Quantum Inverse Scattering Method with anyonic grading

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Abstract. We formulate the Quantum Inverse Scattering Method for the case of anyonic grading. This provides a general framework for constructing integrable models describing interacting hard-core anyons. Through this method we reconstruct the known integrable model of hard core anyons associated with the XXX model, and as a new application we construct the anyonic $t-J$ model. The energy spectrum for each model is derived by means of a generalisation of the algebraic Bethe ansatz. The grading parameters implementing the anyonic signature give rise to sector-dependent phase factors in the Bethe ansatz equations.

AMS classification scheme numbers: 82B20, 82B23

1. Introduction

The development of the quantum inverse scattering method (QISM) [1] led to the discovery of a number of quantum integrable models. Applications of the QISM to physical systems such as the Bloch electron problem [2], BCS models [3] and Bose-Einstein condensates [4] have opened up new applications of well developed mathematical techniques to describe low-dimensional many-body physics [5, 6, 7]. However, the main objects of the QISM are restricted to spin and fermion models (or two-dimensional classical vertex models) which are closely related to representations of Lie algebras and Lie superalgebras in finite Hilbert spaces. Although the Jordan-Wigner transformation can be implemented between different statistics languages, such

as the transmutation from bosons to fermions [8, 9], little attention has been paid to integrable models with generalized statistics. Some integrable lattice models with anyon-like commutation relations have been constructed [10, 11]. The anyonic statistical parameters result in global phase factors acting as a gauge potential in the Bethe ansatz equations. These phase factors lead to magnetic flux-like effects. However, if anyonic commutation relations are imposed on the 1D continuum quantum gases, the dynamical interaction and anyonic statistical interaction are inextricably intertwined [13], resulting in quite different low energy properties and statistical effects [14] than the standard 1D Bose and Fermi gases [6].

It is now well understood that the integrable quantum fermion models can be treated by the graded QISM where the Grassmann parity is adapted to fit the anticommuting property of fermions [15, 16]. It is natural to ask whether one can modify the usual QISM to treat other models with different statistics, like fractional statistics and anyonic statistics [17, 18]. Here we show that integrable lattice models of hard-core anyons can be systematically constructed via the Yang-Baxter equation with $U(1)$ Abelian group-like grading called anyonic grading. This is a generalization of $\mathbb{Z}_2$ grading to a continuous $U(1)$ grading function. The anyonic grading has a similar signature as the color grading invented by Rittenberg and Wyler [19] but is not equivalent.

In this paper, we generalize the QISM to the anyonic grading QISM which can be used to construct quantum integrable models describing hard-core anyons. As a first example of the anyonic QISM, we reconstruct the integrable XXX type model and its exact solution which has previously been studied in [10] via the co-ordinate Bethe ansatz. As a new application, we then consider the anyonic $t-J$ model with the Hamiltonian written in terms of hard-core anyon operators, and exactly solve it by the algebraic Bethe ansatz. This gives the energy spectrum in terms of the Bethe ansatz equations. The anyonic grading functions appear in the Bethe ansatz equations resulting in anyonic signature.

Our motivation for developing the QISM with anyonic grading is that it leads to wider application for studying exactly solvable one-dimensional lattice models than the co-ordinate Bethe ansatz approach, which has previously been discussed in [10, 11]. There are two main reasons for this. The first is that in the co-ordinate Bethe ansatz approach the anyonic statistics are introduced through the use of canonical operators which are anyonic deformations of the familiar canonical fermion operators. The algebraic approach we describe is more general and does not depend on the existence of such a representation of the Hilbert space of states. Secondly, the algebraic approach is more accessible for extending the analysis toward the computation of form factors and correlation functions. For example, for the anyonic $t-J$ model we will construct below, the form factors and correlation functions can be determined following the procedure used in [26] for the supersymmetric $t-J$ model.

This paper is organized as follows. In section 2 we introduce some basic concepts for the generalized grading and present the anyonic grading QISM. We give an explicit expression for the Hamiltonian and derive the Bethe ansatz solution for the XXX

model of hard core anyons in section 3. In section 4 the $t-J$ model of hard core anyons is constructed and the exact solution is obtained by the algebraic Bethe ansatz. Concluding remarks are given in section 5.

2. QISM with anyonic grading

The standard colour algebras [19, 20] are defined through the notion of colour graded vector spaces. The colour structures are a generalization of supersymmetric structures in that the grading with respect to $\mathbb{Z}_2$ is generalized to an arbitrary Abelian group Γ. For anyonic grading with the Abelian group being $U(1)$, we can directly define operations in a parallel way to colour grading. Due to the grading being associated with $U(1)$, we always consider cases where the underlying fields for the vector spaces are $\mathbb{C}$.

Letting U, V denote complex vector spaces with bases $\{u^i\}$, $\{v^j\}$, the anyonic permutation operator $P:\ U\otimes V\ \rightarrow\ V\otimes U$ is defined by the action on the basis vectors

$$P(u^i\otimes v^j)=w(i,j)(v^j\otimes u^i) \tag{1}$$

where $w(i,j)\in U(1)$ are the anyonic grading parameters. The inverse operator $P^{-1}:\ V\otimes U\ \rightarrow\ U\otimes V$ has the action

$$\begin{aligned}P^{-1}(v^j\otimes u^i)&=\tilde{w}(j,i)(u^i\otimes v^j)\\&=w^{-1}(i,j)(u^i\otimes v^j).\end{aligned}$$

This implies that in the special case where $U=V$, which occurs when dealing with indistinguishable particles, the anyonic grading parameters must possess the symmetry

$$w(i,j)=w(j,i).$$

Other than this there are no constraints imposed on the choice of the $w(i,j)$. A significant difference between our formulation of anyonic grading and that of colour grading is that for colour grading the constraint

$$w(i,i)=\pm 1$$

is imposed, whereas for anyonic grading we relax this condition. For each choice of anyonic grading it is natural to also define the dual grading with permutation operator P^* acting as

$$P^*(u^i\otimes v^j)=w^{-1}(i,j)(v^j\otimes u^i).$$

Using the fundamental basis of linear operators $\{e^i_j\}$ acting on U and $\{f^k_l\}$ acting on V such that

$$e^i_j u^m=\delta^m_j u^i,\qquad f^k_l v^n=\delta^n_l v^k,$$

the basis for the anyonic graded tensor product $\mathrm{End}(U)\otimes_{\mathrm{a}}\mathrm{End}(V)$ is defined by

$$e^i_j\otimes_{\mathrm{a}} f^k_l=w(j,k)w^{-1}(j,l)e^i_j\otimes f^k_l. \tag{2}$$

We say that the basis operator $e^i_j \otimes_{\rm a} f^k_l$ is *even* if

$$w(j,k)w^{-1}(j,l) = 1$$

and more generally an operator is even if it is a linear combination of even basis operators. If we introduce a third vector space W with basis $\{g^r_s\}$ then it follows from (2) that the anyonic graded tensor product is associative:

$$(e^i_j \otimes_{\rm a} f^k_l) \otimes_{\rm a} g^r_s = e^i_j \otimes_{\rm a} (f^k_l \otimes_{\rm a} g^r_s).$$

The basis for the opposite anyonic graded tensor product $\mathrm{End}(V) \otimes_{\rm a} \mathrm{End}(U)$ is defined in terms of P^{-1}

$$\begin{aligned} f^k_l \otimes_{\rm a} e^i_j &= \tilde{w}(l,i)\tilde{w}^{-1}(l,j) f^k_l \otimes e^i_j \\ &= w^{-1}(i,l)w(j,l) f^k_l \otimes e^i_j. \end{aligned}$$

Now we define the twist map $T : \mathrm{End}(V) \otimes_{\rm a} \mathrm{End}(U) \to \mathrm{End}(U) \otimes_{\rm a} \mathrm{End}(V)$. It is defined through the inverse anyonic permutation operator and its dual as

$$\begin{aligned} T(f^k_l \otimes_{\rm a} e^i_j) &= (P^*)^{-1}(f^k_l \otimes_{\rm a} e^i_j)P^{-1} \\ &= w(j,l)w^{-1}(i,l)(P^*)^{-1}(f^k_l \otimes e^i_j)P^{-1} \\ &= w(j,l)w^{-1}(i,l)w(i,k)w^{-1}(j,l)(e^i_j \otimes f^k_l) \\ &= w(i,k)w(j,l)w^{-1}(i,l)w^{-1}(j,k)(e^i_j \otimes_{\rm a} f^k_l). \end{aligned}$$

Through the twist map T and the usual matrix multiplication $m_U : \mathrm{End}(U) \otimes \mathrm{End}(U) \to \mathrm{End}(U)$ the anyonic graded tensor product multiplication is formally defined as

$$\begin{aligned} &(e^i_j \otimes_{\rm a} f^k_l)(e^p_q \otimes_{\rm a} f^r_s) \\ &\quad = (m_U \otimes m_V)(\mathrm{id} \otimes T \otimes \mathrm{id})(e^i_j \otimes_{\rm a} f^k_l \otimes_{\rm a} e^p_q \otimes_{\rm a} f^r_s) \\ &\quad = w(p,k)w(q,l)w^{-1}(p,l)w^{-1}(q,k)(m_U \otimes m_V)(e^i_j \otimes_{\rm a} e^p_q \otimes_{\rm a} f^k_l \otimes_{\rm a} f^r_s) \\ &\quad = w(p,k)w(q,l)w^{-1}(p,l)w^{-1}(q,k)(e^i_j e^p_q \otimes_{\rm a} f^k_l f^r_s) \\ &\quad = w(p,k)w(q,l)w^{-1}(p,l)w^{-1}(q,k)(e^i_q \otimes_{\rm a} f^k_s). \end{aligned}$$

On the other hand working directly with the definition (2) we have

$$\begin{aligned} (e^i_j \otimes_{\rm a} f^k_l)(e^p_q \otimes_{\rm a} f^r_s) &= w(j,k)w^{-1}(j,l)w(q,r)w^{-1}(q,s)(e^i_j \otimes f^k_l)(e^p_q \otimes f^r_s) \\ &= w(p,k)w(q,l)w^{-1}(p,l)w^{-1}(q,s)e^i_q \otimes f^k_s \\ &= w(p,k)w(q,l)w^{-1}(p,l)w^{-1}(q,k)e^i_q \otimes_{\rm a} f^k_s \end{aligned} \tag{3}$$

which shows that the definitions for the anyonic graded tensor product and its multiplication are consistent.

The $\mathbb{Z}_2$ graded QISM was set up in [16]. Here we establish an analogous anyonic graded QISM. A matrix $R(\lambda)$ is said to fulfill the Yang–Baxter equation (YBE) with anyonic grading if the identity

$$\begin{aligned} &\left(I \otimes_{\rm a} \overset{\vee}{R}(\lambda-\mu)\right)\left(\overset{\vee}{R}(\lambda) \otimes_{\rm a} I\right)\left(I \otimes_{\rm a} \overset{\vee}{R}(\mu)\right) \\ &= \left(\overset{\vee}{R}(\mu) \otimes_{\rm a} I\right)\left(I \otimes_{\rm a} \overset{\vee}{R}(\lambda)\right)\left(\overset{\vee}{R}(\lambda-\mu) \otimes_{\rm a} I\right) \end{aligned} \tag{4}$$

acting on $V_1 \otimes_{\rm a} V_2 \otimes_{\rm a} V_3$ holds. We will impose that the $\overset{\vee}{R}$-matrix is chosen to be even. Thus the YBE with anyonic grading can be written in component form

$$\overset{\vee}{R}(\lambda-\mu)^{a_2a_3}_{c_2c_3}\overset{\vee}{R}(\lambda)^{a_1c_2}_{b_1d_2}\overset{\vee}{R}(\mu)^{d_2c_3}_{b_2b_3}=\overset{\vee}{R}(\mu)^{a_1a_2}_{c_1c_2}\overset{\vee}{R}(\lambda)^{c_2a_3}_{d_2b_3}\overset{\vee}{R}(\lambda-\mu)^{c_1d_2}_{b_1b_2}. \quad (5)$$

The summation convention is implied for the repeated indices a_j, b_j, c_j and d_j. We notice that despite the fact that the tensor product (4) carries the anyonic grading, there are no extra anyonic grading parameters in (5) compared to the standard one. This is because we consider the case where the R-matrix is even.

With the help of the anyonic permutation operator (1) we may, from (4), prove the anyonic graded YBE in the form

$$R_{12}(\lambda-\mu)R_{13}(\lambda)R_{23}(\mu)=R_{23}(\mu)R_{13}(\lambda)R_{12}(\lambda-\mu), \quad (6)$$

where $R(\lambda)=P\overset{\vee}{R}(\lambda)$. Similarly, in component form it reads

$$\begin{aligned}&R(\lambda-\mu)^{f_1e_2}_{c_1c_2}R(\lambda)^{c_1f_3}_{b_1c_3}R(\mu)^{c_2c_3}_{b_2b_3}w(b_1,c_2)w^{-1}(c_1,c_2)\\&R(\mu)^{f_1f_3}_{c_1c_3}R(\lambda)^{e_2c_3}_{c_2b_3}R(\lambda-\mu)^{c_1c_2}_{b_1b_2}w(c_1,e_2)w^{-1}(f_1,e_2).\end{aligned} \quad (7)$$

If we choose the spaces V_1, V_2 as auxiliary spaces, the space V_3 as the quantum space, then letting $L_n(\lambda)=R_{0n}(\lambda)$ the anyonic graded YBE (6) becomes

$$R_{00'}(\lambda-\mu)R_{0n}(\lambda)R_{0'n}(\mu)=R_{0'n}(\mu)R_{0n}(\lambda)R_{00'}(\lambda-\mu), \quad (8)$$

or equivalently

$$\overset{\vee}{R}(\lambda-\mu)L_n(\lambda)\otimes_{\rm a}L_n(\mu)=L_n(\mu)\otimes_{\rm a}L_n(\lambda)\overset{\vee}{R}(\lambda-\mu). \quad (9)$$

In component form

$$\begin{aligned}&\overset{\vee}{R}(\lambda-\mu)^{a_1a_2}_{c_1c_2}L_n(\lambda)^{a_1a_n}_{b_1r_n}L_n(\mu)^{c_2r_n}_{b_2b_n}w(b_1,c_2)w^{-1}(c_1,c_2)=\\&L_n(\mu)^{a_1a_n}_{c_1r_n}L_n(\lambda)^{a_2r_n}_{c_2b_n}\overset{\vee}{R}(\lambda-\mu)^{c_1c_2}_{b_1b_2}w(c_1,a_2)w^{-1}(a_1,a_2).\end{aligned} \quad (10)$$

Let us define the monodromy matrix $T(\lambda)$ as the matrix product over the Lax operators on all sites of the lattice, i.e.

$$T(\lambda)=L_N(\lambda)L_{N-1}(\lambda)\cdots L_1(\lambda). \quad (11)$$

Here $T(\lambda)$ is a quantum operator valued matrix that acts non-trivially in the anyonic tensor product of a whole quantum space of the lattice and satisfies the global anyonic graded YBE

$$R(\lambda-\mu)T(\lambda)\otimes_{\rm a}T(\mu)=T(\mu)\otimes_{\rm a}T(\lambda)R(\lambda-\mu). \quad (12)$$

Consequently the transfer matrix $\tau(\lambda)=\mathrm{atr}[T(\lambda)]=\sum_{a=1}^{n}w(a,a)^{-1}T(\lambda)^a_a$ forms a commuting family for all values of the spectral parameters. Here atr is the anyonic graded trace carried out in the auxiliary space with n the dimension of the auxiliary space. It follows that the transfer matrix can be considered as the generating functional of the Hamiltonian and of an infinite number of higher conservation laws of the model.

3. The XXX model of hard core anyons

As a first step we consider the integrable hard-core anyon model with the Hamiltonian

$$H = \eta^{-1} \sum_{j=1}^{L} \left(a_{j+1}^{\dagger} a_j + a_j^{\dagger} a_{j+1} + 2 n_{j+1} n_j - 2 n_j \right), \tag{13}$$

where the operator $n_j = a_j^{\dagger} a_j$ is the number operator of hard-core anyons and $a_j^{\dagger}$ and a_j are the creation and annihilation hard-core anyon operators satisfying the commutation relations

$$\{a_j, a_j\} = \left\{a_j^{\dagger}, a_j^{\dagger}\right\} = 0 \quad \left\{a_j, a_j^{\dagger}\right\} = 1 \tag{14}$$

$$a_i^{\dagger} a_j = q a_j a_i^{\dagger}, \quad a_j^{\dagger} a_i = q^{-1} a_i a_j^{\dagger}. \tag{15}$$

$$a_j^{\dagger} a_i^{\dagger} = q a_i^{\dagger} a_j^{\dagger}, \quad a_j a_i = q a_i a_j. \tag{16}$$

Here we assume $i > j$ with $\{\ \}$ denoting the anticommutator as usual. We mention that the on-site hard-core anyons [8] preserve the Pauli exclusion principle whereas the off-site ones keep a free anyonic parameter when two particles exchange their positions. This model (more specifically the XXZ generalisation) was previously solved in [10] using the co-ordinate Bethe ansatz. Below, we will confirm that the same model arises through the anyonic QISM with the same solution obatined by the algebraic Bethe ansatz.

Consider the quantum R-matrix of the XXX model

$$\check{R}(\lambda) = \begin{pmatrix} \lambda+\eta & 0 & 0 & 0 \\ 0 & \eta & \lambda & 0 \\ 0 & \lambda & \eta & 0 \\ 0 & 0 & 0 & \lambda+\eta \end{pmatrix}, \tag{17}$$

where η is a quasiclassical parameter. If we choose the anyonic parity

$$w(1,1) = w(1,2) = w(2,1) = 1; \quad w(2,2) = q, \tag{18}$$

the anyonic grading Lax operator on site j is given by

$$L_j(\lambda) = \begin{pmatrix} \lambda + \eta(1-n_j) & \eta a_j^{\dagger} \\ \eta a_j & \lambda + (\lambda(q-1) + q\eta) n_j \end{pmatrix}. \tag{19}$$

It is shown that the Lax operator (19) constitutes the anyonic grading YBE (9). As a consequence, the monodromy matrix generates the global anyonic grading YBE (12). Then the integrals of motion of the model can be consequently obtained from the expansion of the transfer matrix in the spectral parameter λ. Explicitly,

$$\tau(\lambda) = (1 + H + \cdots)\tau(0), \tag{20}$$

where the Hamiltonian reads

$$H = \sum_{i=1}^{N-1} H_{ii+1} + H_{N1}. \tag{21}$$

Here

$$H_{ii+1} = L_{j+1}(0)L_j^{'}(0)L_j^{-1}(0)L_{j+1}^{-1}(0),$$
$$H_{N1} \ = \mathrm{atr}\left(L_N^{'}(0)L_N^{-1}(0)L_1^{-1}(0)\right). \tag{22}$$

The properties

$$P_{12}^{-1}\ \overset{1}{X}\ P_{12} = \overset{2}{X}, \quad P_{0j}^{-1}P_{0k}P_{0j} = P_{jk} \tag{23}$$

are applied in the above derivation. These properties imply a constraint on the grading function such that $w(\alpha,\beta) = w(\beta,\alpha)$. After a lengthy algebraic calculation, the explicit expression for the Hamiltonian density and the boundary terms is be given by (up to a constant)

$$H_{jj+1} = (1-n_{j+1})(1-n_j) + n_{j+1}n_j + a_{j+1}^{\dagger}a_j + a_j^{\dagger}a_{j+1}, \tag{24}$$
$$H_{N1} \ = (1-n_1)(1-n_N) + n_1n_N + a_1^{\dagger}a_N + a_N^{\dagger}a_1, \tag{25}$$

which constitute the periodic boundary condition for the model (13). To keep the Hamiltonian (13) hermitian, we restrict ourselves to $q^{\dagger} = q^{-1}$. We remark that this model covers the hard-core boson model and the fermion model as special choices of the anyonic gradings. For example, for $q = 1$ the model corresponds to a hard-core boson XXX model. Using the Matsubara and Matsuda transformations [22, 8], this hard-core model becomes the standard XXX vertex model. For $q = -1$ it is the $su(2)$ XXX fermion chain. After performing the standard algebraic Bethe ansatz, the transfer matrix eigenvalues are of the form

$$\Lambda(\lambda,\lambda_1\cdots\lambda_M) = (\lambda+\eta)^N\prod_{\alpha=1}^{M}\frac{\lambda-v_\alpha-\eta}{\lambda-v_\alpha} + \lambda^N q^{M-1}\prod_{\alpha=1}^{M}\frac{\lambda-v_\alpha+\eta}{\lambda-v_\alpha} \tag{26}$$

provided that

$$\left(\frac{v_\alpha+\eta}{v_\alpha}\right)^N = q^{M-1}\prod_{\beta\neq\alpha}^{M}\frac{v_\alpha-v_\beta+\eta}{v_\alpha-v_\beta-\eta}. \tag{27}$$

Here $\alpha = 1,\ldots,N$. If we perform a rescaling of the spectral parameter such that $v_\alpha \to v_\alpha/\mathrm{i} - \frac{\eta}{2}$ the energy spectrum is

$$E = -\eta\sum_{\alpha=1}^{M}\frac{1}{v_\alpha+\eta^2/4}, \tag{28}$$

where now the parameters v_α satisfy

$$\left(\frac{v_\alpha+\mathrm{i}\eta/2}{v_\alpha-\mathrm{i}\eta/2}\right)^N = q^{M-1}\prod_{\beta\neq\alpha}^{M}\frac{v_\alpha-v_\beta+\mathrm{i}\eta}{v_\alpha-v_\beta-\mathrm{i}\eta}. \tag{29}$$

Note the way in which the anyonic grading parameter q appears in the Bethe ansatz equations. It results in different distributions for the v_α than those for the standard XXX model, leading to subtle physical properties [10].

4. The $t-J$ model of hard-core anyons

Much work has been devoted during the last few decades towards a better understanding of integrable models of strongly correlated electrons. A prototypical model in this field is the integrable $t-J$ model [23], which is integrable for *supersymmetric* coupling [24, 25]. Here we present an integrable $t-J$ model of interacting hard-core anyons related to anyonic grading. The Hamiltonian reads

$$\eta H = t\sum_{j=1}^{L}\sum_{\alpha=\uparrow,\downarrow}\left(\tilde{a}^{\dagger}_{j\alpha}\tilde{a}_{j+1,\alpha}+h.c.\right) + J\left\{\sum_{j=1}^{L}\vec{S}_j\cdot\vec{S}_{j+1}+\frac{1}{4}n_jn_{j+1}\right\}+\sum_{j=1}^{L}(1-n_j)(1-n_{j+1}), \tag{30}$$

where $n_j = n_{j\uparrow}+n_{j\downarrow}$ is the number operator of single hard-core anyons with up and down spins. Here $\tilde{a}^{\dagger}_{j\alpha} = a^{\dagger}_{j\alpha}(1-n_{j,-\alpha})$, which prohibits double occupancy. The model is integrable when $t=2J$. These operators satisfy the commutation relations

$$\{a_{j\alpha},a_{j\alpha}\} = \left\{a^{\dagger}_{j\alpha},a^{\dagger}_{j\alpha}\right\} = 0 \quad \left\{a_{j\alpha},a^{\dagger}_{j\alpha}\right\} = 1, \tag{31}$$

$$\begin{aligned}
&a^{\dagger}_{i\downarrow}a_{j\downarrow} = q_1a_{j\downarrow}a^{\dagger}_{i\downarrow}, \quad a^{\dagger}_{i\uparrow}a_{j\uparrow} = q_2a_{j\uparrow}a^{\dagger}_{i\uparrow},\\
&a^{\dagger}_{i\uparrow}a_{j\downarrow} = q_3a_{j\downarrow}a^{\dagger}_{i\uparrow}, \quad a^{\dagger}_{i\downarrow}a_{j\uparrow} = q_3a_{j\uparrow}a^{\dagger}_{i\downarrow},\\
&a^{\dagger}_{j\downarrow}a^{\dagger}_{i\downarrow} = q_1a^{\dagger}_{i\downarrow}a^{\dagger}_{j\downarrow}, \quad a^{\dagger}_{j\uparrow}a^{\dagger}_{i\uparrow} = q_2a^{\dagger}_{i\uparrow}a^{\dagger}_{j\uparrow},\\
&a^{\dagger}_{j\downarrow}a^{\dagger}_{i\uparrow} = q_3a^{\dagger}_{i\uparrow}a^{\dagger}_{j\downarrow}, \quad a^{\dagger}_{j\downarrow}a^{\dagger}_{i\uparrow} = q_3a^{\dagger}_{i\downarrow}a^{\dagger}_{j\uparrow},
\end{aligned} \tag{32}$$

where $i>j$ is assumed. The spin operator is denoted by $\vec{S} = \frac{1}{2}a^{\dagger}_{\alpha}\vec{\sigma}_{\alpha\beta}c_{\beta}$, i.e.

$$S^+ = a^{\dagger}_{\uparrow}a_{\downarrow},\ S^- = a^{\dagger}_{\downarrow}a_{\uparrow},\ S^z = \frac{1}{2}(n_{\uparrow}-n_{\downarrow}). \tag{33}$$

However, the spin exchange interaction is given by

$$\vec{S}_j\cdot\vec{S}_{j+1} = \frac{1}{2}\left(\frac{q_3}{q_1}S^+_{j+1}S^-_j+\frac{q_1}{q_3}S^+_jS^-_{j+1}\right)+S^z_jS^z_{j+1} \tag{34}$$

which evidently depends on the commutation parameters of the hard-core anyons. Of course, we can present another equivalent form of the spin exchange terms,

$$\vec{S}_j\cdot\vec{S}_{j+1} = \frac{1}{2}\left(\frac{q_3}{q_2}S^+_{j+1}S^-_j+\frac{q_2}{q_3}S^+_jS^-_{j+1}\right)+S^z_jS^z_{j+1}, \tag{35}$$

with the operators

$$S^+ = a^{\dagger}_{\downarrow}a_{\uparrow},\ S^- = a^{\dagger}_{\uparrow}a_{\downarrow},\ S^z = \frac{1}{2}(n_{\downarrow}-n_{\uparrow}). \tag{36}$$

In the above q_i $i=1,2,3$ are arbitrary anyonic parameters with the property $q_i^{\dagger} = q_i^{-1}$ for keeping the Hamiltonian (30) hermitian. In this model, on-site interaction between the hard-core anyons preserves the Pauli exclusion principle. But anyonic phases associated with the exchange of two particles at different sites depend on their positions. We also see that anisotropic spin exchange interaction in the hard-core anyon $t-J$ model replaces the antiferromagnetic spin exchange in the standard supersymmetric $t-J$ model. These

free parameters act as anisotropic parameters characterizing the anyon spin interaction. They lead to new phase factors in the Bethe ansatz equations.

In order to link the anyonic grading supersymmetric $t-J$ model to Hamiltonian (30), we need to employ the $su(3)$ R-matrix [25]

$$\check{R}(u) = \begin{pmatrix} a(u) & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & c(u) & 0 & b(u) & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & c(u) & 0 & 0 & 0 & b(u) & 0 & 0 \\ 0 & b(u) & 0 & c(u) & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & a(u) & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & c(u) & 0 & b(u) & 0 \\ 0 & 0 & b(u) & 0 & 0 & 0 & c(u) & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & b(u) & 0 & c(u) & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & a(u) \end{pmatrix} \tag{37}$$

where

$$a(u) = u + \eta, \;\; b(u) = u, \;\; c(u) = \eta.$$

Given the anyonic grading

$$\begin{aligned} &w(1,1) = q_1, \;\; w(2,2) = q_2, \;\; w(1,2) = w(2,1) = q_3, \\ &w(3,1) = w(1,3) = w(3,2) = w(2,3) = w(3,3) = 1 \end{aligned} \tag{38}$$

one can show that the Lax operator

$$L_j(u) = \begin{pmatrix} \begin{array}{c} q_1(\eta+u)n_{j\downarrow} \\ +u(q_3 n_{j\uparrow} + 1 - n_j) \end{array} & q_3\eta a^\dagger_{j\uparrow} a_{j\downarrow} & \eta(1-n_{j\uparrow})a_{j\downarrow} \\ q_3\eta a^\dagger_{j\downarrow} a_{j\uparrow} & \begin{array}{c} u(q_3 n_{j\downarrow} + 1 - n_j) \\ +q_2(u+\eta)n_{j\uparrow} \end{array} & \eta(1-n_{j\downarrow})a_{j\uparrow} \\ \eta a^\dagger_{j\downarrow}(1-n_{j\uparrow}) & \eta a^\dagger_{j\uparrow}(1-n_{j\downarrow}) & u + \eta(1-n_j) \end{pmatrix} \tag{39}$$

generates the local anyonic grading YBE (9). As a consequence, the integrals of motion of the model can be obtained from the expansion of the transfer matrix in the spectral parameter u. Using expressions (21) and (22) with the above anyonic grading, the Hamiltonian (30) can be derived from the relation

$$\tau(\lambda) = (1 + H + \cdots)\tau(0). \tag{40}$$

In this way the integrability of the supersymmetric $t-J$ model of hard-core anyons (30) is guaranteed by the anyonic grading Yang-Baxter equations (4). Special choices of the grading parameters characterize different statistical mechanical models. For example, if $q_i = 1$ the model becomes the $su(3)$ Heisenberg model [25] in terms of hard-core bosons [8]. Whereas if $q_i = -1$ the model becomes a $su(2|1)$ fermion model. We see then that these parameters characterize different statistics and will result in different physical properties. We now turn to the nested algebraic Bethe ansatz [24] to derive the exact solution of the model.

Define

$$T(u) = L_L(u)\cdots L_1(u) = \begin{pmatrix} A_{11}(u) & A_{12}(u) & B_1(u) \\ A_{21}(u) & A_{22}(u) & B_2(u) \\ C_1(u) & C_2(u) & D(u) \end{pmatrix} \tag{41}$$

acting on the anyonic Hilbert space. We choose the vacuum state $|0\rangle = \prod_{i=1}^{L} \otimes_{\mathrm{c}} |0\rangle_i$ with $a_{i\alpha}|0\rangle = 0$. The nested algebraic Bethe ansatz solution of the usual supersymmetric $t-J$ model has been discussed at length in the literature [24], so here we highlight only the differences in the nesting structure caused by the anyonic grading. The commutation relations between the diagonal fields and the creation fields are

$$D(u_1)C_a(u_2) = \frac{a(u_2-u_1)}{b(u_2-u_1)}C_a(u_2)D(u_1) - \frac{c(u_2-u_1)}{b(u_2-u_1)}C_a(u_1)D(u_2) \tag{42}$$

$$\begin{aligned} A_{ab}(u_1)C_f(u_2) = {} & \frac{a(u_1-u_2)}{b(u_1-u_2)}\left\{r^{(1)}(u_1-u_2)^{bf}_{ed}w(e,a)C_e(u_2)A_{ad}(u_1)\right\} \\ & - \frac{c(u_1-u_2)}{b(u_1-u_2)}w(b,a)C_b(u_1)A_{ac}(u_2) \end{aligned} \tag{43}$$

with

$$\begin{aligned} & r^{aa}_{aa} = 1,\ a = 1,2,\ \ r^{ab}_{ab} = \frac{c(u)}{a(u)},\ a \neq b = 1,2, \\ & r^{ab}_{ba} = \frac{b(u)}{a(u)},\ a \neq b = 1,2. \end{aligned} \tag{44}$$

The anyonic grading functions appearing in the commutation relation (43) are kept in the nested transfer matrix for the spin degree of freedom. This makes the nested algebraic Bethe ansatz very complicated. We see however, that the first term in each of the commutation relations (42)–(43) contribute to the eigenvalues of the transfer matrix which should be analytic functions of the spectral parameter u. Consequently, the residues at singular points must vanish. This yields the Bethe-ansatz equations which in turn assure the cancellation of the unwanted terms in the eigenvalues of the transfer matrix. To this end, we choose Bethe state $|\Phi\rangle$ as

$$|\Phi\rangle = C_{g_1}(u_1)\cdots C_{g_N}(u_N)|0\rangle F^{g_N\cdots g_1}. \tag{45}$$

Following the standard procedure of the algebraic Bethe ansatz, the eigenvalue of the monodromy matrix acting on the state (45) is obtained as

$$\begin{aligned} \tau(u)|\Phi\rangle = \Lambda(u,\{u_i\})|\Phi\rangle = (u+\eta)^L \prod_{i=1}^{N}\frac{(u-u_i-\eta)}{(u-u_i)}|\Phi\rangle \\ + u^L \prod_{i=1}^{N}\frac{u-u_i+\eta}{u-u_i}\prod_{l=1}^{N}C_{gl}(u_l)|0\rangle \left[\tau^{(1)}(u)\right]^{h_1\cdots h_N}_{g_1\cdots g_N} F^{g_N\cdots g_1} \end{aligned} \tag{46}$$

provided that

$$\frac{(u+\eta)^L}{u^L}\prod_{\substack{l=1\\ l\neq i}}^{N}\frac{u_i-u_l-\eta}{u_i-u_l+\eta} = \left[\tau^{(1)}(u)\right]^{h_1\cdots h_N}_{g_1\cdots g_N}\ |_{u=u_i}\ . \tag{47}$$

In the above the nested transfer matrix is given by

$$\left[\tau^{(1)}(u)\right]_{g_1\cdots g_N}^{h_1\cdots h_N} = \mathrm{atr}_0\left(L_N^{(1)}(u-u_N)_{ah_N}^{d_{N-1}g_N} L_{N-1}^{(1)}(u-u_{N-1})_{d_{N-1}h_{N-1}}^{d_{N-2}g_{N-1}} \cdots L_2^{(1)}(u-u_2)_{d_2h_2}^{d_1g_2} L_1^{(1)}(u-u_1)_{d_1h_1}^{ag_1}\right). \tag{48}$$

where the local Lax operator reads

$$L_j^{(1)}(u) = \begin{pmatrix} q_1 n_\downarrow + \frac{b(u)}{a(u)} q_3 n_\uparrow & \frac{c(u)}{a(u)} q_3 a_\uparrow^\dagger a_\downarrow \\ \frac{c(u)}{a(u)} q_3 a_\downarrow^\dagger a_\uparrow & q_2 n_\uparrow + \frac{b(u)}{a(u)} q_3 n_\downarrow \end{pmatrix}. \tag{49}$$

This satisfies the anyonic grading YBE (12) with grading $w(1,1) = q_1$, $w(2,2) = q_2$, $w(1,2) = w(2,1) = q_3$. This realization of the nested Lax operator (49) paves the way to diagonalize the transfer matrix of the model. After some algebra, we obtain the eigenvalue of the transfer matrix in the form

$$\Lambda(u,\{u_i\}\{v_i\}) = (u+\eta)^L \prod_{i=1}^N \frac{(u-u_i-\eta)}{(u-u_i)} + u^L q_1^{M-1} q_3^{N-M} \prod_{l=1}^M \frac{u-v_l+\eta}{u-v_l} + u^L q_2^{N-M-1} q_3^M \prod_{i=1}^M \frac{u-u_i+\eta}{u-u_i} \prod_{l=1}^M \frac{u-v_l-\eta}{u-v_l}. \tag{50}$$

Here the quantum numbers N and M are the total number of hard-core anyons and the number of hard-core anyons with down spin, respectively. The parameters u_i and v_l characterize the charge and spin rapidities of the model. If making a rescaling $u_i \to u_i - \eta/2$, $v_i \to v_i - \eta$, the Bethe ansatz equations are given by

$$\frac{(u_i+\frac{\mathrm{i}\eta}{2})^L}{(u_i-\frac{\mathrm{i}\eta}{2})^L} = q_2^{N-M-1} q_3^M \prod_{\substack{l=1\\ l\neq i}}^N \frac{u_i-u_l+\mathrm{i}\eta}{u_i-u_l-\mathrm{i}\eta} \prod_{l=1}^M \frac{u_i-v_l-\frac{\mathrm{i}\eta}{2}}{u_i-v_l+\frac{\mathrm{i}\eta}{2}}, \tag{51}$$

$$q_1^{M-1} q_2^{-(N-M-1)} q_3^{N-2M} \prod_{i=1}^N \frac{v_j-u_i-\frac{\mathrm{i}\eta}{2}}{v_j-u_i+\frac{\mathrm{i}\eta}{2}} = \prod_{\substack{l=1\\ l\neq j}}^M \frac{v_j-v_l-\mathrm{i}\eta}{v_j-v_l+\mathrm{i}\eta},$$

for $i = 1,\ldots,N$ and $j = 1,\ldots,M$. In this way we have the energy spectrum

$$E = L - \eta^2 \sum_{i=1}^N \frac{1}{u_i^2+\frac{\eta^2}{4}}. \tag{52}$$

5. Conclusion

In summary, we have constructed a class of integrable models associated with anyonic grading. We found that these integrable models may be used to describe the interaction of hard-core anyons. With regard to the anyonic grading supersymmetric structure, we presented a unifying approach – the anyonic grading QISM – to treat this class of integrable models. We explicitly constructed integrable models of hard-core anyons associated with the XXX model and the $t-J$ model with anyonic grading. The exact solutions of these models were obtained by means of the algebraic Bethe ansatz. It

is seen that the phase functions associated with the exchange of two hard-core anyons at different sites lead to nontrivial phase factors in the Bethe ansatz equations. These phase factors encode the anyonic effects in these models. We hope to consider their ground state properties and thermodynamics elsewhere.

Acknowledgments

The authors thank the Australian Research Council for support. A Foerster thanks CNPq (Conselho Nacional de Desenvolvimento Científico e Tecnológico) and FAPERGS (Fundação de Amparo ã Pesquisa do Estado do Rio Grande do Sul) for financial support. H-Q Zhou acknowledges support from the National Natural Science Foundation of China (grant No. 10774197) and from the Natural Science Foundation of Chongqing (CSTC2008BC2023).

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