Exact Results for Curvature-Driven Coarsening in Two Dimensions

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We consider the statistics of the areas enclosed by domain boundaries ("hulls") during the curvature-driven coarsening dynamics of a two-dimensional nonconserved scalar field from a disordered initial state. We show that the number of hulls per unit area that enclose an area greater than A has, for large time t, the scaling form $N_h(A, t) = 2c/(A + \lambda t)$, demonstrating the validity of dynamical scaling in this system, where $c = 1/8\pi\sqrt{3}$ is a universal constant. Domain areas (regions of aligned spins) have a similar distribution up to very large values of $A/\lambda t$. Identical forms are obtained for coarsening from a critical initial state, but with c replaced by $c/2$.

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Coarsening dynamics has attracted enormous interest over the last 40 years. The classic scenario concerns a system that in equilibrium exhibits a phase transition from a disordered high-temperature phase to an ordered low-temperature phase with a broken symmetry of the high-temperature phase. The simplest example is, perhaps, the Ising ferromagnet. When the system is cooled rapidly through the transition temperature, domains of the two ordered phases form and grow ("coarsen") with time under the influence of the interfacial surface tension, which acts as a driving force for the domain growth [1–3].

While phase transitions provide the traditional arena for coarsening dynamics, there are many other examples, including soap froths [4], breath figures [5], granular media [6], and interfacial fluctuations [7]. A common feature of nearly all such coarsening systems is that they are well described by a dynamical scaling phenomenology in which there is a single characteristic length scale, $R(t)$, which grows with time. If dynamical scaling holds, the domain morphology is statistically the same at all times when all lengths are measured in units of $R(t)$. The assumption of dynamical scaling also makes possible the determination of the length scale $R(t)$ for a large class of coarsening systems [3,8].

Despite the success of the scaling hypothesis in describing experimental and simulation data, its validity has only been proved for very simple models, including the $1d$ Glauber-Ising model [9] and the nonconserved $O(n)$ model in the limit $n \rightarrow \infty$ [10]. Another noteworthy exact result is the Lifshitz-Slyozov derivation of the domain-size distribution for a conserved scalar field in the limit where the minority phase occupies a vanishingly small volume fraction [11]. The only other exact results, to our knowledge, for domain-size distributions in coarsening dynamics are for the zero-temperature Glauber-Potts [12] and time-dependent Ginzburg-Landau [13] models in $1d$.

In this work we obtain some exact results for the coarsening dynamics of a nonconserved scalar field in $2d$, demonstrating, *en passant*, the validity of the scaling hypothesis. To do this, we use a continuum model in which the velocity, v, of each element of a domain boundary is proportional to the local interfacial curvature, κ:

$$v = -(\lambda/2\pi)\kappa, \qquad (1)$$

where λ is a material constant with the dimensions of a diffusion constant, and the factor $1/2\pi$ is for later convenience. The Allen-Cahn Eq. (1) may be derived from the zero-temperature time-dependent Ginzburg-Landau equation for the order-parameter field [2,3].

From Eq. (1), we can immediately deduce the time-dependence of the area contained within any finite hull (i.e., the interior of a domain boundary) by integrating the velocity around the hull: $dA/dt = \oint v dl = -(\lambda/2\pi) \times \oint \kappa dl = -\lambda$, the final equality following from the Gauss-Bonnet theorem. At any given time t, therefore, hulls with original enclosed area smaller than λt will have disappeared, and the enclosed areas of surviving hulls will have decreased by λt. In other words, the entire distribution of hull enclosed areas is advected uniformly to the left at rate λ. If $N_h(A, t)$ is the number of hulls per unit area of the system with enclosed-area greater than A, it follows that

$$N_h(A, t) = N_h(A + \lambda t, 0), \quad \forall \ A > 0. \qquad (2)$$

To determine the initial condition we note that, shortly after the quench from the high-temperature phase, the system is at the critical point of continuum percolation. Cardy and Ziff [14] have shown that the number of percolation hulls, per unit area of the system, with area greater than A has, for large A, the universal asymptotic form

$$N_p(A) \sim c/A, \qquad (3)$$

where $c = 1/8\pi\sqrt{3}$ is a universal constant. This result provides the desired initial condition, $N_h(A, 0) = 2N_p(A)$, in Eq. (2), giving

$$N_h(A, t) = 2c/(A + \lambda t), \qquad (4)$$

where the factor 2 arises from fact that there are two types of hull, corresponding to the two phases, while the Cardy-Ziff result accounts only for clusters of occupied sites (and not clusters of unoccupied sites). From this result one immediately derives the hull-enclosed area density function, $n_h(A, t) = -\partial N_h(A, t)/\partial A$, where $n_h(A, t)dA$ is the number of hulls, per unit area of the system, having area in the interval $(A, A + dA)$:

$$n_h(A, t) = 2c/(A + \lambda t)^2. \qquad (5)$$

Equation (4) has the expected scaling form $N_h(A, t) = t^{-1}f(A/t)$ corresponding to a system with characteristic area proportional to t. This corresponds to characteristic length scale $R(t) \sim t^{1/2}$, which is the known result if scaling is *assumed* [3]. Here, however, we do not assume scaling—rather, it emerges from the calculation. Furthermore, the conventional scaling phenomenology is restricted to the "scaling limit": $A \to \infty$, $t \to \infty$ with A/t fixed. Equation (4), by contrast, is valid whenever t is sufficiently large, and does not (at least on the continuum) require large A. This follows from the fact that, for large t, the form (4) probes, for any A, the tail (i.e., the large-A regime) of the Cardy-Ziff result (3), which is just the regime in which the latter is valid.

It is, however, instructive to consider what can be deduced from scaling alone, augmented by the drift Eq. (2). The general scaling form corresponding to a scale area t is $N_h(A, t) = t^{-\phi}f(A/t)$, with arbitrary exponent ϕ. Consistency with (2) requires that $N_h(A, t)$ depends on A only through the combination $A + \lambda t$, forcing $N_h(A, t) \propto (A + \lambda t)^{-\phi}$. Finally, $\phi = 1$ is fixed by the requirement that there be of order one hull per scale area, i.e. $N_h(0, t) \propto 1/\lambda t$. Thus, for an internally consistent picture, one requires $N_h(A, 0) \propto A^{-1}$ for large A, and it is gratifying that the Cardy-Ziff result not only has this form but also provides the exact value of the proportionality constant.

The argument above relies on the $T = 0$ Allen-Cahn Eq. (1). Temperature fluctuations have a twofold effect. On the one hand they generate *equilibrium* thermal domains that are not related to the coarsening process. On the other hand they roughen the domain walls thus opposing the curvature-driven growth and slowing it down. Once equilibrium thermal fluctuations are subtracted—hulls associated to the coarsening process are correctly identified—the full temperature dependence enters only through the value of λ, which sets the time scale. λ monotonically decreases from $\lambda(T = 0)$ to $\lambda(T_c) = 0$. For simplicity we focus here on zero working temperature. In [15] we shall show the finite T effects.

To test the above result we carried out numerical simulations on the 2d square-lattice Ising model (2dIM) with periodic boundary conditions using a heat-bath algorithm with random sequential updates. All data have been obtained using systems with size $L^2 = 10^3 \times 10^3$ and 2×10^3 runs using independent initial conditions. Domain areas are identified with the Hoshen-Kopelman algorithm

while hull-enclosed ones are measured by performing a directed walk along the interfaces, in analogy with the algorithm in [16]. We mimic an instantaneous quench from infinite temperature with a random initial condition with spins pointing up or down with probability 1/2. The data are plotted in log-log form to test the prediction $n_h(A, t) \propto A^{-2}$ for large A. The data are in remarkably good agreement with the prediction (5)—shown as a continuous curve in Fig. 1—over the whole range of A and t. The downward deviations from the scaling curve are due to finite size effects. The latter are shown in more detail in the inset where we display the $t = 16$ Monte Carlo (MC) results for four linear sizes. Finite size effects appear only when the weight of the distribution has fallen by many orders of magnitude (7 for a system with $L = 10^3$) and are thus quite irrelevant. The only fitting parameter is λ, which has the value $\lambda = 2.1$ in Fig. 1. In the tail of the probability distribution function the numerical error is smaller than the size of the data points, the agreement with the analytic prediction being nearly perfect. Even at small values of A/t, where the lattice and continuous descriptions are expected to differ most, the difference is only a few percent (for a detailed analysis, see [15]). The agreement between theory and data is all the more impressive given that the curvature-driven growth underlying the prediction (5) only holds in a statistical sense for the lattice Ising model [17].

The mean hull-enclosed area, per unit area of the system, $\langle A \rangle = \int_0^{L^2} dA A n_h(A, t) \sim 2c \ln(L^2/\lambda t)$, diverges with the system size. The fact that the total hull-enclosed area exceeds L^2 seems paradoxical until one notes that a given point in space belongs to many hulls.
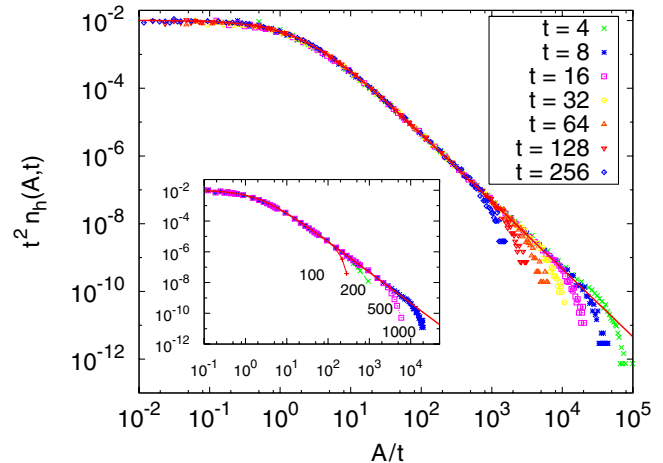

FIG. 1 (color online). Number density of hulls per unit area for the zero-temperature dynamics of the 2d Ising model evolving from an infinite temperature initial condition. The full line is the prediction (5) with $c = 1/8\pi\sqrt{3}$ and $\lambda = 2.1$. Inset: finite size effects at $t = 16$ MC results; four linear sizes of the sample are used and indicated by the data points. The value of A/t at which the data separate from the master curve grows very fast with L with an exponent close to 2.
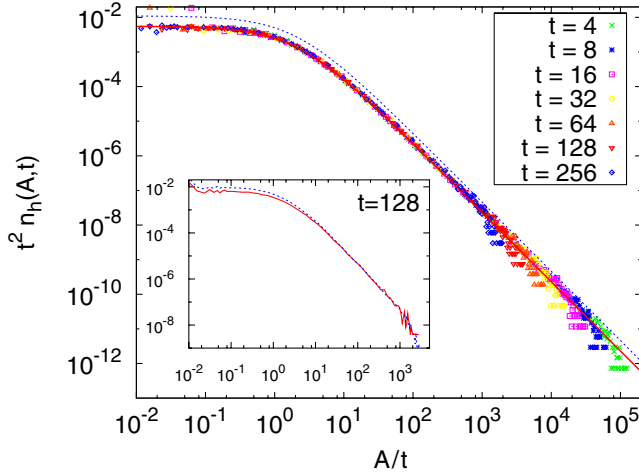
FIG. 2 (color online).   Number density of hulls per unit area for the zero-temperature 2dIM evolving from critical initial conditions. We obtained the initial states after running 10^3 Swendsen-Wang algorithm steps. The full (red) line represents (5) with $c \to c/2$ and $\lambda = 2.1$. The dashed (blue) line is (5) with c. Inset: comparison between the hull-enclosed area distribution at $t = 128$ MC results for equilibrated initial conditions at $T_0 = 2.5$ and $T_0 \to \infty$.

It is clear that the evolution of the hull-enclosed-area distribution follows the same "advection law" (2), with the same value of λ, for other initial conditions. Moreover, Eq. (5) applies to *any* $T_0 > T_c$ equilibrium initial condition asymptotically. Equilibrium initial conditions at different $T_0 > T_c$ show only a different transient behavior; initial states that are closer to T_c take longer to reach the asymptotic law (2) (see the inset in Fig. 2) [15].

An equilibrium state at the critical temperature, $T_0 = T_c$, is, as expected, different. This case has already been addressed, for general space dimension, in the context of coarsening from an initial state with long-ranged spatial correlations [18]. It was argued that the characteristic scale $R(t)$ still grows as $t^{1/2}$, since the growth is curvature driven, but the space and time correlation functions are modified if the spatial correlations in the initial state are sufficiently

long ranged [18]. The statistics of hull and domain areas, however, were not discussed. In place of the continuum percolation statistics that characterize the initial state *shortly* after a quench from high temperature, the initial state for a quench from T_c is characterized by the statistics of Ising cluster hulls at the critical point. The area distribution of the hulls of these clusters has been studied by Cardy and Ziff [14]. The result has a form identical to Eq. (3), but with c replaced by $c/2$. We predict, therefore, that our results for the hull-enclosed-area distributions can be generalized to this case by simply making the replacement $c \to c/2$ everywhere, and keeping the value of λ unchanged. In Fig. 2 we compare this prediction to the number density of hull-enclosed areas in the 2dIM evolving at $T = 0$ from a critical initial condition, once more obtaining excellent agreement.

Also interesting is the distribution of *domain* areas, $n_d(A, t)$, which are the areas of regions of aligned spins [19]. Domains are obtained from hulls by removing any interior hulls. The domain area distribution $n_d(A, t)$ (number density of domains with area A, per unit area) of the 2dIM evolving at zero temperature after a quench from infinite temperature is shown in Fig. 3. The downward deviations from the scaling curve in the main panel as well as the bumps on the tail are finite-size effects due to the percolating clusters and, as for the hull-enclosed areas (see Fig. 1), are not very important.

Remarkably, the domain area distribution, n_d, seems to be almost identical to the hull-enclosed area distribution n_h, i.e., $n_d \sim 2c_d/(A + \lambda_d t)^\tau$ with a prefactor $2c_d$, a parameter λ_d and an exponent τ taking approximately the same values as for the hull-enclosed areas. An argument that treats interior domain walls in a mean-field approximation and uses the exact result for n_h derived above, allows one to derive [15]

$$n_d(A, t) \sim 2c_d[\lambda_d(t + t_0)]^{\tau-2}/[A + \lambda_d(t + t_0)]^\tau, \quad (6)$$

with $\lambda_d = \lambda + O(c)$, $c_d = c + O(c^2)$, and $\tau \sim 2 + O(c)$ for infinite temperature initial conditions and, $c_d \to c_d/2$ for initial conditions equilibrated at $T_0 = T_c$. t_0 is such that
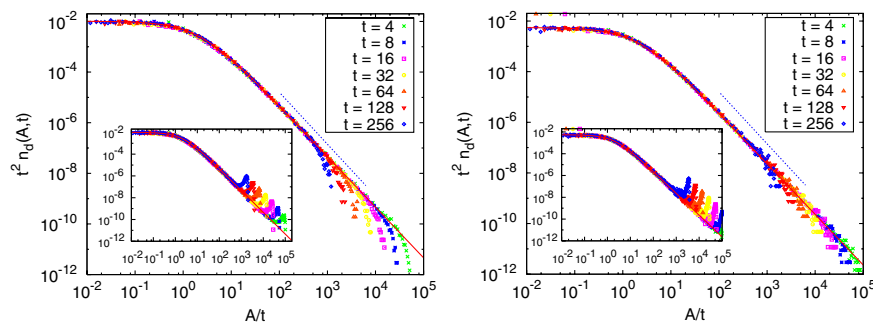

FIG. 3 (color online).   Number density of domains per unit area for the zero-temperature 2dIM evolving from $T_0 \to \infty$ (left panel) and $T_0 = T_c$ (right panel) initial conditions. In the main panels the percolating domains have been extracted from the analysis while in the insets we show the same data including the percolating clusters. We obtained the initial states after running 10^3 Swendsen-Wang algorithm steps. The full (red) line represents (5) with $c = 1/8\pi\sqrt{3}$ (left) and $c \to c/2$ (right), and $\lambda = 2.1$ in both cases. The dotted (blue) lines have slopes -2.03 (left) and -2.05 (right).
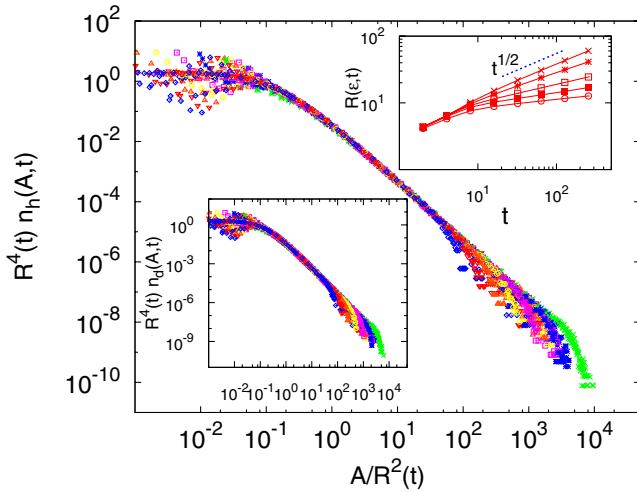
FIG. 4 (color online). Number density of hull-enclosed and domain (lower inset) areas per unit area for the $2d$ random ferromagnetic Ising model with a uniform distribution of exchanges in the interval $[2 - \epsilon, 2 + \epsilon]$ with $\epsilon = 0, 0.5, 1, 1.5, 2$. The system evolves at $T = 0.4$ after a quench from $T_0 \rightarrow \infty$. Upper inset: the evolution of $R(\epsilon, t)$ used to scale the pdfs and its comparison to the pure $R(0, t) \propto t^{1/2}$ law.

$\lambda_d t_0 = a^2$, with a a microscopic length scale, and sets the time scale. The constant c_d and τ characterize the initial distribution of domain areas, $n_d(A, 0) \sim 2c_d a^{\tau-2} A^{-\tau}$. The exponent τ is known for the $2d$IM critical geometrical clusters, $\tau = 379/187 \approx 2.03$ [20], and for cluster masses in $2d$ percolation, $\tau = 187/91 \approx 2.05$ [21]; the constant c_d has not been computed analytically. $\tau > 2$ allows one to satisfy the sum rule establishing that the total domain area, per unit area of the system, must be equal to unity since every spin belongs to one, and only one, domain. Unfortunately, it is hard to distinguish a power law with $\tau = 2$ or 2.03–2.05 from our numerical data: both exponents describe the power-law tail equally well, as shown in Fig. 3, with a (blue) dotted line. It is, however, possible to put to the numerical test the value of the prefactor c_d (or $2c_d$) and λ_d. We shall present this analysis elsewhere [15].

Interestingly, similar results are obtained for the $2d$ *random* ferromagnet, at least when activation is not too important. In Fig. 4 we display data obtained at $T = 0.4$ after a quench from $T_0 \rightarrow \infty$. This low but finite working temperature is enough to avoid the complete pinning of domain walls by quenched disorder. Good scaling with the typical domain area $R^2(t)$ is obtained for $A/R^2(t) \gtrsim 10^{-1}$ and the master curve resembles strongly the one found in the pure Ising case also included in Fig. 4.

In summary, we obtained exact results for the statistics of hull-enclosed areas during the curvature-driven phase-ordering dynamics of $2d$ systems. Notably, these results include a proof of the scaling hypothesis for these systems and are strongly supported by simulations of Ising systems, suggesting a strong degree of universality. The domain area distribution satisfies a very similar law.

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