Chapter 8

Exact solvability in contemporary physics

Angela Foerster[†]*, Jon Links*[‡] *and Huan-Qiang Zhou*[‡]
[†]*Instituto de Física da UFRGS, Porto Alegre, Brasil*
[‡]*Department of Mathematics, The University of Queensland, Australia*

8.1 Introduction

The current realization of nanotechnology as a viable industry is presenting a wealth of challenging problems in theoretical physics. Phenomena such as Bose–Einstein condensation, entanglement and decoherence in the context of quantum information, superconducting correlations in metallic nanograins, soft condensed matter, the quantum Hall effect, nano-optics, the Kondo effect and Josephson tunnelling phenomenon are all emerging to paint a vast canvas of interwoven physical theories which provide hope and expectation that the emergence of new nanotechnologies will be rapid in the short-term future. A significant tool in the evolution of the theoretical aspects of these studies has been the development and application of potent mathematical techniques, which are becoming ever increasingly important as our understanding of the complexities of these physical systems matures.

One approach that has recently been raised to prominence in this regard is that of the exact solution of a physical model. The necessity of studying the exact solution has been demonstrated through the experimental research on aluminium grains with dimensions at the nanoscale level. The work of Ralph, Black and Tinkham (RBT) [1] in 1996 detected the presence of superconducting pairing correlations in metallic nanograins which manifest as a parity effect in the energy spectrum dependent on whether the number of valence electrons on each grain is even or odd. A naïve approach to describing these systems theoretically is to apply the theory of superconductivity due to Bardeen, Cooper and Schrieffer (BCS) [2]. Indeed, the BCS model is the appropriate model for these systems but the associated mean field treatment fails. This is because a mean

field theory approximates certain operators in the model by an average value. At the nanoscale level, the quantum fluctuations are sufficiently large enough that this approximation is invalid. In fact, there had been a long harboured notion that superconductivity would break down for systems where the mean single particle energy level spacing, which is inversely proportional to the volume, is comparable to the superconducting gap, as in the case of metallic nanograins. This was conjectured by Anderson [3] in 1959 on the basis of the BCS theory but the experiments by RBT show this to not be the case. Consequently, an exact solution is highly desired, a view that has been promoted in [4].

The study of exact solutions of quantum mechanical models has its origins in the work of Bethe in 1931 on the Heisenberg model [5]. The field received a tremendous impetus in the 1960s with the work of McGuire [6], Yang [7], Baxter [8] and Lieb and Wu [9] and it has prospered ever since. The work of RBT cited earlier has brought the discipline to a new audience, when it was realized that the exact solution of the BCS model had been obtained, although largely ignored, by Richardson in 1963 [10]. The reason that Richardson's work was overlooked for so long is because the theory that had been proposed by BCS was so spectacularly successful that there had never been a need to use an alternative approach. Once the results of RBT were communicated however, it was clear that a new viewpoint was needed. When the condensed matter physics community became aware of Richardson's work, his results were promptly adopted and it was shown that the analysis of the exact solution gave agreement with the experiments [11]. A concise yet informative account of the developments is given in [12].

In this review, we will recount the quantum inverse scattering method and the associated algebraic Bethe ansatz method for the exact solution of integrable quantum Hamiltonians. We then show how this procedure can be applied for the analysis of three models which are the focus of many current theoretical studies: a model for two Bose–Einstein condensates coupled via Josephson tunnelling, a model for atomic–molecular Bose–Einstein condensation and the BCS model. In each case, we undertake an asymptotic analysis of the solution and demonstrate how this can be applied to extract the asymptotic behaviour of certain correlation functions at zero temperature through use of the Hellmann–Feynman theorem [13].

8.2 Quantum inverse scattering method

First we will review the basic features of the quantum inverse scattering method [14, 15]. The theory of exactly solvable quantum systems in this setting relies on the existence of a solution $R(u) \in \mathrm{End}(V \otimes V)$, where V denotes a vector space, which satisfies the Yang–Baxter equation acting on the three-fold tensor product space $V \otimes V \otimes V$:

$$R_{12}(u-v)R_{13}(u)R_{23}(v) = R_{23}(v)R_{13}(u)R_{12}(u-v). \tag{8.1}$$

Here $R_{jk}(u)$ denotes the matrix in $\mathrm{End}(V \otimes V \otimes V)$ acting non-trivially on the jth and kth spaces and as the identity on the remaining space. The R-matrix solution may be viewed as the structural constants for the Yang–Baxter algebra which is generated by the monodromy matrix $T(u)$ whose entries generate the algebra

$$R_{12}(u-v)T_1(u)T_2(v) = T_2(v)T_1(u)R_{12}(u-v). \tag{8.2}$$

We note that as a result of (8.1) the Yang–Baxter is necessarily associative. In component form, we may write

$$\sum_{p,q} R_{ik}^{pq}(u-v)T_p^j(u)T_q^l(v) = \sum_{p,q} T_k^p(v)T_i^q(u)R_{qp}^{jl}(u-v)$$

so the $R_{ij}^{kl}(u)$ give the structure constants of the algebra.

Here, we will only concern ourselves with the $su(2)$ invariant R-matrix which has the form

$$\begin{aligned} R(u) &= \frac{1}{u+\eta}(u \cdot I \otimes I + \eta P) \\ &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & b(u) & c(u) & 0 \\ 0 & c(u) & b(u) & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \end{aligned} \tag{8.3}$$

with $b(u) = u/(u+\eta)$ and $c(u) = \eta/(u+\eta)$. Here, P is the permutation operator which satisfies

$$P(x \otimes y) = y \otimes x \qquad \forall\, x,\, y \in V.$$

In this case, the Yang–Baxter algebra has four elements which we express as

$$T(u) = \begin{pmatrix} A(u) & B(u) \\ C(u) & D(u) \end{pmatrix}. \tag{8.4}$$

Next suppose that we have a representation, which we denote π, of the Yang–Baxter algebra. For later convenience, we set

$$L(u) = \pi(T(u))$$

which we refer to as an L-operator. Defining the transfer matrix through

$$t(u) = \mathrm{tr}\,\pi((T(u))) = \pi(A(u) + D(u)) \tag{8.5}$$

it follows from (8.1) that the transfer matrices commute for different values of the spectral parameters; viz.

$$[t(u), t(v)] = 0 \qquad \forall u,\, v. \tag{8.6}$$

There are two significant consequences of (8.6). The first is that $t(u)$ may be diagonalized independently of u, that is the eigenvectors of $t(u)$ do not depend on u. Secondly, taking a series expansion

$$t(u) = \sum_k c_k u^k$$

it follows that

$$[c_k, c_j] = 0 \qquad \forall\, k,\ j.$$

Thus, for any Hamiltonian which is expressible as a function of the operators c_k only, then each c_k corresponds to an operator representing a constant of the motion since it will commute with the Hamiltonian. When the number of conserved quantities is equal to the number of degrees of freedom of the system, the model is said to be integrable.

An important property of the Yang–Baxter algebra is that it has a co-multiplication structure which allows us to build tensor product representations. In particular, given two L-operators L^U, L^W acting on $V \otimes U$ and $V \otimes W$ respectively, then $L = L^U L^W$ is also an L-operator as can be see from

$$\begin{aligned}
R_{12}(u-v)L_1(u)L_2(v) &= R_{12}(u-v)L_1^U(u)L_1^W L_2^U(v)L_2^W(v) \\
&= R_{12}(u-v)L_1^U(u)L_2^U(v)L_1^W(u)L_2^W(v) \\
&= L_2^U(v)L_1^U(u)R_{12}(u-v)L_1^W(u)L_2^W(v) \\
&= L_2^U(v)L_1^U(u)L_2^W(v)L_1^W(u)R_{12}(u-v) \\
&= L_2^U(v)L_2^W(v)L_1^U(u)L_1^W(u)R_{12}(u-v) \\
&= L_2(v)L_1(u)R_{12}(u-v).
\end{aligned}$$

Furthermore, if $L(u)$ is an L-operator, then so is $L(u+\alpha)$ for any α since the R-matrix depends only on the difference of the spectral parameters.

8.2.1 Realizations of the Yang–Baxter algebra

In order to construct a specific model, we must address the question of determining a realization of the Yang–Baxter algebra. Here we will present several examples which will all be utilized later. The first realization comes from the R-matrix itself, since it is apparent from (8.1) that we can make the identification $L(u) = R(u)$ such that a representation of (8.2) is obtained. This is the realization used in the construction of the Heisenberg model [14, 15]. A second realization is given by $L(u) = G$ (c-number realization), where G is an arbitrary 2×2 matrix whose entries do not depend on u. This follows from the fact that $[R(u), G \otimes G] = 0$.

There is a realization in terms of canonical boson operators b, $b^\dagger$ with the relations $[b, b^\dagger] = 1$ which reads [16, 17, 31] as:

$$L^b(u) = \begin{pmatrix} u + \eta\hat{N} & b \\ b^\dagger & \eta^{-1} \end{pmatrix} \tag{8.7}$$

where $\hat{N} = b^\dagger b$. There also exists a realization in terms of the $su(2)$ Lie algebra with generators S^z and $S^\pm$ [14, 15]:

$$L^S(u) = \frac{1}{u}\begin{pmatrix} u - \eta S^z & -\eta S^+ \\ -\eta S^- & u + \eta S^z \end{pmatrix} \tag{8.8}$$

with the commutation relations $[S^z, S^\pm] = \pm S^\pm$, $[S^+, S^-] = 2S^z$. It is worth noting that in the case when the $su(2)$ algebra takes the spin-$\frac{1}{2}$ representation, the resulting L-operator is equivalent to that given by the R-matrix. Another is realized in terms of the $su(1, 1)$ generators K^z and $K^\pm$ [18, 19]:

$$L^K(u) = \begin{pmatrix} u + \eta K^z & \eta K^- \\ -\eta K^+ & u - \eta K^z \end{pmatrix} \tag{8.9}$$

with the commutation relations $[K^z, K^\pm] = \pm K^\pm$, $[K^+, K^-] = -2K^z$.

Here we will use these realizations to construct a variety of exactly solvable models. First, however, we will introduce the algebraic Bethe ansatz which provides the exact solution.

8.3 Algebraic Bethe ansatz method of solution

For a given realization of the Yang–Baxter algebra, the solution to the problem of finding the eigenvalues of the transfer matrix (8.5) via the algebraic Bethe ansatz is obtained by utilizing the commutation relations of the Yang–Baxter algebra. We have from the defining relations (8.2) that (among other relations)

$$\begin{aligned} [A(u), A(v)] &= [D(u), D(v)] = 0 \\ [B(u), B(v)] &= [C(u), C(v)] = 0 \\ A(u)C(v) &= \frac{u - v + \eta}{u - v} C(v)A(u) - \frac{\eta}{u - v} C(u)A(v) \\ D(u)C(v) &= \frac{u - v - \eta}{u - v} C(v)D(u) + \frac{\eta}{u - v} C(u)D(v). \end{aligned} \tag{8.10}$$

A key step in successfully applying the algebraic Bethe ansatz approach is finding a suitable pseudovacuum state, $|0\rangle$, which has the properties

$$\begin{aligned} A(u)|0\rangle &= a(u)|0\rangle \\ B(u)|0\rangle &= 0 \\ C(u)|0\rangle &\neq 0 \\ D(u)|0\rangle &= d(u)|0\rangle \end{aligned}$$

where $a(u)$ and $d(u)$ are scalar functions.

Assuming the existence of such a pseudovacuum state, choose the Bethe state

$$|\boldsymbol{v}\rangle \equiv |v_1, \ldots, v_M\rangle = \prod_{i=1}^{M} C(v_i)|0\rangle. \tag{8.11}$$

Note that because $[C(u), C(v)] = 0$, the ordering is not important in (8.11). The approach of the algebraic Bethe ansatz is to use the relations (8.10) to determine the action of $t(u)$ on $|\boldsymbol{v}\rangle$. The result is

$$\begin{aligned} t(u)|\boldsymbol{v}\rangle &= \Lambda(u, \boldsymbol{v})|\boldsymbol{v}\rangle \\ &\quad - \left(\sum_{i}^{N} \frac{\eta a(v_i)}{u - v_i} \prod_{j \neq i}^{M} \frac{v_i - v_j + \eta}{v_i - v_j} \right) |v_1, \ldots, v_{i-1}, u, v_{i+1}, \ldots, v_M\rangle \\ &\quad + \left(\sum_{\alpha}^{M} \frac{\eta d(v_i)}{u - v_i} \prod_{j \neq i}^{M} \frac{v_i - v_j - \eta}{v_i - v_j} \right) |v_1, \ldots, v_{i-1}, u, v_{i+1}, \ldots, v_M\rangle \end{aligned} \tag{8.12}$$

where

$$\Lambda(u, \boldsymbol{v}) = a(u) \prod_{i=1}^{M} \frac{u - v_i + \eta}{u - v_i} + d(u) \prod_{i=1}^{M} \frac{u - v_i - \eta}{u - v_i}. \tag{8.13}$$

This shows that $|\boldsymbol{v}\rangle$ becomes an eigenstate of the transfer matrix with eigenvalue (8.13) whenever the Bethe ansatz equations

$$\frac{a(v_i)}{d(v_i)} = \prod_{j \neq i}^{M} \frac{v_i - v_j - \eta}{v_i - v_j + \eta} \qquad i = 1, \ldots, M \tag{8.14}$$

are satisfied. Note that in the derivation of the Bethe ansatz equations, it is required that $v_i \neq v_j, \ \forall\, i, j$. This is a result of the Pauli principle for Bethe ansatz solvable models as developed in [20] for the Bose gas. We will not reproduce the proofs for the present cases, as they follow essentially the same argument as that in [20].

8.3.1 Scalar products of states

One of the important applications of the previous discussion is that there exists a formula due to Slavnov [14, 21, 22] for the scalar product of states obtained via the algebraic Bethe ansatz for the R-matrix (8.3). The formula reads

$$\begin{aligned} S(\boldsymbol{v} : \boldsymbol{u}) &= \langle 0|B(v_1) \ldots B(v_M) C(u_1) \ldots C(u_M)|0\rangle \\ &= \frac{\det F(\boldsymbol{u} : \boldsymbol{v})}{\det V(\boldsymbol{u} : \boldsymbol{v})} \end{aligned}$$

where

$$F_{ij} = \frac{\partial}{\partial v_i} \Lambda(u_j, \boldsymbol{v}) \qquad V_{ij} = \frac{1}{u_j - v_i}$$

the parameters $\{v_i\}$ satisfy the Bethe ansatz equations (8.14) and $\{u_j\}$ are arbitrary. The significance of this result is that it opens up the possibility of determining form factors and correlation functions for any model which can be derived in this manner. Although we will not go into any details here, we wish to point out that explicit results for two of the models which we will discuss subsequently can be found in [23, 24].

8.4 A model for two coupled Bose–Einstein condensates

Experimental realization of Bose–Einstein condensates in dilute atomic alkali gases has stimulated a diverse range of theoretical and experimental research activity [25–29]. A particularly exciting possibility is that a pair of Bose–Einstein condensates (such as a Bose–Einstein condensate trapped in a double-well potential) may provide a model tunable system in which to observe macroscopic quantum tunnelling. Here we will show that a model Hamiltonian for a pair of coupled Bose–Einstein condensates admits an exact solution. The model is also realizable in Josephson coupled superconducting metallic nanoparticles [30], which has applications in the implementation of solid-state quantum computers.

The canonical Hamiltonian which describes tunnelling between two Bose–Einstein condensates takes the form [27]

$$H = \frac{K}{8}(N_1 - N_2)^2 - \frac{\Delta\mu}{2}(N_1 - N_2) - \frac{\mathcal{E}_{\mathcal{J}}}{2}(b_1^\dagger b_2 + b_2^\dagger b_1). \tag{8.15}$$

where $b_1^\dagger$, $b_2^\dagger$ denote the single-particle creation operators in the two wells and $N_1 = b_1^\dagger b_1$, $N_2 = b_2^\dagger b_2$ are the corresponding boson number operators. The total boson number $N_1 + N_2$ is conserved and set to the fixed value of N. The physical meaning of the coupling parameters for different realizable systems may be found in [27]. It is useful to divide the parameter space into three regimes: Rabi ($K/\mathcal{E}_{\mathrm{J}} \ll N^{-1}$), Josephson ($N^{-1} \ll K/\mathcal{E}_{\mathrm{J}} \ll N$) and Fock ($N \ll K/\mathcal{E}_{\mathrm{J}}$). There is a correspondence between (8.15) and the motion of a pendulum [27]. In the Rabi and Josephson regimes, this motion is semiclassical, unlike the case of the Fock regime. For both the Fock and Josephson regimes, the analogy corresponds to a pendulum with fixed length, while in the Rabi regime the length varies. An important problem is to study the behaviour in the crossover regimes, which is accessible through the exact solution. The exact solvability of (8.15) which we discuss here follows from the fact that it is mathematically equivalent to the discrete self-trapping dimer model studied by Enol'skii *et al* [31], who solved the model through the algebraic Bethe ansatz. We will describe this construction later.

The co-multiplication behind the Yang–Baxter algebra allows us to choose the following representation of the monodromy matrix:

$$\begin{aligned} L(u) &= L_1^b(u+\omega)L_2^b(u-\omega) \\ &= \begin{pmatrix} (u+\omega+\eta N_1)(u-\omega+\eta N_2) + b_2^\dagger b_1 & (u+\omega+\eta N_1)b_2 + \eta^{-1}b_1 \\ (u-\omega+\eta N_2)b_1^\dagger + \eta^{-1}b_2^\dagger & b_1^\dagger b_2 + \eta^{-2} \end{pmatrix}. \end{aligned} \tag{8.16}$$

Defining the transfer matrix as before through $t(u) = \mathrm{tr}(L(u))$, we have explicitly in the present case

$$t(u) = u^2 + u\eta\hat{N} + \eta^2 N_1 N_2 + \eta\omega(N_2 - N_1) + b_2^\dagger b_1 + b_1^\dagger b_2 + \eta^{-2} - \omega^2.$$

Then

$$t'(0) = \left.\frac{\mathrm{d}t}{\mathrm{d}u}\right|_{u=0} = \eta\hat{N}$$

and it is easy to verify that the Hamiltonian is related to the transfer matrix $t(u)$ by

$$H = -\kappa(t(u) - \tfrac{1}{4}(t'(0))^2 - ut'(0) - \eta^{-2} + \omega^2 - u^2)$$

where the following identification has been made for the coupling constants:

$$\frac{K}{4} = \frac{\kappa\eta^2}{2} \qquad \frac{\Delta\mu}{2} = -\kappa\eta\omega \qquad \frac{\mathcal{E}_{\mathcal{J}}}{2} = \kappa.$$

An explicit representation of (8.4) is obtained from (8.16) with the identification

$$\begin{aligned} A(u) &= (u + \omega + \eta N_1)(u - \omega + \eta N_2) + b_2^\dagger b_1 \\ B(u) &= (u + \omega + \eta N_1)b_2 + \eta^{-1}b_1 \\ C(u) &= (u - \omega + \eta N_2)b_1^\dagger + \eta^{-1}b_2^\dagger \\ D(u) &= b_1^\dagger b_2 + \eta^{-2}. \end{aligned}$$

Choosing the Fock vacuum as the pseudovacuum, which satisfies $B(u)|0\rangle = 0$ as required by the Bethe ansatz procedure, the eigenvalues $a(u)$ and $d(u)$ of $A(u)$ and $D(u)$ on $|0\rangle$ are

$$\begin{aligned} a(u) &= (u + \omega)(u - \omega) \\ d(u) &= \eta^{-2}. \end{aligned}$$

The Bethe ansatz equations are then explicitly

$$\eta^2(v_i^2 - \omega^2) = \prod_{j\neq i}^{N} \frac{v_i - v_j - \eta}{v_i - v_j + \eta} \tag{8.17}$$

with the eigenstates of the form (8.11) with $C(u)$ given as before. From the Bethe ansatz equations, we may derive the useful identity

$$\prod_{i=1}^{m} \eta^2(v_i^2 - \omega^2) = \prod_{i=1}^{m}\prod_{j=m+1}^{N} \frac{v_i - v_j - \eta}{v_i - v_j + \eta} \tag{8.18}$$

which will be used later.

It is clear that the Bethe states are eigenstates of $\hat{N}$ with eigenvalue N. As N is the total number of bosons, we expect $N + 1$ solutions of the Bethe ansatz equations. As mentioned earlier, we must exclude any solution in which the roots of the Bethe ansatz equations are not distinct. For example, the solution

$$v_j = \pm\sqrt{\omega^2 - (-1)^N\eta^{-2}} \qquad \forall\ j \tag{8.19}$$

of (8.17) is invalid, except when $N = 1$. (Note the error in [24].) For a given valid solution of the Bethe ansatz equations, the energy of the Hamiltonian is obtained from the transfer matrix eigenvalues (8.13) and reads as

$$E = -\kappa\left(\eta^{-2}\prod_{i=1}^{N}\left(1+\frac{\eta}{v_i-u}\right) - \frac{\eta^2N^2}{4} - u\eta N - u^2 \right.$$
$$\left. - \eta^{-2} + \omega^2 + (u^2-\omega^2)\prod_{i=1}^{N}\left(1-\frac{\eta}{v_i-u}\right)\right). \tag{8.20}$$

Note that this expression is independent of the spectral parameter u which can be chosen arbitrarily. The formula simplifies considerably with the choice $u = \omega$, by employing (8.18), which yields a polynomial form:

$$E = -\kappa\left(\eta^{-2}\prod_{i=1}^{N}\eta^2(v_i-\omega+\eta)(v_i+\omega) - \frac{\eta^2N^2}{4} - \eta\omega N - \eta^{-2}\right).$$

However, for the purpose of an asymptotic analysis in the Rabi regime, it is more convenient to choose $u = 0$, while for the Fock regime we use $u = \eta^2$.

8.4.1 Asymptotic analysis of the solution

Here we will recall the asymptotic analysis of the exact solution that was conducted in [32]. We start the analysis with the Rabi regime where $\eta^2 N \ll 1$. From the Bethe ansatz equations, it is clear that $\eta^2 v_i^2 \to 1$ as $\eta \to 0$, so that $v_i \approx \pm\eta^{-1}$. However, when $\eta = 0$ we know that the Hamiltonian is diagonalizable by using the Bogoliubov transformation, from which we can deduce that the solution of the Bethe ansatz equations corresponding to the ground state must have $v_i \approx \eta^{-1}$. Therefore, it is reasonable to consider the asymptotic expansion

$$v_i \approx \eta^{-1} + \epsilon_i + \eta\delta_i. \tag{8.21}$$

Excitations correspond to changing the signs of the leading terms in the Bethe ansatz roots. To study the asymptotic behaviour for the mth excited state, we set

$$\begin{aligned} v_i &\approx -\eta^{-1} + \epsilon_i + \eta\delta_i \qquad i = 1, \ldots, m \\ v_i &\approx \eta^{-1} + \epsilon_i + \eta\delta_i \qquad i = m+1, \ldots, N \end{aligned} \tag{8.22}$$

with the convention that the ground state corresponds to $m = 0$.

From the leading terms of the Bethe ansatz equations for v_i, $i \le m$, we find

$$\epsilon_i = \sum_{j\neq i}^{m}\frac{1}{\epsilon_i - \epsilon_j} \tag{8.23}$$

which implies

$$\sum_{i=1}^{m} \epsilon_i = 0 \qquad \sum_{i=1}^{m} \epsilon_i^2 = \frac{m(m-1)}{2}.$$

In a similar fashion, we have, for $m < i \leq N$,

$$\epsilon_i = -\sum_{\substack{j=m+1 \\ j\neq i}}^{N} \frac{1}{\epsilon_i - \epsilon_j} \tag{8.24}$$

which implies

$$\sum_{i=m+1}^{N} \epsilon_i = 0 \qquad \sum_{i=m+1}^{N} \epsilon_i^2 = -\frac{(N-m)(N-m-1)}{2}.$$

It is clear from (8.23) and (8.24) why the Pauli exclusion principle applies in the present case. In the asymptotic expansion for v_i, ϵ_i is assumed finite. However, if $v_i = v_j$ for some i, j, then $\epsilon_i = \epsilon_j$ and (8.23) and (8.24) imply that ϵ_i, ϵ_j are infinite which is a contradiction. Hence, v_i must be distinct for different i. Note also that for this approximation to be valid, we require $\eta^{-1} \gg \epsilon_i$. However, we see that $|\epsilon_i|$ is of the order of $N^{1/2}$. Thus, our approximation will be valid for $\eta N^{1/2} \ll 1$, which is precisely the criterion for the Rabi region and, consequently, N cannot be arbitrarily large for fixed η or *vice versa*.

Now we go to the next order. From (8.18), we find

$$\sum_{i=1}^{m} \delta_i = -\frac{m(m-1)}{4} + \frac{m(m-N)}{2} - \frac{m\omega^2}{2}$$

$$\sum_{i=m+1}^{N} \delta_i = -\frac{(N-m)(N-m-1)}{4} + \frac{m(m-N)}{2} + \frac{(N-m)\omega^2}{2}$$

which using (8.20) leads us to the result

$$\frac{E_m}{\kappa} \approx -N + 2m - \frac{\eta^2\omega^2(N-2m)}{2} + \frac{\eta^2 N}{4} + \frac{\eta^2}{2}m(N-m).$$

The energy level spacings $\Delta_m = E_m - E_{m-1}$ are, thus,

$$\Delta_m \approx \kappa \left(2 + \eta^2\omega^2 + \frac{\eta^2}{2}(N-2m+1)\right).$$

One may check that Δ_m/N is of the order of N^{-1}. This indicates that the Rabi regime is semiclassical [27]. This value for the gap between the ground and first excited state agrees, to leading order in $\eta^2 N$, with the Gross–Pitaevskii mean-field theory [33] giving a Josephson plasma frequency of $\omega_J = 2\kappa(1 + \eta^2 N/2)^{1/2}$.

Now we look at the asymptotic behaviour of the Bethe ansatz equations in the Fock regime $\eta^2 \gg N$. It is necessary to distinguish the following cases: (i) $\omega = 0$ and (ii) $\omega \neq 0$.

(i) $\omega = 0$. In this case, it is appropriate to consider the permutation operator P which interchanges the labels 1 and 2 in (8.15). For $\omega = 0$, P commutes with the Hamiltonian and any eigenvector of the Hamiltonian is also an eigenvector of P with eigenvalue ± 1. Therefore, the Hilbert space splits into the direct sum of two subspaces corresponding to the symmetric and antisymmetric wavefunctions. From now on, we restrict ourselves to the case when N is even, i.e. $N = 2M$, although a similar calculation is also applicable to the case when N is odd. A careful analysis leads us to conclude that the ground state lies in the symmetric subspace. The asymptotic form of the roots of the Bethe ansatz equations for the ground state takes the 'string'-like structure

$$v_{j\pm} \approx -(M-j)\eta \pm \mathrm{i}\frac{C_M^j}{(j-1)!}\eta^{-(2j-1)} + M(M+1)\eta^{-3}\delta_{j1} \qquad j = 1, \ldots, M$$

where C_M^j is a binomial coefficient. For this asymptotic ansatz to be valid, we require that any term in the asymptotic expansion should be much smaller than those preceding it. This yields $\eta^2 \gg N$ which coincides with the defining condition for the Fock region. Throughout, the Pauli exclusion principle has been taken into account to exclude any possible spurious solutions of the Bethe ansatz equations.

This structure clearly indicates that in the ground state the N bosons fuse into M 'bound' states and excitations correspond to a breakdown of these bound states. Specifically, the first and second excited states correspond to the breakdown of the bound state at $-(M-1)\eta$, with the first excited state in the antisymmetric subspace and the second excited state in the symmetric subspace. Explicitly, we can write down the spectral parameter configurations for the first two excited states:

$$v_{1+} \approx -M\eta + a_{1+}\eta^{-3} \qquad v_{1-} \approx -(M-1)\eta + a_{1-}\eta^{-3}$$
$$v_{j\pm} \approx -(M-j)\eta + a_{j\pm}\eta^{-(2j-1)} \qquad j = 2, \ldots, M$$

with

$$a_{1+} = -\frac{M+1}{2} \qquad a_{1-} = \frac{M(M+1)}{2}$$
$$a_{2\pm} = \frac{-(M-1)^2 \pm (M-1)\sqrt{13M^2+10M+1}}{12}$$
$$a_{3\pm} = \pm\frac{(M-1)(M-2)\sqrt{2M(M+1)}}{24}$$
$$a_{j\pm} = \frac{M-j+1}{\sqrt{(j+1)j(j-1)(j-2)}}a_{j-1,\pm} \qquad j = 3, \ldots, M$$

for the (antisymmetric) first excited state and

$$a_{1+} = -\frac{(M+1)(2M+1)}{2} \qquad a_{1-} = -\frac{M(M+1)}{2}$$
$$a_{2\pm} = \frac{-(M-1)^2 \pm \mathrm{i}(M-1)\sqrt{11M^2+14M-1}}{12}$$
$$a_{3\pm} = \pm\mathrm{i}\frac{(M-1)(M-2)\sqrt{2M(M+1)}}{24}$$
$$a_{j\pm} = \frac{M-j+1}{\sqrt{(j+1)j(j-1)(j-2)}}a_{j-1,\pm} \qquad j = 3, \ldots, M$$

for the (symmetric) second excited state. The breakdown of the bound state at $-(M-j)\eta$, $j = 2, \ldots, M$ results in the higher excited states.

Substituting these results into (8.20) leads us to the asymptotic ground-state energy

$$E_0 \approx -2\kappa\eta^{-2}M(M+1)$$

while for the first and second excited states, we have

$$E_1 \approx \kappa\eta^2 - \kappa\eta^{-2}\frac{M^2+M-2}{3}$$
$$E_2 \approx \kappa\eta^2 + \kappa\eta^{-2}\frac{5M^2+5M+2}{3}.$$

In contrast to the Rabi regime, the Fock regime is not semiclassical, as the ratio of the gap Δ and N is of finite order when N is large.

We can perform a similar analysis for odd N. In this case, the gap between the ground and the first excited states is proportional to $\kappa\eta^{-2}$ instead of $\kappa\eta^2$. Furthermore, the ground-state root structure is different in the odd case since not all the bosons can be bound in pairs. This indicates there is a strong parity effect in the Fock regime, in contrast to the Rabi regime.

(ii) $\omega \neq 0$. In this case the root structure is somewhat more complicated than for $\omega = 0$, so we will not present the details. We remark, however, that our calculations show that up to order η^{-2} the ground-state energy eigenvalue takes the same form as in the case $\omega = 0$. Actually, the leading contribution arising from the ω term appears only as $\omega^2\eta^{-4}$. This means that the results presented here are applicable for all values of ω (or, equivalently, $\Delta\mu$).

Although it is difficult to define rigorously [27, 34], the relative phase between Bose–Einstein condensates is useful in understanding interference experiments [28, 29, 35]. Recall that in Josephson's original proposal [36] for Cooper pair tunnelling through an insulating barrier between macroscopic superconductors, the current is a manifestation of the relative phase between the wavefunctions of the superconductors. By definition, the relative phase Φ is conjugate to the relative number of atoms in the two condensates $n \equiv N_1 - N_2$.

Using the Hellmann–Feynman theorem, we find that

$$\langle \Delta n^2 \rangle = 8\frac{\partial E_0}{\partial K} - 4\left(\frac{\partial E_0}{\partial \Delta\mu}\right)^2.$$

For the ground state in the limit of strong tunnelling (i.e. the Rabi regime), $\langle \Delta n^2 \rangle \approx N - (\Delta\mu N/\mathcal{E}_J)^2$. In the case of weak tunnelling (i.e. the Fock regime), $\langle \Delta n^2 \rangle \approx 2N(N+2)(\mathcal{E}_J/K)^2$. The degree of coherence between the two Bose–Einstein condensates can be discussed in terms of [27]

$$\alpha \equiv \frac{1}{2N}\langle a_1^\dagger a_2 + a_2^\dagger a_1 \rangle = -\frac{1}{N}\frac{\partial E_0}{\partial \mathcal{E}_J}.$$

In the strong coupling limit, $\alpha \approx 1 - N^{-1}(\Delta\mu)^2/(8\mathcal{E}_J)^2$, indicating very close to full coherence in the ground state. In the opposite limit, we have $\alpha \approx 2(N+2)\mathcal{E}_J/K \ll 1$, indicating the absence of coherence. These results give the first-order corrections to the results presented in [28, 37] for the number fluctuations and the coherence factor at zero temperature.

8.5 A model for atomic–molecular Bose–Einstein condensation

After the experimental realization of Bose–Einstein condensation in dilute alkali gases, many physicists started to consider the possibility of producing a molecular Bose–Einstein condensate from photoassociation and/or the Feshbach resonance of an atomic Bose–Einstein condensate of a weakly interacting dilute alkali gas [38, 39]. This novel area has attracted considerable attention from both experimental and theoretical physicists and, in particular, it has recently been reported that a Bose–Einstein condensate of rubidium has been achieved comprised of a coherent superposition of atomic and molecular states [40, 41]. As stressed in [42], even in the ideal two-mode limit, mean field theory fails to provide long-term predictions due to strong interparticle entanglement near the dynamically unstable molecular mode. The numerical results have shown that the large-amplitude atom–molecular coherent oscillations are damped by the rapid growth of fluctuations near the unstable point, which contradicts the mean field theory predictions. In order to clarify the controversies raised by these investigations, one can appeal to the exact solution of the two-mode model, the derivation of which we will now present.

The two-mode Hamiltonian takes the form

$$H = \frac{\omega}{2}a^\dagger a + \frac{\Omega}{2}(a^\dagger a^\dagger b + b^\dagger a a) \tag{8.25}$$

where $a^\dagger$ and $b^\dagger$ denote the creation operators for atomic and molecular modes respectively. Note that the total atom number operator $\hat{N} = N_a + 2N_b$ where $N_a = a^\dagger a$, $N_b = b^\dagger b$ provides a good quantum number since $[H, \hat{N}] = 0$.

In order to derive this Hamiltonian through the quantum inverse scattering method, we take the following L-operator

$$L(u) = GL^b(u - \delta - \eta^{-1})L^K(u)$$

with the matrix G given by

$$G = \begin{pmatrix} -\eta^{-1} & 0 \\ 0 & \eta^{-1} \end{pmatrix}.$$

This gives us the explicit realization of the Yang–Baxter algebra:

$$\begin{aligned} A(u) &= -\eta^{-1}(u + \eta K^z)(u - \delta - \eta^{-1} + \eta N_b) + bK^+ \\ B(u) &= -K^-(u - \delta - \eta^{-1} + N_b) - \eta^{-1}b(u - \eta K^z) \\ C(u) &= \eta^{-1}b^\dagger(u + \eta K^z) - \eta^{-1}K^+ \\ D(u) &= b^\dagger K^- + \eta^{-2}(u - \eta K^z) \end{aligned}$$

and

$$t(0) = \delta K^z + b^\dagger K^- + bK^+ - \eta K^z N_b. \tag{8.26}$$

Let $|0\rangle$ denote the Fock vacuum state and let $|k\rangle$ denote a lowest weight state of the $su(1, 1)$ algebra with weight k, i.e. $K^z|k\rangle = k|k\rangle$. On the product state $|\Psi\rangle = |0\rangle|k\rangle$, it is clear that $B(u)|\Psi\rangle = 0$ and

$$\begin{aligned} a(u) &= -\eta^{-1}(u + \eta k)(u - \delta - \eta^{-1}) \\ d(u) &= \eta^{-2}(u - \eta k). \end{aligned}$$

We can immediately conclude that the eigenvalues of (8.26) are given by

$$\Lambda(0) = k(\delta + \eta^{-1}) \prod_{i=1}^{M} \frac{v_i - \eta}{v_i} - k\eta^{-1} \prod_{i=1}^{M} \frac{v_i + \eta}{v_i} \tag{8.27}$$

subject to the Bethe ansatz equations

$$\frac{(v_i + \eta k)(1 - \eta v_i + \eta\delta)}{(v_i - \eta k)} = \prod_{j \neq i}^{M} \frac{v_i - v_j - \eta}{v_i - v_j + \eta}. \tag{8.28}$$

Realizing the $su(1, 1)$ algebra in terms of canonical boson operators through

$$K^+ = \frac{(a^\dagger)^2}{2} \qquad K^- = \frac{a^2}{2} \qquad K^z = \frac{2N_a + 1}{4}$$

we then find that the Hamiltonian (8.25) is related to (8.26) through

$$H = \lim_{\eta \to 0} \Omega(t(0) - \delta/4)$$

with $\omega = \Omega\delta$. Note that, in this case, the possible lowest weight states for the $su(1, 1)$ algebra are

$$|k = 1/4\rangle \equiv |0\rangle \qquad |k = 3/4\rangle \equiv a^\dagger|0\rangle.$$

Moreover, we have $N = 2M + 2k - 1/2$.

It is worth mentioning at this point that another realization of the $su(1, 1)$ algebra is given in terms of two sets of boson operators by

$$K^+ = a^\dagger c^\dagger \qquad K^- = ac \qquad K^z = \frac{N_a + N_c + 1}{2}$$

with $J = N_a - N_c$ a central element commuting with the $su(1, 1)$ algebra in this representation. Due to the symmetry $a^\dagger \leftrightarrow c^\dagger$ we may assume $J \geq 0$. For this case we define the Hamiltonian

$$\begin{aligned} H &= \lim_{\eta\to 0} \Omega\,(t(0) - \delta/2) + \beta J \\ &= \alpha N_a + \gamma N_c + \Omega\,(a^\dagger c^\dagger b + b^\dagger ac) \end{aligned} \tag{8.29}$$

with $\alpha = \delta\Omega/2 + \beta$ and $\gamma = \delta\Omega/2 - \beta$. This model has a natural interpretation for atomic–molecular Bose–Einstein condensation for two distinct atomic species which can bond to form a di-atomic molecule. In this case, the possible lowest weight states for the $su(1, 1)$ algebra are

$$|k = (m + 1)/2\rangle \equiv (a^\dagger)^m|0\rangle$$

and $J = 2k - 1$. A detailed analysis of this model through the exact solution will be given at a later date.

For the exact solution of the Hamiltonian (8.25) it is necessary to take the *quasi-classical limit* $\eta \to 0$ in the Bethe ansatz equations (8.28). The resulting Bethe ansatz equations take the form

$$\delta - v_i + \frac{2k}{v_i} = 2\sum_{j\neq i}^{M} \frac{1}{v_j - v_i}. \tag{8.30}$$

Also, in this limit the corresponding energy eigenvalue is

$$\begin{aligned} E &= \omega(M + k - 1/4) - \Omega\sum_{i=1}^{M} v_i \\ &= \omega(k - 1/4) - 2k\Omega\sum_{i=1}^{M} \frac{1}{v_i}. \end{aligned} \tag{8.31}$$

The equivalence of the two energy expressions can be deduced from (8.30). The eigenstates too are obtained by this procedure. Consider the following class of

states:

$$|v_1, \ldots, v_M\rangle = \prod_{i=1}^{M} c(v_i)|\Psi\rangle \tag{8.32}$$

where $c(v) = (vb^\dagger - a^\dagger a^\dagger/2)$, $|\Psi\rangle = |0\rangle$ for $k = 1/4$ and $|\Psi\rangle = a^\dagger|0\rangle$ for $k = 3/4$. In the case when the set of parameters $\{v_i\}$ satisfy the Bethe ansatz equations (8.30), then (8.32) are precisely the eigenstates of the Hamiltonian.

8.5.1 Asymptotic analysis of the solution

In the limit of large $|\delta|$, we can perform an asymptotic analysis of the Bethe ansatz equations to determine the asymptotic form of the energy spectrum. We choose the following ansatz for the Bethe roots:

$$v_i \approx \begin{cases} \delta^{-1}\mu_i & i \le m \\ \delta + \epsilon_i + \delta^{-1}\mu_i & i > m. \end{cases}$$

For $i > m$, we obtain, from the zero-order terms in the Bethe ansatz equations,

$$\epsilon_i = 2 \sum_{\substack{j=m+1 \\ j\neq i}}^{M} \frac{1}{\epsilon_i - \epsilon_j}$$

which implies

$$\sum_{i=m+1}^{M} \epsilon_i = 0.$$

From the terms in δ^{-1}, we find

$$\mu_i = 2(k+m) + 2 \sum_{\substack{j=m+1 \\ j\neq i}}^{M} \frac{\mu_j - \mu_i}{(\epsilon_j - \epsilon_i)^2}$$

and, thus,

$$\sum_{i=m+1}^{M} \mu_i = 2(k+m)(M-m).$$

Next we look at the Bethe ansatz equations for $i \le m$. The terms in δ give

$$1 + \frac{2k}{\mu_i} = 2 \sum_{j\neq i}^{m} \frac{1}{\mu_j - \mu_i}$$

which implies

$$\sum_{i=1}^{m} \mu_i = -2km - m(m-1).$$

This gives the energy levels

$$E_m \approx \omega(M + (k - 1/4)) - \omega(M - m) - \Omega \sum_{i=m+1}^{M} \epsilon_i - \frac{\Omega^2}{\omega} \sum_{i=1}^{M} \mu_i$$
$$= \omega(m + k - 1/4) + \frac{\Omega^2}{\omega}(3m^2 - m + 4km - 2kM - 2mM).$$

The level spacings are

$$\Delta_m = E_m - E_{m-1}$$
$$\approx \omega - \frac{2\Omega^2}{\omega}(M + 2 - 3m - 2k)$$

from which we conclude that, in this limit, the model is semi-classical.

Let $\mathcal{E}$ denote the ground-state energy ($\mathcal{E} = E_0$ for $\Omega\delta \gg 0$, $\mathcal{E} = E_M$ for $\Omega\delta \ll 0$) and Δ the gap to the first excited state. Employing the Hellmann–Ferynman theorem, we can determine the asymptotic form of the following zero-temperature correlations

$$\langle N_a \rangle = 2\frac{\partial \mathcal{E}}{\partial \omega} \qquad \theta = -2\frac{\partial \mathcal{E}}{\partial \Omega}$$

where $\theta = -\langle a^\dagger a^\dagger b + b^\dagger a a \rangle$ is the coherence correlator. For large N, we introduce the rescaled variables

$$\delta^* = \frac{\delta}{N^{1/2}} \qquad \Delta^* = \frac{\Delta}{\Omega N^{1/2}} \qquad \langle N_a \rangle^* = \frac{\langle N_a \rangle}{N} \qquad \theta^* = \frac{\theta}{N^{3/2}}. \tag{8.33}$$

We then have, for $\delta^* \gg 0$,

$$\Delta^* \approx \delta^* - \frac{1}{\delta^*} \qquad \langle N_a \rangle^* \approx 0 \qquad \theta^* \approx 0$$

while, for $\delta^* \ll 0$,

$$\Delta^* \approx -\delta^* - \frac{2}{\delta^*} \qquad \langle N_a \rangle^* \approx 1 - \frac{1}{2(\delta^*)^2} \qquad \theta^* \approx -\frac{1}{\delta^*}.$$

This shows that the model has scale invariance in the asymptotic limit. The scaling properties actually hold for a wide range of values of the scaled detuning parameter δ^*, which is established through numerical analysis [43].

8.5.2 Computing the energy spectrum

For this model, there is a convenient method to determine the energy spectrum without solving the Bethe ansatz equations (cf [19]). This is achieved by introducing the polynomial function whose zeros are the roots of the Bethe ansatz

equations, i.e.

$$G(u) = \prod_{i=1}^{M} (1 - u/v_i).$$

It can be shown from the Bethe ansatz equations that G satisfies the differential equation

$$uG'' - (u^2 - \delta u - 2k)G' + (Mu - E/\Omega + \delta(k - 1/4))G = 0 \tag{8.34}$$

subject to the initial conditions

$$G(0) = 1 \qquad G''(0) = \frac{E - \omega(k - 1/4)}{2k\Omega}.$$

In order to show this, we set

$$F(u) = uG' - (u^2 - \delta u - 2k)G'.$$

As a result of the Bethe ansatz equations (8.30), it is deduced that $F(v_i) = 0$. Given that $F(u)$ is a polynomial of degree $(M + 1)$, we then conclude that $F(u) = (\alpha u + \beta)G(u)$ for some constants α, β, which are determined by the asymptotic limits $u \to 0$ and $u \to \infty$. Equation (8.34) then follows.

By setting $G(u) = \sum_n g_n u^n$, the recurrence relation

$$g_{n+1} = \frac{E - \omega(n + k - 1/4)}{\Omega(n + 1)(n + 2k)} g_n + \frac{n - M - 1}{(n + 1)(n + 2k)} g_{n-1} \tag{8.35}$$

is readily obtained. It is clear from this relation that g_n is a polynomial in E of degree n. We also know that G is a polynomial function of degree M and so we must have $g_{M+1} = 0$. The $(M + 1)$ roots of g_{M+1} are precisely the energy levels E_m. Moreover, the eigenstates (8.32) are expressible as (up to overall normalization)

$$|v_1, \ldots, v_M\rangle = \sum_{n=1}^{M} g_n (b^\dagger)^{(M-n)} \left(\frac{a^\dagger a^\dagger}{2}\right)^n |\Psi\rangle.$$

The recurrence relation (8.35) can be solved as follows (cf [19]). Setting

$$g_{n+1} = g_0 \prod_{j=0}^{n} x_j y_j$$

with

$$x_j = \frac{E - \omega(j + k - 1/4)}{\Omega(j + 1)(j + 2k)}$$

and substituting into the recurrence relation (8.35), we have

$$x_j x_{j-1} y_{j-1}(y_j - 1) = \frac{j - M - 1}{(j + 1)(j + 2k)}.$$

This yields $y_j = 1 + c_{j-1}/y_{j-1}$ with

$$c_j = \frac{\Omega^2(j+1)(j+2k)(j-M)}{(E-\omega(j+k+3/4))(E-\omega(j+k-1/4))}$$

which means y_j can be expressed as a continued fraction. The requirement that G is a polynomial function of order M decrees $y_M = 0$, in turn implying

$$y_{M-1} = \frac{\Omega^2 M(M+2k-1)}{(E-\omega(M+k-1/4))(E-\omega(n+k-5/4))}$$

which is an algebraic equation that determines the allowed energy levels E_m. This procedure can easily be employed to determine the energy spectrum numerically, without resorting to solving the Bethe ansatz equations. Explicit results can be found in [43].

8.6 The BCS Hamiltonian

The experimental work of Ralph, Black and Tinkham [1] on the discrete energy spectrum in small metallic aluminium grains generated interest in understanding the nature of superconducting correlations at the nanoscale level. Their results indicate significant parity effects due to the number of electrons in the system. For grains with an odd number of electrons, the gap in the energy spectrum reduces with the size of the system, in contrast to the case of a grain with an even number of electrons, where a gap larger than the single-electron energy levels persists. In the latter case, the gap can be closed by a strong applied magnetic field. The conclusion drawn from these results is that pairing interactions are prominent in these nanoscale systems. For a grain with an odd number of electrons, there will always be at least one unpaired electron, so it is not necessary to break a Cooper pair in order to create an excited state. For a grain with an even number of electrons, all excited states have a least one broken Cooper pair, resulting in a gap in the spectrum. In the presence of a strongly applied magnetic field, it is energetically more favourable for a grain with an even number of electrons to have broken pairs and, hence, in this case there are excitations which show no gap in the spectrum.

The physical properties of a small metallic grain are described by the reduced BCS Hamiltonian [11]

$$H = \sum_{j=1}^{\mathcal{L}} \epsilon_j n_j - g \sum_{j,k}^{\mathcal{L}} c_{k+}^{\dagger} c_{k-}^{\dagger} c_{j-} c_{j+}. \tag{8.36}$$

Here, $j = 1, \ldots, \mathcal{L}$ labels a shell of doubly degenerate single-particle energy levels with energies ϵ_j and n_j is the fermion number operator for level j. The operators $c_{j\pm}$, $c_{j\pm}^{\dagger}$ are the annihilation and creation operators for the fermions at level j. The labels $\pm$ refer to time-reversed states.

One of the features of the Hamiltonian (8.36) is the *blocking effect*. For any unpaired electron at level j, the action of the pairing interaction is zero since only paired electrons are scattered. This means that the Hilbert space can be decoupled into a product of paired and unpaired electron states in which the action of the Hamiltonian on the subspace for the unpaired electrons is automatically diagonal in the natural basis. In view of the blocking effect, it is convenient to introduce hard-core boson operators $b_j = c_{j-}c_{j+}$, $b_j^\dagger = c_{j+}^\dagger c_{j-}^\dagger$ which satisfy the relations

$$(b_j^\dagger)^2 = 0 \qquad [b_j, b_k^\dagger] = \delta_{jk}(1 - 2b_j^\dagger b_j) \qquad [b_j, b_k] = [b_j^\dagger, b_k^\dagger] = 0$$

on the subspace excluding single-particle states. In this setting, the hard-core boson operators realize the $su(2)$ algebra in the pseudo-spin representation, which will be utilized later.

The original approach of BCS [2] to describing the phenomenon of superconductivity was to employ a mean field theory using a variational wavefunction for the ground state which has an undetermined number of electrons. The expectation value for the number operator is then fixed by means of a chemical potential term μ. One of the predictions of the BCS theory is that the number of Cooper pairs in the ground state of the system is given by the ratio Δ/d where Δ is the BCS 'bulk gap' and d is the mean level spacing for the single-electron eigenstates. For nanoscale systems, this ratio is of the order of unity, in seeming contradiction with the experimental results discussed earlier. The explanation for this is that the mean field approach is inappropriate for nanoscale systems due to large superconducting fluctuations.

As an alternative to the BCS mean field approach, one can appeal to the exact solution of the Hamiltonian (8.36) derived by Richardson [10] and developed by Richardson and Sherman [44]. It has also been shown by Cambiaggio *et al* [45] that (8.36) is integrable in the sense that there exists a set of mutually commutative operators which commute with the Hamiltonian. These features have recently been shown to be a consequence of the fact that the model can be derived in the context of the quantum inverse scattering method using the L-operator (8.8) with a c-number L-operator [23, 46], which we will now explicate.

8.6.1 A universally integrable system

In this case, we use a c-number realization G of the L-operator as well as (8.8) to construct the transfer matrix

$$t(u) = \mathrm{tr}_0(G_0 L_{0\mathcal{L}}(u - \epsilon_{\mathcal{L}}) \cdots L_{01}(u - \epsilon_1)) \tag{8.37}$$

which is an element of the $\mathcal{L}$-fold tensor algebra of $su(2)$. Here tr_0 denotes the trace taken over the auxiliary space labelled 0 and $G = \exp(-\alpha\eta\sigma)$ with $\sigma = \mathrm{diag}(1, -1)$. Defining

$$T_j = \lim_{u \to \epsilon_j} \frac{u - \epsilon_j}{\eta^2} t(u)$$

for $j = 1, 2, \ldots, \mathcal{L}$, we may write, in the quasi-classical limit, $T_j = \tau_j + o(\eta)$ and it follows from the commutivity of the transfer matrices that $[\tau_j, \tau_k] = 0, \ \forall\, j, k$. Explicitly, these operators read as

$$\tau_j = 2\alpha S_j^z + \sum_{k\neq j}^{\mathcal{L}} \frac{\theta_{jk}}{\epsilon_j - \epsilon_k} \tag{8.38}$$

with $\theta = S^+ \otimes S^- + S^- \otimes S^+ + 2S^z \otimes S^z$.

We define a Hamiltonian through

$$H = -\frac{1}{\alpha}\sum_{j=1}^{\mathcal{L}} \epsilon_j \tau_j + \frac{1}{4\alpha^3}\sum_{j,k=1}^{\mathcal{L}} \tau_j \tau_k + \frac{1}{2\alpha^2}\sum_{j=1}^{\mathcal{L}} \tau_j - \frac{1}{2\alpha}\sum_{j=1}^{\mathcal{L}} C_j \tag{8.39}$$

$$= -\sum_{j=1}^{\mathcal{L}} 2\epsilon_j S_j^z - \frac{1}{\alpha}\sum_{j,k=1}^{\mathcal{L}} S_j^- S_k^+ \tag{8.40}$$

where

$$C = S^+S^- + S^-S^+ + 2(S^z)^2$$

is the Casimir invariant for the $su(2)$ algebra. The Hamiltonian is universally integrable since it is clear that $[H, \tau_j] = 0, \ \forall j$ irrespective of the realizations of the $su(2)$ algebra in the tensor algebra.

In order to reproduce the Hamiltonian (8.36), we realize the $su(2)$ generators through the hard-core boson (spin-$\frac{1}{2}$) representation, i.e.

$$S_j^+ = b_j \qquad S_j^- = b_j^\dagger \qquad S_j^z = \tfrac{1}{2}(I - n_j). \tag{8.41}$$

In this instance, one obtains (8.36) (with the constant term $-\sum_j^{\mathcal{L}} \epsilon_j$) where $g = 1/\alpha$ as shown by Zhou *et al* [23] and von Delft and Poghossian [46].

For each index k in the tensor algebra in which the transfer matrix acts, and accordingly in (8.40), suppose that we represent the $su(2)$ algebra through the irreducible representation with spin s_k. Thus $\{S_k^+, S_k^-, S_k^z\}$ act on a $(2s_k + 1)$-dimensional space. In employing the method of the algebraic Bethe ansatz discussed earlier, we find that

$$a(u) = \exp(-\alpha\eta) \prod_{k=1}^{\mathcal{L}} \frac{u - \epsilon_k - \eta s_k}{u - \epsilon_k}$$

$$d(u) = \exp(\alpha\eta) \prod_{k=1}^{\mathcal{L}} \frac{u - \epsilon_k + \eta s_k}{u - \epsilon_k}$$

which gives the eigenvalues of the transfer matrix (8.37) as

$$\Lambda(u) = \exp(\alpha\eta) \prod_{k=1}^{\mathcal{L}} \frac{u - \epsilon_k + \eta s_k}{u - \epsilon_k} \prod_{j=1}^{M} \frac{u - v_j - \eta}{u - v_j} + \exp(-\alpha\eta) \prod_{k=1}^{\mathcal{L}} \frac{u - \epsilon_k - \eta s_k}{u - \epsilon_k} \prod_{j=1}^{M} \frac{u - v_j + \eta}{u - v_j}.$$

The corresponding Bethe ansatz equations read as

$$\exp(2\alpha\eta) \prod_{k=1}^{\mathcal{L}} \frac{v_l - \epsilon_k + \eta s_k}{v_l - \epsilon_k - \eta s_k} = -\prod_{j=1}^{M} \frac{v_l - v_j + \eta}{v_l - v_j - \eta}.$$

The eigenvalues of the conserved operators (8.38) are obtained through the appropriate terms in the expansion of the transfer matrix eigenvalues in the parameter η. This yields the following result for the eigenvalues λ_j of τ_j:

$$\lambda_j = \left(2\alpha + \sum_{k \neq j}^{\mathcal{L}} \frac{2s_k}{\epsilon_j - \epsilon_k} - \sum_{i=1}^{M} \frac{2}{\epsilon_j - v_i} \right) s_j \tag{8.42}$$

such that the parameters v_j now satisfy the Bethe ansatz equations

$$2\alpha + \sum_{k=1}^{\mathcal{L}} \frac{2s_k}{v_j - \epsilon_k} = \sum_{i \neq j}^{M} \frac{2}{v_j - v_i}. \tag{8.43}$$

Through (8.42) we can now determine the energy eigenvalues of (8.40). It is useful to note the following identities:

$$2\alpha \sum_{j=1}^{M} v_j + 2 \sum_{j=1}^{M} \sum_{k=1}^{\mathcal{L}} \frac{v_j s_k}{v_j - \epsilon_k} = M(M-1)$$

$$\alpha M + \sum_{j=1}^{M} \sum_{k=1}^{\mathcal{L}} \frac{s_k}{v_j - \epsilon_k} = 0$$

$$\sum_{j=1}^{M} \sum_{k=1}^{\mathcal{L}} \frac{v_j s_k}{v_j - \epsilon_k} - \sum_{j=1}^{M} \sum_{k=1}^{\mathcal{L}} \frac{s_k \epsilon_k}{v_j - \epsilon_k} = M \sum_{k=1}^{\mathcal{L}} s_k.$$

Employing these, it is deduced that

$$\sum_{j=1}^{\mathcal{L}} \lambda_j = 2\alpha \sum_{j=1}^{\mathcal{L}} s_j - 2\alpha M$$

$$\sum_{j=1}^{\mathcal{L}} \epsilon_j \lambda_j = 2\alpha \sum_{j=1}^{\mathcal{L}} \epsilon_j s_j + \sum_{j=1}^{\mathcal{L}} \sum_{k \neq j}^{\mathcal{L}} s_j s_k - 2M \sum_{k=1}^{\mathcal{L}} s_k - 2\alpha \sum_{j=1}^{M} v_j + M(M-1)$$

which, combined with the eigenvalues $2s_j(s_j+1)$ for the Casimir invariants C_j, yields the energy eigenvalues

$$E = 2\sum_{j=1}^{M} v_j. \tag{8.44}$$

From this expression, we see that the quasi-particle excitation energies are given by twice the Bethe ansatz roots $\{v_j\}$ of (8.43). In order to specialize this result to the case of the BCS Hamiltonian (8.36), it is a matter of setting $s_k = 1/2,\ \forall\, k$. Finally, let us remark that in the quasi-classical limit the eigenstates assume the form

$$|\Psi\rangle = \prod_{i=1}^{M}\sum_{j=1}^{\mathcal{L}} \frac{b_j^\dagger}{v_i - \epsilon_j}|0\rangle.$$

The construction given here can also be applied on a more general level. Taking higher spin representations of the $su(2)$ algebra produces models of BCS systems which are coupled by Josephson tunnelling, as described in [47, 48]. One can also employ higher-rank Lie algebras, such as $so(5)$ [49] and $su(4)$ [50,51] which produce coupled BCS systems which model pairing interactions in nuclear systems. For the general case of an arbitrary Lie algebra, we refer to [52]. Finally, let us mention that if one reproduces this construction with the $su(1,1)$ L-operator (8.9) in place of the $su(2)$ L-operator (8.8), the pairing model for bosonic systems introduced by Dukelsky and Schuck [53] is obtained.

8.6.2 Asymptotic analysis of the solution

In the limit $g \to 0$, we can easily determine the ground-state energy of (8.36): it is given by filling the Fermi sea. Here, we will assume that the number of fermions is even. Thus, for small $g > 0$, it is appropriate to consider the asymptotic solution

$$v_i \approx \epsilon_i + g\delta_i + g^2\mu_i \qquad i = 1, \ldots, M.$$

Substituting this into (8.43) and equating the different orders in g yields

$$v_i \approx \epsilon_i - \frac{g}{2} + \frac{g^2}{4}\left(\sum_{k=m+1}^{\mathcal{L}} \frac{1}{\epsilon_j - \epsilon_k} - \sum_{i\neq j}^{M} \frac{1}{\epsilon_j - \epsilon_i}\right)$$

which immediately gives us the asymptotic ground-state energy

$$E_0 \approx 2\sum_{j=1}^{M}\epsilon_j - gM + \frac{g^2}{2}\sum_{j=1}^{M}\sum_{k=M+1}^{\mathcal{L}} \frac{1}{\epsilon_j - \epsilon_k}.$$

Next we look at the first excited state. In the $g = 0$ case, this corresponds to breaking the Cooper pair at level ϵ_M and putting single unpaired electrons in the

levels ϵ_M and ϵ_{M+1}. Now these two levels become blocked. Solving equations (8.43) for this excited state is the same as for the ground state except that there are now $(M-1)$ Cooper pairs and we have to exclude the blocked levels. We can, therefore, write down the energy

$$E_1 \approx \epsilon_M + \epsilon_{M+1} + 2\sum_{j=1}^{M-1}\epsilon_j - g(M-1) + \frac{g^2}{2}\sum_{j=1}^{M-1}\sum_{k=M+2}^{\mathcal{L}}\frac{1}{\epsilon_j - \epsilon_k}.$$

The gap is found to be

$$\Delta \approx \epsilon_{M+1} - \epsilon_M + g + \frac{g^2}{2}\left(\sum_{j=1}^{M-1}\frac{1}{\epsilon_{M+1}-\epsilon_j} + \sum_{k=M+1}^{\mathcal{L}}\frac{1}{\epsilon_k - \epsilon_M}\right).$$

As in previous examples, we can calculate some asymptotic correlation functions for zero temperature by using the Hellmann–Feynman theorem. In particular,

$$\langle n_i\rangle = \frac{\partial E_0}{\partial \epsilon_i}$$

which, for $i \leq M$, gives

$$\langle n_i\rangle \approx 2 - \frac{g^2}{2}\sum_{k=M+1}^{\mathcal{L}}\frac{1}{(\epsilon_i - \epsilon_k)^2}$$

while, for $i > M$, we get

$$\langle n_i\rangle \approx \frac{g^2}{2}\sum_{j=1}^{M}\frac{1}{(\epsilon_j - \epsilon_i)^2}.$$

We, can also determine the asymptotic form of the Penrose–Onsager–Yang off-diagonal long-range order parameter [54, 55] to be

$$\begin{aligned}\frac{1}{\mathcal{L}}\sum_{i,j=1}^{\mathcal{L}}\langle b_i^\delta b_j\rangle &= -\frac{1}{\mathcal{L}}\frac{\partial E_0}{\partial g}\\ &\approx \frac{M}{\mathcal{L}} - \frac{g}{\mathcal{L}}\sum_{j=1}^{M}\sum_{k=M+1}^{\mathcal{L}}\frac{1}{\epsilon_j - \epsilon_k}.\end{aligned}$$

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