, as the activity a decreases, the bowls grow deeper, more energy-decreasing paths lead from an initial state to the center of the bowl and the initial configuration should be nearer to a given pattern in order to retrieve it : more information is required to distinguish between correlated patterns. Nevertheless it still represents an enhancement in comparison to the original or modified Hopfield model [4].



a)



b)

Fig. 4. — $f(m_0)$ *versus* m_0 for the p.a.s. case with activities : a) $a = 0.4$ and b) $a = 0.2$ for two different values of P and for $N = 128$ (○), 256 (△) and 512 (□).

In figure 5, we present the results for the o.p.s. configuration for low activity patterns and the only effect when comparing to the $a = 0.5$ case is the reduction of the size of the basins of attraction.

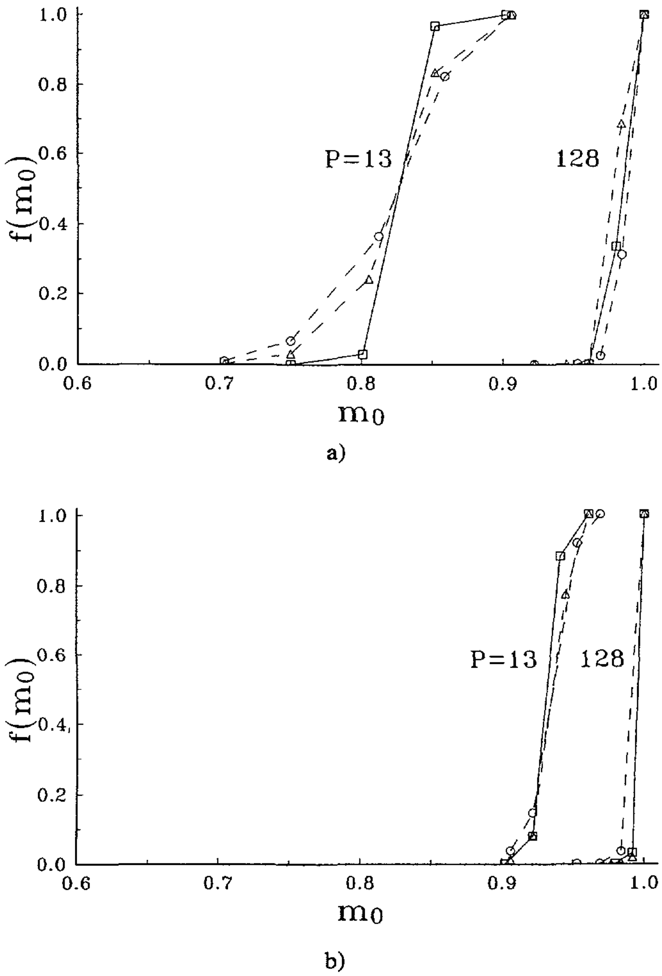


Fig. 5. — *$f(m_0)$ versus m_0* for the o.p.s. case with activities : a) $a = 0.4$ and b) $a = 0.2$ for two different values of P and for $N = 128$ (○), 256 (△) and 512 (□). Note the different scale on the horizontal axis.

In conclusion, we presented here the results of numerical simulations for the Hamiltonian equation (5), considering two limiting cases : when both patterns and antipatterns are stored (p.a.s.) and when only the patterns are considered (o.p.s.). In the case of $a = 0.5$, the capacity of the net is greatly improved, specially for the p.a.s. nets, compared to the Hopfield model. Also the retrieval of memories are always perfect, i.e., the final overlap of a retrieved state is always 1, reflecting the fact that the patterns are always minima of the energy function. The Hamiltonian function can be expanded in a sum of different orders of interactions and, in the p.a.s. case, it can be truncated at the second order term, recovering the Hopfield model. As this cut-off is reasonable only when the load parameter is low and the patterns are uncorrelated [12, 16], the critical value $\alpha_c = 0.14$ may be regarded as a limiting value for the validity of the approximation implied by the cut-off at the second order term.

A novelty is that, in the o.p.s. configuration, the size of the basins of attraction scales with the number P of stored patterns, because of the bowl-like shape of the energy in phase space. The p.a.s. configuration seems to occupy more efficiently the phase space. On the other hand, in the o.p.s. case the basins of attraction may become too small. This reduction in the size of the basins is also verified for low activity patterns, for both p.a.s. and o.p.s. nets.

A limit value for the load parameter α for $a = 0.5$, p.a.s. nets has not been found. Anyway, the patterns are always zero-energy states of a non-negative Hamiltonian and hence the definition of a limit value for α may require some adjustment in this model : for example, a limitation may be imposed by the velocity (or some other parameter) in retrieving the patterns rather than by the ability in addressing them, although up to now we did not detect such effect. Work in this direction is presented elsewhere [17].

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